Hybrid Control Lyapunov Functions for the Stabilization of Hybrid Systems

Sterano Di Cairano
Mechatronics
Mitsubishi Electric Research Laboratories
Cambridge, MA 02139
dicairano@ieee.org

Maurice Heemels
Dept. Mechanical Engineering
Technical University of Eindhoven
P.O. Box 513
5600 MB Eindhoven, The Netherlands
m.heemels@tue.nl

Mircea Lazar
Dept. Electrical Engineering
Technical University of Eindhoven
P.O. Box 513
5600 MB Eindhoven, The Netherlands
m.lazar@tue.nl

Alberto Bemporad
IMT Institute for Advanced Studies
Piazza San Ponziano 6 55100
Lucca, Italy
alberto.bemporad@imtlucca.it

ABSTRACT

The design of stabilizing controllers for hybrid systems is particularly challenging due to the heterogeneity present within the system itself. In this paper we propose a constructive procedure to design stabilizing dynamic controllers for a fairly general class of hybrid systems. The proposed technique is based on the concept of a hybrid control Lyapunov function (hybrid CLF) that was previously introduced by the authors. In this paper we generalize the concept of hybrid control Lyapunov function, and we show that the existence of a hybrid CLF guarantees the existence of a standard control Lyapunov function (CLF) for the hybrid system. We provide a constructive procedure to design a hybrid CLF and the corresponding dynamic control law, which is stabilizing because of the established connection to a standard CLF that becomes a Lyapunov function for the closed-loop system. The obtained control law can be conveniently implemented by constrained predictive control in the form of a receding horizon control strategy. A numerical example highlighting the features of the proposed approach is presented.

HSCC'13, April 8–11, 2013, Philadelphia, Pennsylvania, USA.

Categories and Subject Descriptors

I.2.8 [**Artificial Intelligence**]: Problem Solving, Control Methods, and Search—*Control theory*; G.1.6 [**Numerical Analysis**]: Optimization—*Constrained Optimization*

General Terms

Theory, Algorithms

Keywords

Hybrid system, control Lyapunov function, dynamic controller, receding horizon control

1. INTRODUCTION

Hybrid systems [1,47] are powerful models for describing physical processes interacting with computer systems and embedded controllers, since they allow to formulate continuous and discrete dynamics in a unified framework. However, such modeling power is also associated with an inherent complexity in analyzing and manipulating the hybrid system, due to its heterogenous nature. While the concept of hybrid system was already introduced by Witsenhausen in the sixties [48], only in recent years tools to address stability analysis and control of certain classes of hybrid systems [8,21,23,25,34,38] have been developed. For the problem of stabilizing a hybrid system equilibrium, several techniques have been proposed for particular classes of hybrid systems, see, e.g., [11]. However, most of the available techniques can be applied only to switched systems [23,34], and piecewise affine systems [25,38,43], where the discrete dynamics are trivial, since the discrete state is uniquely determined by the current input, continuous state, and possibly external disturbances. In order to allow control applications such as robot tasks [5,17], industrial batch processes [18,33], and program executions in embedded and software-enabled control systems [4, 42] stabilizing controller design techniques for hybrid systems with more general discrete dynamics are needed. Unfortunately, the existing methodologies for designing stabilizing controllers in this case are still limited.

Some previous works on stability and stabilization of hybrid systems with discrete dynamics are the following. A stability analysis based on the *hybrid distance* was developed in [35] for hybrid automata [22]. In [7] an approach based on model predictive control guaranteeing attractivity of the equilibrium is presented. More recently, in [19] a novel perspective on the stability of hybrid systems is presented, based on the concepts of *hybrid time* and *graphic convergence*, together with conditions that guarantee stability of the closed-loop system. Existence results for stabilizing controllers are given in [19], but the problem on how to synthesize stabilizing controllers for hybrid systems still remains widely open.

In [14], the authors have proposed a design approach for stabilizing controllers for a hybrid system based on receding horizon control and on the concept of *hybrid control Lyapunov function*, that resulted in a specific notion of closed-loop stability. The control design was implementable for classes of systems that have been used in practical applications [6, 9, 13, 15, 40, 41]. In this paper we extend the ideas of [14] towards a general control design framework based on a formal notion of hybrid control Lyapunov function. We derive a fundamental result that shows that a hybrid CLF, which is significantly simpler to construct than a standard CLF, guarantees the existence of a standard control Lyapunov function [2, 26, 44] for the hybrid system. As a result, a controller synthesized based on the hybrid control Lyapunov function achieves (asymptotic) Lyapunov stability of the hybrid system in closed loop. A major advantage of the hybrid CLF approach is that we can derive a systematic procedure to construct the hybrid CLF and the related control law, thereby obtaining a systematic design procedure for stabilizing controllers for hybrid systems, without the need of constructing a standard CLF, which may be extremely difficult. By combining the hybrid CLF conditions with ideas from receding horizon control, a feasible implementation of the controller for a large class of systems [12, 20, 46] is also obtained.

The paper is structured as follows. In Section 2, we briefly review the basic notions of stability, control Lyapunov function and some notions of graph theory, and we formulate the stabilizing control design problem. In Section 3 we introduce the concept of hybrid control Lyapunov function, and we show that its existence guarantees the existence of a standard control Lyapunov function for the closed-loop system. In Section 4, starting from our previous results in [14], we propose a construction for the hybrid CLF and the consequent design of the stabilizing control law, and we implement the stabilizing controller by using a receding horizon constrained control strategy. In Section 5 we present a numerical example that highlights the features of the proposed approach, and in Section 6 we summarize the conclusions.

2. PRELIMINARIES AND PROBLEM DEFINITION

$\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{0+}$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_{0+}$ denote the set of real, positive real, and non-negative real, integer, positive integer, and non-negative integer numbers, respectively. For a countable set $\mathcal{S}$, $|\mathcal{S}|$ denotes its cardinality. We use the notation $\mathbb{Z}_{(c_1,c_2]}$, where $c_1, c_2 \in \mathbb{Z}$, (and similarly with $\mathbb{R}$) to denote the set $\{k \in \mathbb{Z} : c_1 < k \leq c_2\}$. The Hölder p-norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\|_p \triangleq (|[x]_1|^p + \ldots + |[x]_n|^p)^{\frac{1}{p}}$, if $p \in \mathbb{Z}_{[1,\infty)}$ and $\|x\|_\infty \triangleq \max_{i=1,\ldots,n} |[x]_i|$, where $[x]_i$, $i = 1, \ldots, n$, is the i-th component of x and $|\cdot|$ is the absolute value. By $\|\cdot\|$ we denote an arbitrary p-norm.

We denote a function from set $\mathcal{A}$ to set $\mathcal{B}$ by $\phi : \mathcal{A} \to \mathcal{B}$, and a set-valued function by $\phi : \mathcal{A} \rightrightarrows \mathcal{B}$. Given a system $x(k+1) = \phi(x(k), u(k))$, an initial state $x(0)$ and an input sequence $\mathbf{u}_N = (u_0, \ldots, u_{N-1})$, $N \in \mathbb{Z}_+$, $\mathbf{x}_N = (x_0, \ldots, x_N)$ is the sequence of states obtained from $x(0)$ following the application of the input sequence $\mathbf{u}_N$. For the simplicity of notation, we denote $\phi^j(x(0), \mathbf{u}_N) \triangleq x(j)$ for $j \in \mathbb{Z}_{[0,N]}$, where we use the convention $\phi^0(x(0), \mathbf{u}_N) \triangleq x(0)$. For two vectors $u \in \mathbb{R}^{n_u}$ and $v \in \mathbb{R}^{n_v}$, we sometimes write $(u, v) = [u' \ v']' \in \mathbb{R}^{n_u+n_v}$. In addition, with a little abuse of notation, we sometimes separate the discrete valued and the real (continuous) valued arguments of a function $f(x, u)$, i.e., given $x = [x_c' \ x_d']'$, $u = [u_c' \ u_d']'$, where x_c, u_c are the continuous components, and x_d, u_d are the discrete components of x and u, respectively, we write sometimes $f(x_c, x_d, u_c, u_d) \triangleq f(x, u)$.

2.1 Stability Notions

A function $\varphi : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. It *belongs to class* $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and $\varphi(s) \to \infty$ when $s \to \infty$. A function $\beta : \mathbb{R}_{0+} \times \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ *belongs to class* $\mathcal{KL}$ if for each $k \in \mathbb{R}_{0+}$, $\beta(\cdot, k) \in \mathcal{K}$, for each $s \in \mathbb{R}_{0+}$, $\beta(s, \cdot)$ is decreasing, and $\lim_{k\to\infty} \beta(s, k) = 0$.

Consider the hybrid system

$$x(k+1) = \begin{bmatrix} x_c(k+1) \\ x_d(k+1) \end{bmatrix} \in \begin{bmatrix} \Phi_c(x(k)) \\ \Phi_d(x(k)) \end{bmatrix} = \Phi(x(k)), \quad (1)$$

where $x(k) = [x_c(k)' \ x_d(k)']' \in \mathbb{X} \subseteq \mathbb{X}_c \times \mathcal{E}$ is the hybrid state at time $k \in \mathbb{Z}_{0+}$, with $x_c(k) \in \mathbb{R}^n$ the continuous part and $x_d(k) \in \mathcal{E}$ the discrete part, and $\mathcal{E} \triangleq \{\epsilon_1, \ldots, \epsilon_{n_d}\}$ is a finite set of symbols. The mapping $\Phi_c : \mathbb{X} \rightrightarrows \mathbb{X}_c$ is an arbitrary, possibly discontinuous, nonlinear set-valued function, that defines the continuous state dynamics of the hybrid system (1), and $\Phi_d : \mathbb{X} \rightrightarrows \mathcal{E}$ is an arbitrary set-valued function that defines the discrete state dynamics of the hybrid system (1)

Let $x^e = [x_c^{e\prime} \ x_d^{e\prime}]' \in \mathbb{R}^n \times \mathcal{E}$. If $\Phi(x^e) = \{x^e\}$, $x^e \in \mathbb{X}$ is an equilibrium for (1). In order to define asymptotic stability for discrete-time hybrid systems that exhibit both discrete and continuous dynamics, we introduce a distance function d_h in a hybrid state space. We first introduce a discrete distance function d_d for purely discrete state spaces.

Definition 1 [35] *Given a finite set* $\mathcal{E}$ *the* discrete distance *is the function* $d_d : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_{0+}$ *defined by*

$$d_d(x_d, y_d) \triangleq \begin{cases} 0 & \text{if} \quad x_d = y_d \\ 1 & \text{if} \quad x_d \neq y_d \end{cases}. \quad (2)$$

for $x_d, y_d \in \mathcal{E}$.

Definition 2 [35] *Given a hybrid state space* $\mathbb{X}$ *the* hybrid distance *is the function* $d_h : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_{0+}$ *defined by*

$$d_h(x, \chi) = \|x_c - \chi_c\| + d_d(x_d, \chi_d), \quad (3)$$

for $x = \left[\begin{smallmatrix} x_c \\ x_d \end{smallmatrix}\right] \in \mathbb{X}$ *and* $\chi = \left[\begin{smallmatrix} \chi_c \\ \chi_d \end{smallmatrix}\right] \in \mathbb{X}$.

Definition 3 *Consider hybrid system* (1) *and let* $x^e \in \mathbb{X}$ *satisfy* $\Phi(x^e) = \{x^e\}$. *The equilibrium* x^e *is called* asymptotically stable (AS) in $\mathbb{X}$ *for* (1), *if there exists a* $\mathcal{KL}$-*function* β *such that for any* $x(0) \in \mathbb{X}$ *all the trajectories generated by* (1) *satisfy*

$$d_h(x(k), x^e) \leq \beta(d_h(x(0), x^e), k), \ \forall k \in \mathbb{Z}_+. \tag{4}$$

Definition 3 is consistent with [35], and it coincides with the stability definition of purely continuous or purely discrete systems in case (1) is a purely continuous or purely discrete system, respectively.

2.2 Lyapunov Functions and Control Lyapunov Functions

Definition 4 *A set* $\mathcal{P} \subseteq \mathbb{X}_c \times \mathcal{E}$ *is called* positively invariant (PI) *for system* (1) *if for all* $x \in \mathcal{P}$, $\Phi(x) \subseteq \mathcal{P}$.

Theorem 1 *Let* $\mathbb{X}$ *be a PI set for* (1) *with* $x^e \in \mathbb{X}$. *Let* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathbb{R}_{[0,1)}$, *and let* $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}_{0+}$ *be a function such that*

$$\alpha_1(d_h(x, x^e)) \leq \mathcal{V}(x) \leq \alpha_2(d_h(x, x^e)) \tag{5a}$$

$$\mathcal{V}(x^+) \leq \rho \mathcal{V}(x) \tag{5b}$$

for all $x \in \mathbb{X}$, *and all* $x^+ \in \Phi(x)$. *Then,* x^e *is AS for* (1) *in* $\mathbb{X}$.

The proof of Theorem 1 is similar in nature to the proof given in [29,32] by replacing the continuous difference *equation* with the hybrid difference *inclusion* (1), and hence it is omitted here. The proof can also be obtained by following [27], which discusses robust stability of discrete-time difference inclusions. A function $\mathcal{V}$ that satisfies the hypothesis of Theorem 1 is called a *Lyapunov function* for hybrid system (1).

Consider now the discrete-time hybrid system with control inputs described by the difference equation

$$\begin{aligned} x(k+1) &= \begin{bmatrix} x_c(k+1) \\ x_d(k+1) \end{bmatrix} \\ &= \begin{bmatrix} \phi_c(x(k), u(k)) \\ \phi_d(x(k), u(k)) \end{bmatrix} = \phi(x(k), u(k)), \end{aligned} \tag{6}$$

where $x(k) \in \mathbb{X} \subseteq \mathbb{X}_c \times \mathcal{E}$, $u(k) \in \mathbb{U} \subseteq \mathbb{U}_c \times \mathcal{E}_u$ are the state and input at $k \in \mathbb{Z}_{0+}$, and $\mathcal{E}_u \triangleq \{\epsilon_{u_1}, \dots, \epsilon_{u_{m_d}}\}$ is a finite set of input symbols. In (6), $\phi : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is an arbitrary, possibly discontinuous, nonlinear function. Assume that for a desired equilibrium $x^e = [x_c^{e\prime} \ x_d^{e\prime}]' \in \mathbb{X}$ there exists $u^e = [u_c^{e\prime} \ u_d^{e\prime}]' \in \mathbb{U}$ such that $\phi(x^e, u^e) = x^e$.

Definition 5 *A function* $\mathcal{V}_h : \mathbb{X} \to \mathbb{R}_{0+}$ *that satisfies* (5a) *for some* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and for which there exists* $\rho \in \mathbb{R}_{[0,1)}$ *such that for all* $x \in \mathbb{X}$, *there exists* $u \in \mathbb{U}$ *such that* $\phi(x, u) \in \mathbb{X}$ *and*

$$\mathcal{V}_h(\phi(x, u)) \leq \rho \mathcal{V}_h(x), \tag{7}$$

is called a control Lyapunov function *(CLF) for* x^e *in* $\mathbb{X}$ *for* (6).

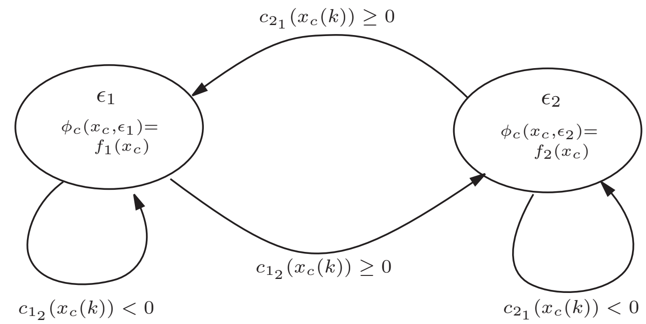


Figure 1: Graphical representation of a simple hybrid system with a graph associated to the discrete dynamics.

Given the CLF $\mathcal{V}_h$, define the control law

$$u(k) \in R(x(k)), \ k \in \mathbb{Z}_{0+}, \tag{8}$$

where $R : \mathbb{X} \rightrightarrows \mathbb{U}$ satisfies for all $x \in \mathbb{X}$,

$$\emptyset \neq R(x) \subseteq \Gamma(x) := \{u \in \mathbb{U} : \phi(x, u) \in \mathbb{X}, \text{ and (7) hold}\}. \tag{9}$$

By using (8) in (6), we obtain the closed-loop system

$$x(k+1) \in \phi(x(k), R(x(k))) \triangleq \{\phi(x(k), u) : u \in R(x(k))\}. \tag{10}$$

Theorem 2 *Consider* (6) *and* $x^e = [x_c^{e\prime} \ x_d^{e\prime}]' \in \mathbb{X}$, *where there exists* $u^e \in \mathbb{U}$ *such that* $\phi_c(x^e, u^e) = x^e$. *Suppose that there exists a CLF for* x^e *in* $\mathbb{X}$ *for* (6). *Then,* x^e *is asymptotically stable in* $\mathbb{X}$ *for* (10).

Theorem 2 is a consequence of Theorem 1 as $\mathbb{X}$ is PI for (10) by the definition of R. Theorem 2 shows that once a CLF is found, the construction of controller (8) satisfying (9) for all $x \in \mathbb{X}$ is immediate. If (6) consists only of continuous dynamics, we obtain a continuous CLF as defined in [26].

2.3 Graph Notions

A directed graph $G = (V, E)$ is described by the set of *nodes* $V = \{v_1, \dots, v_s\}$ and the set of *edges* $E \subseteq (V \times V)$, where $e_{ij} \in E$, $e_{ij} = (v_i, v_j)$ is the edge from node $v_i \in V$ to node $v_j \in V$.

In this paper, the discrete dynamics of (6) are considered from the perspective of an automaton (see the example in Figure 1) with states $\mathcal{E} = \{\epsilon_1, \dots, \epsilon_{n_d}\}$ and transitions $T \subseteq \mathcal{E} \times \mathcal{E}$, where $(\epsilon_i, \epsilon_j) \in T$ if and only if there exists $u \in \mathbb{U}$ and $x = [x_c' \ x_d']'$, such that $x_d = \epsilon_i$, $\phi(x, u) \in \mathbb{X}$ and $\phi_d(x, u) = \epsilon_i$. A graph is associated to the automaton, where each node $v \in V$ is bijectively associated to a state $\epsilon \in \mathcal{E}$ and each edge $e \in E$ is bijectively associated to a transition $t \in T$. The discrete distance (2) can be used as a distance measure between the nodes of the graph. However, for the graph associated to ϕ_d, when using the discrete distance (2) all the states appear to be equally far from the target state x_d^e, except the target state itself. Thus, this distance is less useful for measuring how close (in terms of discrete transitions) to the target state a certain state is. Therefore, we introduce a different notion of distance, the *graph distance*.

Definition 6 *Given a graph $G = (V, E)$, a* graph path *from $v_r \in V$ to $v_t \in V$, is a sequence of vertices $\tau = (\nu^{(0)}, \dots, \nu^{(\ell)})$, $\ell \in \mathbb{Z}_{0+}$, where $\nu^{(j)} \in V$ for $j \in \mathbb{Z}_{[0,\ell]}$, $(\nu^{(j)}, \nu^{(j+1)}) \in E$ for $j \in \mathbb{Z}_{[0,\ell-1]}$, and $\nu^{(0)} = v_r$, $\nu^{(\ell)} = v_t$. The length of the path is $\mathcal{L}(\tau) \triangleq \ell$, i.e., the number of edges traversed along τ from v_r to v_s.*

For $v_r, v_t \in V$, let $\mathcal{T}_{r,t}$ denote the set of all graph paths from v_r to v_t.

Definition 7 *Given a directed graph $G = (V, E)$, the* graph distance *between $v_r, v_t \in V$ is the length of the shortest graph path between them, i.e., for $v_r \neq v_t$, if $\mathcal{T}_{r,t} \neq \emptyset$, $d(v_r, v_t) \triangleq \min_{\tau \in \mathcal{T}_{r,t}} \mathcal{L}(\tau)$, and if $\mathcal{T}_{r,t} = \emptyset$, $d(v_r, v_t) \triangleq \infty$. For $v_r = v_t$, $d(v_r, v_t) \triangleq 0$.*

The graph distance, which represents the minimum number of edges to travel between two nodes is a proper distance function on undirected graphs, but it lacks the symmetry property on directed graphs, since in general $d(v_r, v_t) \neq d(v_t, v_r)$. However, this does not impact our use of the graph distance. For a given graph $G(V, E)$, for all $v_r, v_t \in V$ the graph distance $d(v_r, v_t)$ can be computed using, for instance, Dijkstra's algorithm [16].

2.4 Problem Definition

Consider hybrid system (6), where, for $k \in \mathbb{Z}_{0+}$, $x(k) \in \mathbb{X} \subseteq \mathbb{X}_c \times \mathcal{E}$, $\mathbb{X}_c \subseteq \mathbb{R}^n$ and $\mathcal{E} = \{\varepsilon_1, \dots, \varepsilon_{n_d}\}$, and $u(k) \in \mathbb{U} \subseteq \mathbb{U}_c \times \mathcal{E}_u$ with $\mathbb{U}_c \subseteq \mathbb{R}^m$ and $\mathcal{E}_u = \{\epsilon_{u_1}, \dots, \epsilon_{u_{m_d}}\}$. The sets $\mathbb{X}$ and $\mathbb{U}$ define the ranges of states and inputs, respectively, which possibly model system constraints. While state and input constraints defined by $\mathbb{X}$ and $\mathbb{U}$ are independent from each other, this condition is introduced here only to simplify the notation, and it can be easily relaxed. Given $\epsilon \in \mathcal{E}$, $\mathcal{X}_h(\epsilon) \triangleq \{x \in \mathbb{X} : x_d = \epsilon\}$ is the set of hybrid states where the discrete state is ϵ, and obviously $\mathbb{X} = \bigcup_{\epsilon_i \in \mathcal{E}} \mathcal{X}_h(\epsilon)$. Furthermore, $\mathcal{X}_h(\epsilon) = \mathcal{X}_c(\epsilon) \times \{\epsilon\}$, where $\mathcal{X}_c(\epsilon) \triangleq \{x_c \in \mathbb{R}^n : \left[\begin{smallmatrix} x_c \\ \epsilon \end{smallmatrix}\right] \in \mathbb{X}\}$ is the set of continuous states compatible with discrete state ϵ, sometimes referred to as the *domain of* ϵ.

We consider the stabilization of the desired (closed-loop) equilibrium $x^e = [x_c^{e\prime}\ x_d^{e\prime}]' \in \mathbb{X}$, for which there exists $u^e \in \mathbb{U}$ such that $\phi(x^e, u^e) = x^e$. The general problem that this paper addresses is to provide a *constructive* procedure to design a controller such that x^e is asymptotically stable for (6) in closed loop with the designed controller. In Section 2.2 we have described how such a control law can be obtained from a CLF. However, the direct derivation of a CLF for (6) is far from trivial.

In order to stabilize the desired equilibrium x^e of hybrid system (6), we consider the following class of *dynamic* controllers

$$z(k+1) = \psi(x(k), z(k), u(k), v(k)), \tag{11a}$$

$$\begin{bmatrix} u(k) \\ v(k) \end{bmatrix} \in R(x(k), z(k)), \tag{11b}$$

where $z \in \mathcal{Z} \subseteq \mathbb{R}^{n_z}$ is the controller state with dynamics defined by (11a), $v \in \mathbb{V}$ is an additional (endogenous) control input, and (11b) defines the set-valued command as a function of x and z. The problem addressed in this paper is formulated as follows.

Problem: Stabilizing Feedback Control Design. Given a desired equilibrium $x^e \in \mathbb{X}$ for (6) with $u^e \in \mathbb{U}$ satisfying $\phi(x^e, u^e) = x^e$, synthesize (11) such that there exist a non-trivial set $\Xi \subseteq \mathbb{X} \times \mathcal{Z}$, and $z^e \in \mathcal{Z}$, such that (x^e, z^e) is an asymptotically stable equilibrium in Ξ for the closed-loop system (6), (11).

At a conceptual level, the approach that we take in this paper is first to appropriately select the controller dynamics (11a) such that the interconnection of (6) and (11a) allows for a CLF $\mathcal{V}_h : \Xi \to \mathbb{R}_{0+}$, and then to choose the feedback R such that (9) is satisfied for (6), (11). The choice of z and the construction of the dynamics (11a) are major contributions of the paper, next to crafting the CLF in a systematic manner. In fact, the CLF is built in a compositional manner based on a so-called "hybrid CLF". Before presenting the hybrid CLF approach in the next section, note that the proposed approach is different from the standard CLF-based stabilization, which typically results in a *static* state feedback law $u(k) = \gamma(x(k))$, while (11) is a dynamic controller.

3. HYBRID CONTROL LYAPUNOV FUNCTIONS

Given a desired equilibrium $x^e \in \mathbb{X}$, we construct (11) using a so-called hybrid control Lyapuonv function (hybrid CLF). In [14] we introduced a specific type of hybrid CLF for the first time. Here, we generalize the concept of hybrid CLF and we show that it induces a CLF $\mathcal{V}_h$ consistent with Definition 5 for the interconnection of (6) and (11a), and that can be used for constructing R in (11b).

3.1 Definition of Hybrid CLF

A hybrid control Lyapunov function is defined as follows.

Definition 8 *A hybrid CLF for (6), (11a) for $(x^e, z^e) \in \Xi \subset \mathbb{X} \times \mathcal{Z}$ is a triple $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$, where $\mathcal{V}_c : \mathbb{X}_c \to \mathbb{R}_{0+}$, $\mathcal{V}_d : \mathcal{E} \to \mathbb{R}_{0+}$ and $\mathcal{V}_z : \mathcal{Z} \to \mathbb{R}_{0+}$ satisfy the bounds*

$$\alpha_1^c(\|x_c - x_c^e\|) \le \mathcal{V}_c(x_c) \le \alpha_2^c(\|x_c - x_c^e\|),\ \forall x_c \in \mathbb{X}_c, \tag{12a}$$

$$\alpha_1^d(d_d(x_d, x_d^e)) \le \mathcal{V}_d(x_d) \le \alpha_2^d(d_d(x_d, x_d^e)),\ \forall x_d \in \mathcal{E}, \tag{12b}$$

$$\alpha_1^z(\|z - z^e\|) \le \mathcal{V}_z(z) \le \alpha_2^z(\|z - z^e\|),\ \forall z \in \mathcal{Z}, \tag{12c}$$

for some $\alpha_1^c, \alpha_2^c, \alpha_1^d, \alpha_2^d, \alpha_1^z, \alpha_2^z \in \mathcal{K}_\infty$. Moreover, for each $(x, z) \in \Xi$ there must exist $(u, v) \in \mathbb{U} \times \mathbb{V}$ such that

$$(\phi(x, u), \psi(x, z, u, v)) \in \Xi \tag{13}$$

and

$$\begin{cases} \mathcal{V}_c(\phi_c(x, u)) \le \rho_c \mathcal{V}_c(x_c) + M_c \\ \mathcal{V}_z(\psi(x, z, u, v)) \le \mathcal{V}_z(z) - 1 \\ \mathcal{V}_d(\phi_d(x, u)) \le \mathcal{V}_d(x_d) \end{cases} \text{if} \quad x_d \neq x_d^e \tag{14a}$$

$$\begin{cases} \mathcal{V}_c(\phi_c(x, u)) \le \rho_c \mathcal{V}_c(x_c) \\ \mathcal{V}_z(\psi(x, z, u, v)) \le \rho_z \mathcal{V}_z(z) \\ \mathcal{V}_d(\phi_d(x, u)) \le \mathcal{V}_d(x_d) \end{cases} \text{if} \quad x_d = x_d^e, \tag{14b}$$

for some constants $\rho_c, \rho_z \in [0, 1)$, $M_c \in \mathbb{R}_{0+}$.

Roughly speaking, (14) imposes that $\mathcal{V}_c$ is a local CLF for the continuous dynamics of (6) once the discrete state is equal to the desired discrete state (as in (14b)), $\mathcal{V}_z$ is a CLF for the controller dynamics (11a), and $\mathcal{V}_d$ for the discrete dynamics of (6), although no strict decrease is required. The three components of a hybrid CLF can be combined to obtain a standard CLF $\mathcal{V}_h$ for (6) and (11a) in the sense of Definition 5, hence justifying the name "hybrid CLF". In this way, by obtaining a procedure for constructing a hybrid CLF, which will be shown later to be easier than obtaining a standard CLF due to its compositional nature, a constructive procedure for the design of stabilizing controllers is defined.

3.2 Existence of a Hybrid CLF Guarantees Existence of a Standard CLF

Before proving that a hybrid CLF induces a standard CLF, we need the following technical lemma[1].

Lemma 1 *Let a hybrid CLF* $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ *for* $(x^e, z^e) \in \Xi \subset \mathbb{X} \times \mathcal{Z}$ *be given for system* (6), (11a)*, and assume* $\mathcal{Z}$ *is a bounded set. Consider the function* $\mathcal{V}_D : \mathcal{E} \times \mathcal{Z} \to \mathbb{R}_{0+}$,

$$\mathcal{V}_D(x_d, z) = \mathcal{V}_d(x_d) + \mathcal{V}_z(z),\ (x_d, z) \in \mathcal{E} \times \mathcal{Z}.$$

Then, there exist $0 < \lambda_1 < 1$ *and* $0 < \lambda_2 < 1$ *such that for all* $(x, z) \in \Xi$ *with* $x_d \neq x_d^e$ *there exists* $(u, v) \in \mathbb{U} \times \mathbb{V}$ *such that*

$$\mathcal{V}_D(\phi_d(x, u), \psi(x, z, u, v)) \leq \lambda_1 \mathcal{V}_D(x_d, z) - \lambda_2.$$

Theorem 3 *Let a hybrid CLF* $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ *for* $(x^e, z^e) \in \Xi \subseteq \mathcal{X} \times \mathcal{Z}$ *be given, and assume* $\mathcal{Z}$ *is bounded. Then, for a sufficiently large* $\alpha \in \mathbb{R}_+$*, the function* $\mathcal{V}_h : \Xi \to \mathbb{R}_{0+}$*, given by*

$$\mathcal{V}_h(x, z) = \alpha \mathcal{V}_D(x_d, z) + \mathcal{V}(x_c), \tag{15}$$

where $(x, z) \in \Xi$ *and* $\mathcal{V}_D$ *is as in Lemma 1, is a CLF for* (6), (11) *for* (x^e, z^e) *in* Ξ.

From Theorem 3, the next corollary follows immediately.

Corollary 1 *Let a hybrid CLF* $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ *for* $(x^e, z^e) \in \Xi \subseteq \mathbb{X} \times \mathcal{Z}$ *be given, and assume* $\mathcal{Z}$ *is bounded. Consider a CLF* $\mathcal{V}_h$ *for* (6) *and* (11) *for* $(x^e, z^e) \in \Xi$ *obtained as in Theorem 3 for a sufficiently large* $\alpha > 0$*. Then, there exists* $0 \leq \rho_h < 1$ *such that if* $(u, v) \in \mathbb{U} \times \mathbb{V}$ *satisfies* (13)-(14) *for* $(x, z) \in \Xi$*, then* $(u, v) \in \mathbb{U} \times \mathbb{V}$ *satisfies*

$$\mathcal{V}_h(\phi(x, u), \psi(x, z, u, v)) \leq \rho_h \mathcal{V}_h(x, z)), \tag{16a}$$

$$(\phi(x, u), \psi(x, z, u, v)) \in \Xi. \tag{16b}$$

Corollary 1 is instrumental for designing $R : \Xi \to \mathbb{U} \times \mathbb{V}$ in (11b), since it guarantees that for $(x, z) \in \Xi$, if $(u, v) \in \mathbb{U} \times \mathbb{V}$ is chosen such that the hybrid CLF conditions (13)-(14) are satisfied, the standard CLF conditions (16) are satisfied for the standard CLF $\mathcal{V}_h$.

3.3 Stabilizing Dynamic Controller

Due to Theorem 2 and Theorem 3, if $R : \Xi \to \mathbb{U} \times \mathbb{V}$ is chosen according to (9) for $\mathcal{V}_h$, $(x^e, z^e) \in \Xi$ is asymptotically stable for (6), (11). For $(x, z) \in \Xi$, Corollary 1 shows that if R is chosen as

$$R(x, z) := \{(u, v) \in \mathbb{U} \times \mathbb{V} \mid (13) - (14)\}, \tag{17}$$

then (9) is satisfied, and hence $(x^e, z^e) \in \Xi$ is asymptotically stable for (6), (11).

Thus, we obtain the following corollary.

Corollary 2 *Let a hybrid CLF* $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ *for* $(x^e, z^e) \in \Xi \subseteq \mathbb{X} \times \mathcal{Z}$ *be given for the system* (6),(11a)*, and assume* $\mathcal{Z}$ *is bounded. If* $R : \Xi \to \mathbb{U} \times \mathbb{V}$ *is chosen as in* (17)*, then* (x^e, z^e) *is asymptotically stable in* Ξ *for the closed-loop system* (6), (11).

4. DESIGN PROCEDURE FOR STABILIZING CONTROLLERS

In this section we provide a systematic method to design (11) that stabilizes (x^e, z^e) in Ξ, based on appropriately selecting (11a) and choosing the hybrid CLF for the hybrid dynamics (6), (11a). The procedure results in a hybrid CLF similar to the one originally proposed in [14], with some important modifications. Then $R : \Xi \to \mathbb{U} \times \mathbb{V}$ as in (11b) will follow automatically via Corollary 2. The ingredients we have to select for specifying (11) according to this procedure are z, z^e, $\mathcal{Z}$, ψ, Ξ, $\mathcal{V}_c$,$\mathcal{V}_d$, and $\mathcal{V}_z$ in (11).

4.1 Controller Dynamics

To specify the controller dynamics (11a), and in particular z, $\mathcal{Z}$, v, and $\mathbb{V}$, consider a desired equilibrium $x^e \in \mathbb{X}$ with equilibrium input $u^e \in \mathbb{U}$, i.e., $\phi(x^e, u^e) = x^e$, and an input sequence $\mathbf{u}_N(k) = (u_0(k), \ldots, u_{N-1}(k)) \in \mathbb{U}^N$ at time $k \in \mathbb{Z}_{0+}$. In (11), define $v(k)$ such that $(u(k), v(k)) = \mathbf{u}_N(k)$, so that $u(k) = u_0(k) \in \mathbb{U}$, and $v(k) = (u_1(k), \ldots, u_{N-1}(k)) \in \mathbb{U}^{N-1}$. Hence, $\mathbb{V} \triangleq \mathbb{U}^{N-1}$. Then, define the controller dynamics as

$$\psi(x, z, u, v) \triangleq \sum_{j=1}^{N} d(\phi_d^j(x, \mathbf{u}_N), x_d^e). \tag{18}$$

Hence, at time $k \in \mathbb{Z}_{0+}$, the controller state update is

$$z(k+1) = \psi(x(k), u(k), v(k)) = \psi(x(k), \mathbf{u}_N(k)). \tag{19}$$

Equation (18) defines the next controller state $z(k+1)$ as the cumulated graph distance from step 1 to step N along the trajectory generated from $x(k)$ following the application of $\mathbf{u}_N(k)$. Note that $\mathcal{Z} = \mathbb{R}_{[0,c_z]}$, $c_z = N \max_{x_d \in \mathcal{E}} d(x_d, x_d^e)$.

For the subsequent discussion it is important to notice that, by (19), the first element of the summation in (18) for $z(k)$ is $d(x_d(k), x_d^e)$, when $k \in \mathbb{Z}_+$. Hence, if $z(k) = 0$ for $k \in \mathbb{Z}_+$, then necessarily $x_d(k) = x_d^e$. In fact, we take $z^e = 0$, which satisfies for $(u^e, v^e) = \mathbf{u}_N = (u^e, \ldots, u^e)$ that $\psi(x^e, z^e, u^e, v^e) = 0 = z^e$. Hence, (x^e, z^e) serves as the desired equilibrium for (6), (11a), as we already have that $\phi(x^e, u^e) = x^e$.

4.2 Construction of Hybrid CLF

The component $\mathcal{V}_d$ of the hybrid CLF (14) related to the discrete state of the hybrid system is defined by using the discrete distance (2) as

$$\mathcal{V}_d(x_d) = d_d(x_d, x_d^e). \tag{20}$$

Thus, the constraints on $\mathcal{V}_d$ in (14) require now that

$$d_d(\phi_d(x, u), x_d^e) \leq d_d(x_d, x_d^e). \tag{21}$$

This condition is guaranteed by the following.

[1]In this paper the technical proofs are omitted due to space limitations. The statements of lemmas, theorems and corollaries provide an effective guidance towards the rationale of the control design properties.

Assumption 1 *For any* $x \in \mathcal{X}_h(x_d^e)$ *there exists* $u \in \mathbb{U}$ *such that* $\phi(x,u) \in \mathcal{X}_h(x_d^e)$*.*

Assumption 1 requires that for any hybrid state where the discrete state is at the target, there exists an input that keeps the discrete state at the target state.

The component $\mathcal{V}_z$ of the hybrid CLF is defined as

$$\mathcal{V}_z(z) = \|z\| = z. \tag{22}$$

where the second equality holds due to z being always non-negative. Given the controller dynamics (18), the conditions (14) on the hybrid CLF impose the constraints

$$\psi(x,z,\mathbf{u}_N) \leq z-1 \quad \text{if} \quad x_d \neq x_d^e \tag{23a}$$
$$\psi(x,z,\mathbf{u}_N) \leq \quad \rho_z z \quad \text{if} \quad x_d = x_d^e\,, \tag{23b}$$

for all $(x,z) \in \Xi$ and some constant $0 < \rho_z < 1$. Constraint (23) is called the *cumulative graph distance contraction* (CGDC) requirement, and it can be seen as a relaxation of

$$d(\phi_d(x,u),x_d^e) \leq \rho_d d(x_d,x_d^e),\ 0 \leq \rho_d < 1, \tag{24}$$

that would require the discrete state to come closer to the equilibrium at every time step. Constraint (24) would be difficult to enforce in most practical systems the discrete state cannot change at *every* step. In order to guarantee feasibility of (23) we introduce the following assumption.

Assumption 2 *Let* $x^e \in \mathbb{X}$*. For any discrete state* $x_d \in \mathcal{E}\setminus\{x^e\}$ *there exists* $n_g \in \mathbb{Z}_{0+}$ *such that for any* $x \in \mathcal{X}_h(x_d)$*, there exists* $\bar{x}_d \in \mathcal{E}$*, where* $d(\bar{x}_d,x_d^e) < d(x_d,x_d^e)$*, and an input sequence* $\mathbf{u}_\ell \in \mathbb{U}^\ell$*, such that: (i)* $\ell \leq n_g$*; (ii)* $\phi^q(x,\mathbf{u}_\ell) \in \mathbb{X}$*,* $\phi_d^q(x,\mathbf{u}_\ell) = x_d$*,* $q \in \mathbb{Z}_{[1,\ell-1]}$*; (iii)* $\phi_d^{\ell}(x,\mathbf{u}_\ell) = \bar{x}_d$*.*

Assumption 2 requires the existence of a horizon ℓ such that the discrete state gets closer to x_d^e, when in discrete state x_d. In general, if Assumption 2 is satisfied, the horizon n_g required is shorter than the one required by the approach in [7], because reachability of a discrete state closer to the equilibrium, rather than reachability of the equilibrium itself is required.

Definition 9 *Given* $x_d \in \mathcal{E}$*, the minimum graph distance progress horizon* $n(x_d) \in \mathbb{Z}_{0+}$ *for* $x_d \in \mathcal{E}$ *is the minimum value* n_g *for which Assumption 2 holds for* x_d*, where we use* $n(x_d^e) \triangleq 0$*.*

In Definition 9, $n(x_d)$ is the minimum horizon needed for the discrete state to get closer to x_d^e, with respect to the graph distance. The value $n(x_d)$ can be computed by offline backward reachability analysis (see, e.g., [3,10,39,45]). For the proposed approach backward reachability analysis is computationally viable since by Assumption 2 the discrete state remains constant, hence we only have to verify reachability for a constrained continuous system.

The final component in the hybrid CLF $(\mathcal{V}_c,\mathcal{V}_d,\mathcal{V}_z)$ is

$$\mathcal{V}_c : \mathbb{X}_c \to \mathbb{R}_{0+}, \tag{25}$$

which by (14), should satisfy that for x with $(x,z) \in \Xi$ there exists $u \in \mathbb{U}$ such that

$$\mathcal{V}_c(\phi_c(x,u)) \leq \rho_c \mathcal{V}_c(x_c) + M_c \quad \text{if} \quad x_d \neq x_d^e \tag{26a}$$
$$\mathcal{V}_c(\phi_c(x,u)) \leq \rho_c \mathcal{V}_c(x_c) \quad \text{if} \quad x_d = x_d^e\,, \tag{26b}$$

where $\rho_c \in \mathbb{R}_{[0,1)}$ and $M_c \in \mathbb{R}_+$ are appropriately selected constants. In fact, (26a) implies that $\mathcal{V}_c$ is a standard CLF of the continuous dynamics locally around the equilibrium x^e, and only for the dynamics associated to $x_d = x_d^e$. Finding CLFs for continuous dynamics is a well-studied problem [2,26,44], and is significantly simpler than the design of a (global) CLF for the hybrid system. Techniques for calculating local CLFs based on infinity norms are discussed, for instance, in [30,31], while techniques for computing local CLFs based on quadratic forms are discussed, for instance, in [24,28,31]. We adopt the following assumption regarding $\mathcal{V}_c$.

Assumption 3 *There exists* $\mathcal{V}_c$ *as in* (25) *and* $0 \leq \rho_c < 1$*, such that the bounds* (12a) *are satisfied for some* $\alpha_1^c, \alpha_2^c \in \mathcal{K}_\infty$*, and for all* $x \in \mathcal{X}_h(x_d^e)$ *there exists* $u \in \mathbb{U}$ *such that*

$$\phi(x,u) \in \mathcal{X}_h(x_d^e), \tag{27a}$$
$$\mathcal{V}(\phi_c(x,u)) \leq \rho_c \mathcal{V}(x_c). \tag{27b}$$

In addition, we assume that $\sup_{x\in\mathbb{X}} \mathcal{V}_c(x_c) < \infty$*.*

Two observations are in order. First of all, note that Assumption 3 implies Assumption 1. Second, note that to guarantee (26b) we can set $M_c = \sup_{x\in\mathbb{X}} \mathcal{V}_c(x_c)$.

To prove that Assumptions 2, 3 imply that $(\mathcal{V}_c,\mathcal{V}_d,\mathcal{V}_z)$ is indeed a hybrid CLF for (6), (11a), we first state the following lemma.

Lemma 2 *Suppose Assumptions 1 and 2 hold. Given* $x \in \mathbb{X}$ *for any* $\varsigma \in \mathbb{Z}_+$ *there exists* $\mathbf{u}_\varsigma \in \mathbb{U}^\varsigma$*, such that* $\phi^i(x,\mathbf{u}_\varsigma) \in \mathbb{X}$*, for* $i \in \mathbb{Z}_{[1,\varsigma]}$*, and*

$$d(\phi_d^{i+1}(x,\mathbf{u}_\varsigma),x_d^e) \leq d(\phi_d^i(x,\mathbf{u}_\varsigma),x_d^e),\ i \in \mathbb{Z}_{[0,\varsigma-1]}.$$

Theorem 4 *Suppose Assumptions 2, 3 hold and let* $N \geq \max_{x_d\in\mathcal{E}} n(x_d)$*. Define*

$$\Xi = \{(x,z) \in \mathbb{X} \times \mathbb{R}_{[0,c_z]} \ : \exists (u,v) \in \mathbb{U}\times\mathbb{V},\ \ (14) \text{ holds}\}. \tag{28}$$

Then $(\mathcal{V}_c,\mathcal{V}_d,\mathcal{V}_z)$*, defined respectively by* (25), (20), (22) *is a hybrid CLF for* (6), (19) *for* (x^e,z^e) *in* Ξ*.*

In order to prove Theorem 4 we need the following technical Lemma.

Lemma 3 *Suppose Assumptions 1 and 2 hold, and let* $N \geq \max_{x_d\in\mathcal{E}} n(x_d)$*. Let* $(\mathcal{V}_c,\mathcal{V}_d,\mathcal{V}_z)$ *in* (14) *be defined respectively by* (25), (20), (22)*. If* (14) *is feasible for* $(x,z) \in \mathbb{X}\times\mathbb{R}_{[0,c_z]}$ *for* $(u,v) \in \mathbb{U}\times\mathbb{V}$*, then there exists* $(\tilde{u},\tilde{v}) \in \mathbb{U}\times\mathbb{V}$ *such that it is feasible for* $(\phi(x,u),\psi(x,u,v)) \in \mathbb{X}\times\mathbb{R}_{[0,c_z]}$*.*

We can now prove Theorem 4.

Proof. (Theorem 4) Given $(\mathcal{V}_c,\mathcal{V}_d,\mathcal{V}_z)$ defined respectively by (25), (20), (22), the existence of class $\mathcal{K}$ bounds on $\mathcal{V}_c$ is guaranteed by Assumption 3, while for $\mathcal{V}_d$, $\mathcal{V}_z$, it follows by construction since $\mathcal{V}_z(z) = z = \|z\|$ and $\mathcal{V}_d(x_d) = d_d(x_d,x_d^e)$. We only need to prove that for each $(x,z) \in \Xi$ there exists $(u,v) \in \mathbb{U}\times\mathbb{V}$ such that $(\phi(x,u),\psi(x,z,u,v)) \in \Xi$ and (14) is satisfied. Lemma 3 ensures that if there exists $(u,v) \in \mathbb{U}\times\mathbb{V}$, such that (14) is feasible for $(x,z) \in \mathbb{X}\times\mathbb{R}_{[0,c_z]}$, then there exists $(\tilde{u},\tilde{v}) \in \mathbb{U}\times\mathbb{V}$ such that (14) is feasible for $(\phi(x,u),\psi(x,u,v)) \in \mathbb{X}\times\mathbb{R}_{[0,c_z]}$. Hence, by choosing Ξ as in (28), for any $(x,z) \in \Xi$ there is always $(u,v) \in \mathbb{U}\times\mathbb{V}$ such that (14) holds and $(\phi(x,u),\psi(x,z,u,v)) \in \Xi$. □

The next corollary follows directly.

Corollary 3 *Let Assumptions 1 and 2 hold and* $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ *be defined respectively by* (25), (20), (22)*. For any* $x \in \mathbb{X}$*, there exists* $0 \leq \bar{z} \leq c_z$*, such that for any* $z \geq \bar{z}$*,* (14) *is feasible. If* (14) *is feasible for* $(x(0), z(0)) = (x, z)$*, there exists a finite* $\bar{k} \in \mathbb{Z}_{0+}$ *such that* $z(k) = 0$*, and* $x_d(k) = x_d^e$*, for all* $k \geq \bar{k}$*.*

Corollary 3 proves that by initializing the controller state appropriately, convergence to the equilibrium is achieved for any initial state, and that the discrete state converges in finite time to the discrete equilibrium state.

The mapping R in (11) can now be designed according to Corollary 2 providing the complete dynamical controller (11) that stabilizes (x^e, z^e) in Ξ.

Next, we discuss a specific implementation of (17) based on receding horizon control.

4.3 Controller Implementation by Receding Horizon Control

The hybrid CLF (14) results in a controller (11) with R as in (17) that generates a sequence of inputs along a future horizon. As previously shown in [14], such controller can be implemented by using a predictive control strategy.

Corollary 3 guarantees that there exists a finite value $\bar{z} \in \mathbb{R}_{0+}$ such that $(x, z) \in \Xi$, for any $x \in \mathbb{X}$. Hence, with an appropriate initialization of the controller state z, Corollary 2 guarantees convergence to the desired equilibrium for any initial state of the hybrid system. The stabilizing properties established in Corollary 2 are guaranteed for any control input that satisfies (14), i.e., for any $(u, v) \in R(x, z)$ in (17). The actual input (u, v) can be chosen by optimizing the set of feasible inputs with respect to a defined performance criterion. In this way a receding horizon predictive control strategy based on the repetitive solution of an optimization problem is obtained. A common definition of the performance criterion in optimization-based predictive control, such as model predictive control [36], is

$$J(x, \mathbf{u}_N) \triangleq F(\phi^N(x, \mathbf{u}_N)) + \sum_{h=0}^{N-1} L(\phi^h(x, \mathbf{u}_N), u_h), \quad (29)$$

where $F(\cdot)$ and $L(\cdot)$ denote suitable terminal and stage costs, respectively, [37]. Cost (29) typically trades off the regulation performance, in terms of distance from the equilibrium, and the actuation effort.

Constraint (23) can be implemented by a single constraint as

$$\begin{aligned}\mathcal{V}_z(\psi(x(k), \mathbf{u}_N(k))) \quad &\leq \quad (1 - d_d(x_d(k), x_d^e))\rho_z \mathcal{V}_z(z(k)) \\ &\qquad - d_d(x_d(k), x_d^e)\end{aligned}$$

Algorithm 1 *(Hybrid CLF-based Stabilizing Receding Horizon Control)*
Initialization. *Set* $k = 0$*, measure* $x(0) \in \mathbb{X}$ *and set* $z(0) \geq N\, d(x_d(0), x_d^e)$*.*

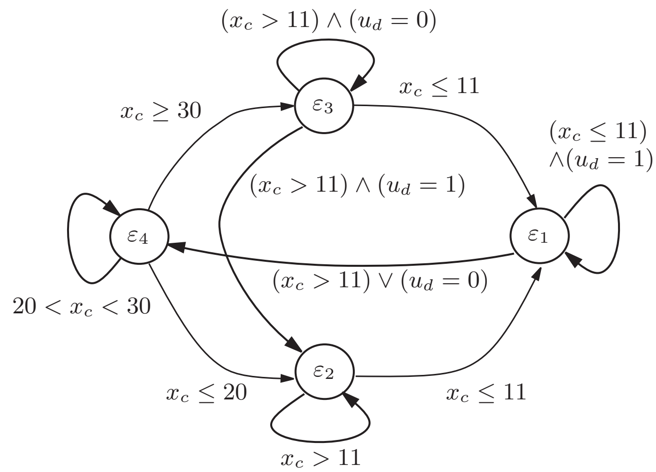


Figure 2: Automaton describing the discrete dynamics of the system in the numerical example.

Step 1. *Solve the optimization problem*

$$\begin{aligned}
&\min_{\mathbf{u}_N(k)} && J(x(k), \mathbf{u}_N(k)) && (30a)\\
\text{s.t. :} \quad & x_{h+1} &=\;& \phi(x_h, u_h(k)), && (30b)\\
& z_1 &=\;& \psi(x_0, \mathbf{u}_N(k)) && (30c)\\
& \mathcal{V}_c(\phi_c^1(x_0, \mathbf{u}_N(k))) &\leq\;& \rho \mathcal{V}_c(x_{0c}(k)) + M d_d(x_{0d}, x_d^e) && (30d)\\
& \mathcal{V}_z(\psi(x_0, \mathbf{u}_N(k))) &\leq\;& (1 - d_d(x_{0d}(k), x_d^e))\rho_z \mathcal{V}_z(z_0) && \\
& && -d_d(x_{0d}, x_d^e) && (30e)\\
& \mathcal{V}_d(\phi_d^1(x_0, \mathbf{u}_N(k))) &\leq\;& \mathcal{V}_d(x_{0d}) && (30f)\\
& && \mathbf{u}_N(k) \in \mathbb{U}^N, && (30g)\\
& && x_h \in \mathbb{X},\ h \in \mathbb{Z}_{[1,N]} && (30h)\\
& && x_0 = x(k),\ z_0 = z(k). && (30i)
\end{aligned}$$

Step 2. *Let* $\bar{\mathbf{u}}_N(k) = (\bar{u}_0(k), \ldots, \bar{u}_{N-1}(k))$ *be a feasible solution of* (30)*, possibly, but not necessarily, the optimal one. Set* $u(k) = \bar{u}_0(k)$*, and* $z(k+1) = \psi(x(k), \bar{\mathbf{u}}_N(k))$*.*
Step 3 *Measure* $x(k+1)$*, set* $k \leftarrow k+1$*, and go to Step* 1*.*

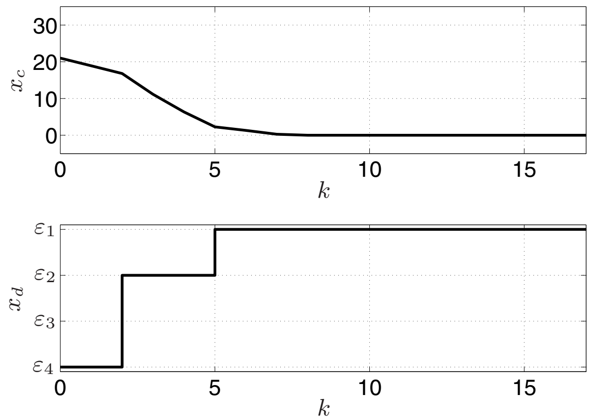
Algorithm 1 implements the constraints as required in the definition of the hybrid CLF and minimizes the performance criterion (29). It is worth noticing that the optimization problem (30) is always feasible because of the hybrid CLF existence results. Also, similarly to what demonstrated in [14], (30) can be formulated as a mixed integer linear/quadratic problem for a large and practically useful class of hybrid systems, for which several high performance numerical algorithms are available.

5. NUMERICAL EXAMPLE

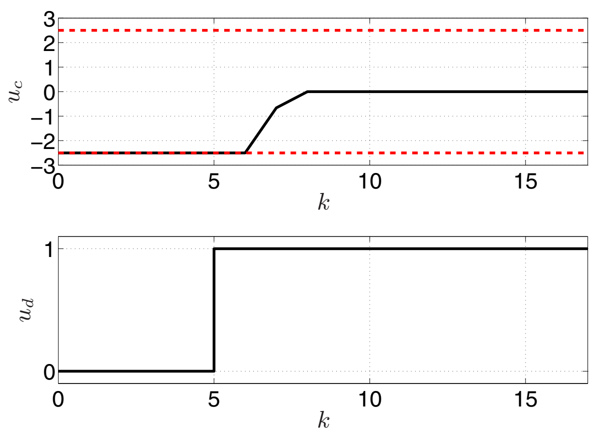
In what follows we present the application of the proposed technique to a numerical. We consider a system with continuous state domain $\mathcal{X}_c := [-5, 30]$, discrete state domain $\mathcal{E} := \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, one continuous input $u_c \in \mathbb{U}_c := [-2.5, 2.5]$ and one discrete input $u_d \in \mathcal{E}_u := \{0, 1\}$. Hence, $\mathbb{U} := [-2.5, 2.5] \times \{0, 1\}$, and $\mathbb{X} \subseteq [-5, 30] \times \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, where in particular $\mathcal{X}_c(\epsilon_1) = [-5, 11]$. The automaton describing the discrete dynamics ϕ_d of the example system is shown in Figure 2. The continuous dynamics are

$$x(k+1) = A_i x(k) + B_i u(k) \quad \text{if} \quad x_d = \epsilon_i,$$

and $(A_1, B_1) = (1.07, 0.4)$, $(A_2, B_2) = (0.85, 1.25)$, $(A_3, B_3) = (0.7, 1.05)$, $(A_4, B_4) = (1.02, 1)$.



(a) Continuous (top) and discrete (bottom) state evolution.



(b) Continuous (top) and discrete (bottom) input evolution.

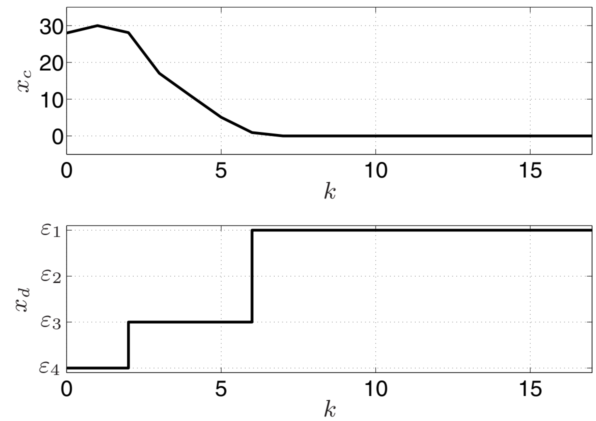
Figure 3: Simulation results of the numerical example for $(x_c(0), x_d(0)) = (21, \epsilon_4)$.

The desired equilibrium is $(x_c^e,\ x_d^e) = (0,\ \epsilon_1)$ for a steady state input $(u_c^e,\ u_d^e) = (0,\ 1)$. The function $\mathcal{V}(x_c) = \|x_c\|_\infty$ is chosen as the local CLF for the continuous state in $\mathcal{X}_c(\varepsilon_1)$ which can be proved to exists in $\mathcal{X}_c(\varepsilon_1)$ by using the auxiliary controller $(u_c, u_d) = (K_c x_c, 1)$, where $K_c = -0.2250$. From $\mathcal{V}_c$ and the construction of z, $\mathcal{V}_z$ (where we have chosen $N = 4$ which satisfies Assumption 2) and $\mathcal{V}_d$ proposed in Section 4 we obtain the hybrid control Lyapunov function (14).

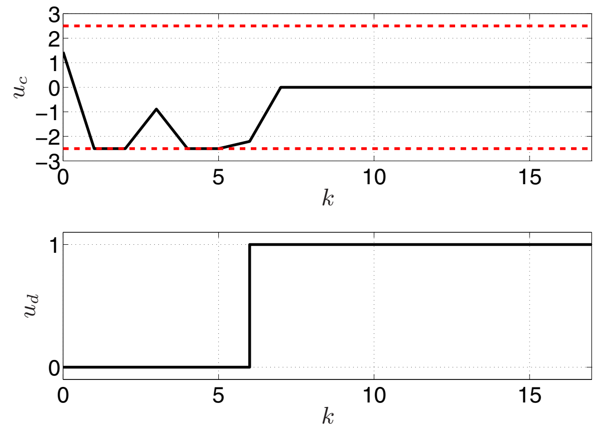
The stabilizing dynamic controller (11a) is obtained by executing Algorithm 1, where problem (30) is implemented with

$$L(x, u) = \|Q_x(x - x^e)\|_\infty + \|Q_u(u - u^e)\|_\infty,$$
$$Q_x = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right],\ Q_u = \left[\begin{smallmatrix} 0.1 & 0 \\ 0 & 0.1 \end{smallmatrix}\right],\ \bar{\rho}_c = 0.98,\ N = 4. \quad (31)$$

The model of the hybrid system dynamics (30c), (30b) is formulated as a discrete hybrid automaton using the language in [46], so that problem (30) results in a mixed-integer linear program.



(a) Continuous (top) and discrete (bottom) state evolution.



(b) Continuous (top) and discrete (bottom) input evolution.

Figure 4: Simulation results of the numerical example for $(x_c(0), x_d(0)) = (28, \epsilon_4)$.

Figure 3 shows the simulation results for initial condition $x(0) = (21,\ \epsilon_4)$. Note that in this simulation inequality (5b) is enforced at every step, which means that the continuous state component of the hybrid CLF, $\mathcal{V}_c(x_c)$ is decreasing along the entire trajectory. In Figure 3 we show the simulation simulation results for the case when the initial condition is $x(0) = (28,\ \epsilon_4)$ are shown by solid lines. In this case $\mathcal{V}_c(x_c)$ is not monotonically decreasing along the whole trajectory. This is according to (26), where the decrease of $\mathcal{V}_c$ is required only in the set $\mathcal{X}_c(\epsilon_1)$. However, the closed-loop system is asymptotically stable, due to Theorem 3 that guarantees the existence of a control Lyapunov function for the hybrid dynamics, which becomes a Lyapunov function for the closed-loop system, is guaranteed. The continuous state trajectories for many initial conditions are shown in Figure 5.

It is worth to point out that for the same setup, the hybrid controller proposed in [7] that guarantees attractivity, but not Lyapunov stability, is infeasible unless a longer horizon

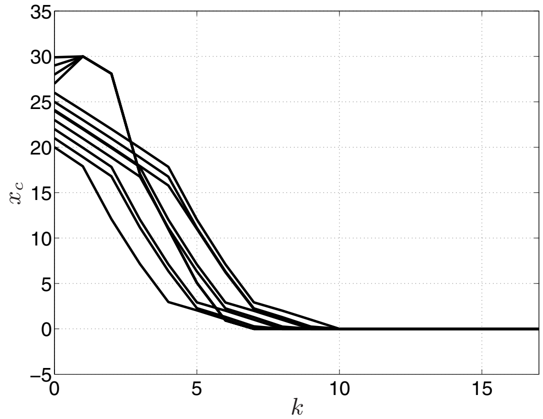


Figure 5: State evolution in the numerical example for different initial conditions.

(at least $N = 9$) is used, due to the required controllability to the equilibrium by the end of the horizon. On the other hand, for the control law proposed here it is enough to enforce a decrease of the cumulated graph distance along the prediction horizon, which is possible for a horizon $N = 4$. Clearly, this indicates also numerical advantages of our novel approach.

6. CONCLUSIONS

In this paper we have provided a general control design framework based on a formal notion of the concept of hybrid control Lyapunov function. We have shown that the existence of a hybrid CLF guarantees the existence of a standard CLF. This result induces a class of dynamic control laws based on the hybrid CLF that stabilize a desired hybrid system equilibrium. Building on our previous research, we have defined a constructive procedure to obtain a hybrid control Lyapunov function and the corresponding control law for a fairly general class of hybrid systems, and we have implemented that by receding horizon constrained control. The proposed approach has been demonstrated on a numerical example.

7. ACKNOWLEDGMENTS

M. Lazar, W.P.M.H. Heemels, and A. Bemporad are partially supported by the European Commission under project FP7-INFSO-ICT-248858 'MOBY-DIC - Model-based synthesis of digital electronic circuits for embedded control'.

8. REFERENCES

[1] P. Antsaklis. A brief introduction to the theory and applications of hybrid systems. *Proc. IEEE, Special Issue on Hybrid Systems: Theory and Applications*, 88(7):879–886, July 2000.

[2] Z. Artstein. Stabilization with relaxed controls. *Nonlinear Analysis*, 7:1163–1173, 1983.

[3] E. Asarin, O. Bournez, T. Dang, and O. Maler. Approximate reachability analysis of piecewise-linear dynamical systems. In B. Krogh and N. Lynch, editors, *Hybrid Systems: Computation and Control*, volume 1790 of *Lecture Notes in Computer Science*, pages 20–31. Springer-Verlag, 2000.

[4] A. Balluchi, L. Benvenuti, M. Di Benedetto, C. Pinello, and A. Sangiovanni-Vincentelli. Automotive engine control and hybrid systems: Challenges and opportunities. *Proceedings of the IEEE*, 88(7):888–912, 2002.

[5] C. Belta, A. Bicchi, M. Egerstedt, E. Frazzoli, E. Klavins, and G. Pappas. Symbolic Planning and Control of Robot Motion. *Robotics & Automation Magazine*, 14:62–66, 2007.

[6] A. Bemporad, S. Di Cairano, E. Henriksson, and K. Johansson. Hybrid model predictive control based on wireless sensor feedback: an experimental study. *Int. J. Robust Nonlinear Control*, 20(2), 2010. special issue "Industrial applications of wireless control".

[7] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3):407–427, 1999.

[8] M. S. Branicky, V. S. Borkar, and S. K. Mitter. A unified framework for hybrid control: model and optimal control theory. *IEEE Trans. Aut. Control*, 43(1):31–45, 1998.

[9] M. Braun and J. Shear. A mixed logical dynamic model predictive control approach for handling industrially relevant transportation constraints. In *IEEE Conf. Automation Science and Engineering*, pages 966–971, Toronto, Canada, 2010.

[10] T. Dang. Approximate reachability computation for polynomial systems. In *Hybrid Systems: Computation and Control*, number 3927, pages 93–107. 2006.

[11] R. DeCarlo, M. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. *Proceedings of the IEEE*, 88(7):1069–1082, 2002.

[12] S. Di Cairano and A. Bemporad. Equivalent piecewise affine models of linear hybrid automata. *IEEE Trans. Aut. Control*, 55(2):498–502, 2010.

[13] S. Di Cairano, A. Bemporad, I. Kolmanovsky, and D. Hrovat. Model predictive control of magnetically actuated mass spring dampers for automotive applications. *Int. J. Control*, 80(11):1701–1716, 2007.

[14] S. Di Cairano, M. Lazar, A. Bemporad, and W. Heemels. A control Lyapunov approach to predictive control of hybrid systems. In B. M. M. Egerstedt, editor, *Hybrid Systems: Computation and Control*, Lect. Not. in Computer Science, pages 130–143. Springer-Verlag, 2008.

[15] S. Di Cairano, H. Tseng, D. Bernardini, and A. Bemporad. Vehicle yaw stability control by coordinated active front steering and differential braking in the tire sideslip angles domain. *IEEE Trans. Contr. Systems Technology*, 2012. Preprints on ieeexplore.org.

[16] E. W. Dijkstra. A note on two problems in connexion with graphs. *Numerische Mathematik*, 1(1):269–271, 1959.

[17] M. Egerstedt. Behavior based robotics using hybrid automata. In B. Krogh and N. Lynch, editors, *Hybrid Systems: Computation and Control*, number 1790 in Lecture Notes in Computer Science, pages 103–116. Springer-Verlag, 2000.

[18] S. Engell and O. Stursberg. Hybrid control techniques for the design of industrial controllers. In *Proc. 44th*

IEEE Conf. on Decision and Control, pages 5612–5617, Seville, Spain, 2005.

[19] R. Goebel, R. Sanfelice, and A. Teel. Hybrid dynamical systems. *Control Systems Magazine*, 29(2):28–93, 2009.

[20] W. Heemels, B. de Schutter, and A. Bemporad. Equivalence of hybrid dynamical models. *Automatica*, 37(7):1085–1091, July 2001.

[21] W. Heemels, B. D. Schutter, J. Lunze, and M. Lazar. Stability analysis and controller synthesis for hybrid dynamical systems. *Philosophical Transactions of the Royal Society A*, 368:4937–4960, 2010.

[22] T. Henzinger. The theory of hybrid automata. In *Proceedings of the 11th Annual IEEE Symposium on Logic in Computer Science (LICS '96)*, pages 278–292, New Brunswick, New Jersey, 1996.

[23] J. Hespanha. Uniform stability of switched linear systems: extensions of LaSalle's invariance principle. *IEEE Trans. Aut. Control*, 49(4):470–482, 2004.

[24] M. Johansson. *Piecewise linear control systems*. PhD thesis, Lund Institute of Technology, Sweden, 1999.

[25] M. Johansson and A. Rantzer. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Trans. Aut. Control*, 43(4):555–559, 1998.

[26] C. Kellett and A. Teel. Discrete-time asymptotic controllability implies smooth control-Lyapunov function. *Systems & Control Letters*, 52(5):349–359, 2004.

[27] C. M. Kellett and A. R. Teel. On the robustness of $\mathcal{KL}$-stability for difference inclusions: Smooth discrete-time Lyapunov functions. *SIAM Journal on Control and Optimization*, 44(3):777–800, 2005.

[28] M. V. Kothare, V. Balakrishnan, and M. Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32(10):1361–1379, 1996.

[29] M. Lazar. *Model predictive control of hybrid systems: Stability and robustness*. PhD thesis, Eindhoven University of Technology, The Netherlands, 2006.

[30] M. Lazar, W. P. M. H. Heemels, B. J. P. Roset, H. Nijmeijer, and P. P. J. van den Bosch. Input-to-state stabilizing sub-optimal NMPC algorithms with an application to DC-DC converters. *International Journal of Robust and Nonlinear Control*, 18:890–904, 2007.

[31] M. Lazar, W. P. M. H. Heemels, S. Weiland, and A. Bemporad. Stabilizing model predictive control of hybrid systems. *IEEE Trans. Aut. Control*, 51(11):1813–1818, 2006.

[32] M. Lazar, D. Muñoz de la Peña, W. P. M. H. Heemels, and T. Alamo. On input-to-state stability of min-max nonlinear model predictive control. *Systems & Control Letters*, 57:39–48, 2008.

[33] B. Lennartson, M. Tittus, B. Egardt, and S. Pettersson. Hybrid systems in process control. *Control Systems Magazine*, 16(5):45–56, 1996.

[34] D. Liberzon and A. Morse. Basic problems in stability and design of switched systems. *Control Systems Magazine*, 19(5):59–70, 2002.

[35] J. Lygeros, K. Johansson, S. Simic, J. Zhang, and S. Sastry. Dynamical properties of hybrid automata. *IEEE Trans. Aut. Control*, 48:2–17, Jan. 2003.

[36] D. Mayne, J. Rawlings, C. Rao, and P. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, June 2000.

[37] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[38] D. Mignone, G. Ferrari-Trecate, and M. Morari. Stability and stabilization of piecewise affine and hybrid systems: An LMI approach. In *Proc. 39th IEEE Conf. on Decision and Control*, volume 1, pages 504–509, 2002.

[39] I. Mitchell, A. Bayen, and C. Tomlin. A time-dependent hamilton-jacobi formulation of reachable sets for continuous dynamic games. *IEEE Trans. Aut. Control*, 50(7):947–957, 2005.

[40] B. Potocnik, A. Bemporad, F. Torrisi, G. Music, and B. Zupancic. Hybrid modelling and optimal control of a multiproduct batch plant. *Control Engineering Practice*, 12(9):1127–1137, 2004.

[41] G. Ripaccioli, A. Bemporad, F. Assadian, C. Dextreit, S. Di Cairano, and I. Kolmanovsky. Hybrid Modeling, Identification, and Predictive Control: An Application to Hybrid Electric Vehicle Energy Management. In *Hybrid Systems: Computation and Control*, volume 5469, pages 321–335. 2009.

[42] M. Roozbehani, E. Feron, and A. Megrestki. Modeling, optimization and computation for software verification. In L. T. M. Morari, editor, *Hybrid Systems: Computation and Control*, number 3414 in Lecture Notes in Computer Science, pages 606–622. Springer-Verlag, 2005.

[43] E. D. Sontag. Nonlinear regulation: the piecewise linear approach. *IEEE Transactions on Automatic Control*, 26(2):346–357, 1981.

[44] E. D. Sontag. A Lyapunov-like characterization of asymptotic controllability. *SIAM Journal of Control and Optimization*, 21:462–471, 1983.

[45] C. Tomlin, I. Mitchell, A. Bayen, and M. Oishi. Computational techniques for the verification of hybrid systems. *Proceedings of the IEEE*, 91(7):986–1001, 2003.

[46] F. Torrisi and A. Bemporad. HYSDEL — A tool for generating computational hybrid models. *IEEE Trans. Contr. Systems Technology*, 12(2):235–249, Mar. 2004.

[47] A. J. van der Schaft and J. M. Schumacher. *An introduction to hybrid dynamical systems*, volume 251 of *Lecture Notes in Control and Information Sciences*. Springer, 2000.

[48] H. Witsenhausen. A class of hybrid-state continuous-time dynamic systems. *IEEE Trans. Aut. Control*, 11(2):161–167, 1966.