Robust Event-Triggered MPC With Guaranteed Asymptotic Bound and Average Sampling Rate

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***Abstract*—We propose a robust event-triggered model predictive control (MPC) scheme for linear time-invariant discrete-time systems subject to bounded additive stochastic disturbances and hard constraints on the input and state. For given probability distributions of the disturbances acting on the system, we design event conditions such that the average frequency of communication between the controller and the actuator in the closed-loop system attains a given value. We employ Tube MPC methods to guarantee robust constraint satisfaction and a robust asymptotic bound on the system state. Moreover, we show that instead of a given periodically updated Tube MPC scheme, an appropriate event-triggered MPC scheme can be applied, with the same guarantees on constraints and region of attraction, but with a reduced number of average communications.**

***Index Terms*—Asymptotic stability, communication networks, control systems, control system synthesis, cost function, discrete-time systems, Lyapunov method, predictive models, predictive control.**

I. Introduction

NETWORKED control systems are distributed control systems in which the communication between sensors, controllers, and actuators takes place over a certain communication network (see, for example, [1]). There are multiple reasons for reducing the overall amount of communication in such a control system. On one hand, communications may induce a nontrivial cost in terms of energy, which is especially the case in wireless communication. On the other hand, communications may take place over a shared, bandwidth-limited communication network. Reducing the bandwidth required by the controller releases resources for other network tasks. One method that is suited to reduce the required communication is event-triggered control. Instead of updating the control input of the plant at periodic time instances, in event-triggered control new inputs are only transmitted to the actuators if certain well-defined events occur in the plant. Typically, these events are defined in terms of the plant output or state leaving a certain set. For a recent overview of event-triggered control, please refer [2].

If the plant is subject to hard constraints on the input and state, model predictive control (MPC) has proven to be viable control method, see [3] for an overview of MPC. In event-triggered MPC, an event is usually triggered if the plant state deviates by a certain amount from the prediction of the state that was computed in the MPC optimization problem at the last event (see, for example, [4]–[11], and the references therein). For linear systems, this approach has the advantage that the event conditions are only dependent on the error dynamics, induced by the disturbances, which are decoupled from the nominal system dynamics. This idea also plays a central role in the event-triggered control schemes for unconstrained systems proposed in [12]–[14], where a model of the nominal closed-loop system is included in the actuators. One major difference between this paper and the cited works above (except for [6], where time-varying thresholds based on gains and Lipschitz constants of nonlinear systems were employed) is that the thresholds used here are explicitly dependent on the time since the last sampling instance, allowing a better tradeoff between the average sampling rate and the asymptotic bound on the system state. Alternatively, the MPC cost function may be used to define the event conditions (see, for example, [6], [15]). These control schemes, including the scheme proposed in this paper, require a whole sequence of predicted inputs to be transmitted to the actuators at a given event. For many communication protocols, the size of communicated packets is fixed, such that it is more efficient to transmit whole sequences of inputs only at infrequent times than to transmit a single input at each point in time, although the overall amount of transmitted *information* is the same (or even higher), see [4] and [16] with reference to [17]. See also [18], where an MPC scheme using packetized communication is proposed. This control structure is illustrated inFig. 1, which is the structure also assumed in [5].

For systems that are additionally subject to uncertainty or disturbances, robust MPC methods must be applied. Here, we consider bounded, additive disturbances, which can be effectively and efficiently handled by a method known as Tube MPC [19], [20]. In particular, the uncertainty in the prediction of the future plant state due to disturbances is described by a sequence

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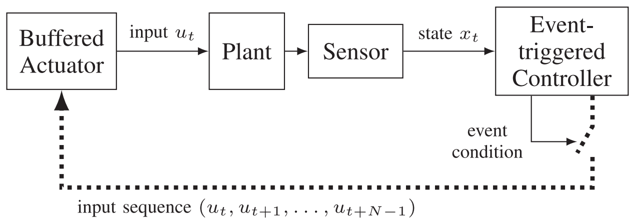

Fig. 1. Control structure with buffered actuator.

of sets (the so called "tube"), which are centered around the prediction of a nominal system, see also [21] and [22]. The main idea in Tube MPC is to assume that feedback is applied to the plant at every time step, which allows us to limit the growth of the uncertainty in the prediction, as the (future) feedback will counteract the effect of the disturbances. Even if no feedback is assumed, bounds on the system dynamics, for example, by assuming the dynamics to be Lipschitz continuous, may be used to obtain the tubes (see, for example, [23]). This technique was also employed in robust event-triggered MPC (see, for example, [6], [9], [10]).

In this paper, based on the Tube MPC approach presented in [19], we propose a robust event-triggered MPC scheme where the main novelty is the following insight. If a disturbance of significantly lower than worst case magnitude affects the system at a given time step, then the deviation of the plant trajectory from the predicted trajectory will not be greater than what was previously predicted as a worst case, even if *no* feedback is applied at the given time step. We design a robust event-triggered controller that does *not* update the inputs of the plant in the very event of such less-than-worst-case disturbances, saving communication, and possibly computational power, in the process. This method allows any periodically updated Tube MPC scheme to be used for designing an event-triggered MPC scheme while retaining the guarantees of its periodically updated counterpart concerning robust constraint satisfaction, region of attraction, and asymptotic bound, with a reduced average amount of communication between the controller and the actuators. Additionally, we present a method of artificially increasing the assumed bound on the disturbances in order to further reduce the communication in the system, leading to a linear scaling of the uncertainty sets involved in the MPC scheme, similar to what was proposed in [22] for a periodically triggered setting. In particular, we show how, based on the knowledge of the distribution of the disturbances, event conditions can be designed such that the time between events is a random variable with a predefined, arbitrary probability distribution with finite and discrete support. Here, we use a Markov-like property of the event-triggered scheme that limits the computational effort involved with computing the appropriate thresholds: the trigger behavior of the scheme does not depend on the state of the system before the last event instance.

Our scheme shares some similarities with [5], which also considers event-triggered MPC of linear discrete-time system subject to bounded disturbances. The major differences are the stochastic viewpoint in this paper and the guarantees on the average amount of communication that are not given in [5]. Furthermore, we provide a tighter estimate of the worst case asymptotic bound on the system state. In order to achieve this tighter bound, we have to modify the standard analysis of recursive feasibility and stability available in Tube MPC, and in particular in [19], as the uncertainty present in the prediction now stems from two sources: the additive disturbances on the one hand, and the permitted deviation of the real trajectory from the predicted trajectory due to the event-triggered scheme on the other hand. Contrary to [5], we do not simply model this deviation as an additional, independent, additive disturbance, allowing us to show that for a certain choice of parameters, it is possible to achieve the same asymptotic bound as a periodically updated Tube MPC scheme. Another event-triggered scheme with guaranteed average communication rate was presented in [24], which was shown to achieve a strictly better performance than a periodically updated scheme with the same communication rate. Therein, stochastic (unbounded) disturbances but no constraints were considered.

This paper is based on the preliminary work presented in [25]. Here, we provide proofs for the statements made in [25] and also provide an additional threshold design method that allows a better tradeoff between the average sampling rate and the guaranteed asymptotic bound. We also present more detailed numerical examples here. Some techniques that appear in this paper were also used in the recent work [26] by the authors, in which *self-triggered* robust MPC is considered. The main difference between self-triggered and event-triggered MPC is that in event-triggered MPC the state is measured at each time instant, potentially allowing the controller to react to disturbances. In self-triggered MPC on the other hand, the plant is controlled in a true open-loop fashion between communication instances, leading to larger and differently shaped uncertainty sets in the predictions. Furthermore, the cost function is self-triggered MPC that has to be designed carefully in accordance with the set that is to be stabilized, leading to a considerably more involved optimization problem.

The remainder of this paper is structured in the following way. Section II contains notes on notation and several preliminary results. The formal problem statement is given in Section III. The robust event-triggered MPC scheme is presented in Section IV and its relevant properties are described in Section V. The design of the event conditions based on the probability density function describing the disturbances is explained in Section VI. Section VII contains numerical examples illustrating our results and Section VIII concludes this paper with an outlook on open questions.

II. Notation and Preliminaries

Notation: Let $\mathbb{N}$ denote the set of nonnegative integers. For $q, s \in \mathbb{N}$, let $\mathbb{N}_{[q,s]}$ denote the set $\{r \in \mathbb{N} \,|\, q \le r \le s\}$. For a given real number $a \in \mathbb{R}$, we use $\mathbb{R}_{>a}$ and $\mathbb{R}_{\ge a}$ to denote the set of real numbers greater than a, or greater than or equal to a, respectively. For symmetric matrices $S = S^\top \in \mathbb{R}^{n\times n}$, we use $S > 0$ and $S \ge 0$ to denote the fact that S is positive definite and positive semidefinite, respectively. Given sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$, a scalar α, and matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$, we define $\alpha\mathcal{X} := \{\alpha x \,|\, x \in \mathcal{X}\}$, $A\mathcal{X} := \{Ax \,|\, x \in \mathcal{X}\}$, and $B^{-1}\mathcal{X} := \{x \in \mathbb{R}^n | Bx \in \mathcal{X}\}$. The Minkowski set addition is defined by $\mathcal{X} \oplus \mathcal{Y} := \{x + y \,|\, x \in \mathcal{X}, y \in \mathcal{Y}\}$. Given a vector $x \in \mathbb{R}^n$, we define $\mathcal{X} \oplus x := x \oplus \mathcal{X} := \{x\} \oplus \mathcal{X}$. The Pontryagin set difference is defined by $\mathcal{X} \ominus \mathcal{Y} := \{z \in \mathbb{R}^n \mid z \oplus \mathcal{Y} \subseteq \mathcal{X}\}$, see [27] and [28]. Given a sequence of sets $\mathcal{X}_i$ for $i \in \mathbb{N}_{[a,b]}$ with $a, b \in \mathbb{N}$, we define $\bigoplus_{i=a}^{b} \mathcal{X}_i = \mathcal{X}_a \oplus \mathcal{X}_{a+1} \oplus \cdots \oplus \mathcal{X}_b$. By convention, the empty sum is equal to $\{0\}$. Similarly, for any vectors $v_i \in \mathbb{R}^n$, $i \in \mathbb{N}$, we define $\sum_{i=a}^{b} v_i = 0$ for any $a, b \in \mathbb{N}$ if $a > b$. We call a compact, convex set containing the origin a C-set. A C-set containing the origin in its (nonempty) interior is called a PC-set. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. If additionally $\alpha(s) \to \infty$ as $s \to \infty$, α is said to belong to class $\mathcal{K}_\infty$. The Euclidean norm of a vector $v \in \mathbb{R}^n$ is denoted by $|v|$, the infinity norm by $\|v\|_\infty$. Given any compact set $\mathcal{Y} \subseteq \mathbb{R}^n$, the distance between v and $\mathcal{Y}$ is defined by $|v|_\mathcal{Y} := \min_{s \in \mathcal{Y}} |v - s|$. The Lebesgue measure of $\mathcal{Y}$ is denoted by $\mathrm{vol}(\mathcal{Y})$. The empty set is denoted by $\emptyset$. Define finally the Euclidean unit ball by $\mathcal{B} := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

Definition 1: Consider a dynamical system of the form $(x_{t+1}^\top, z_{t+1}^\top)^\top = f(x_t, z_t, w_t)$, $t \in \mathbb{N}$, $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathcal{W} \to \mathbb{R}^n$ with a compact set $\mathcal{W} \subseteq \mathbb{R}^n$, and $z_0 = g(x_0)$ for a function $g : \mathbb{R}^n \to \mathbb{R}^p$. A set $\mathcal{Y} \subseteq \mathbb{R}^n$ is *robustly asymptotically stable with region of attraction* $\hat{\mathcal{X}} \subseteq \mathbb{R}^n$ for this system, if there exists a class $\mathcal{K}$-function α, such that $|x_t|_\mathcal{Y} \leq \alpha(|x_0|_\mathcal{Y})$, $t \in \mathbb{N}$, and $\lim_{t\to\infty} |x_t|_\mathcal{Y} = 0$, for all $x_0 \in \hat{\mathcal{X}}$, and all $w_t \in \mathcal{W}$, $t \in \mathbb{N}$, compare [29].

Definition 2: Given a dynamical system described by $x_{t+1} = Ax_t + w_t$ with $x_t \in \mathbb{R}^n$, $w_t \in \mathcal{W}$, $t \in \mathbb{N}$, where $\mathcal{W} \subseteq \mathbb{R}^n$ is a C-set and A is a Schur matrix, the *minimal robust positively invariant set* is the nonempty compact set $\mathcal{Y}^\star \subseteq \mathbb{R}^n$ satisfying $A\mathcal{Y}^\star \oplus \mathcal{W} \subseteq \mathcal{Y}^\star$, which is contained in every compact set $\bar{\mathcal{Y}} \subseteq \mathbb{R}^n$ satisfying $A\bar{\mathcal{Y}} \oplus \mathcal{W} \subseteq \bar{\mathcal{Y}}$, see also [28] and [30].

The following lemma summarizes several properties of the Minkowski set addition and the Pontryagin set difference used in this paper.

Lemma 1: Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^n$ be compact convex sets. Let further $A \in \mathbb{R}^{m\times n}$. Then it holds that $\mathcal{X} \oplus \mathcal{Y} = \mathcal{Y} \oplus \mathcal{X}$, $\mathcal{X} \ominus (\mathcal{Y} \oplus \mathcal{Z}) = (\mathcal{X} \ominus \mathcal{Y}) \ominus \mathcal{Z}$, $(\mathcal{X} \oplus \mathcal{Y}) \ominus \mathcal{Y} = \mathcal{X}$, $(\mathcal{X} \ominus \mathcal{Y}) \oplus \mathcal{Y} \subseteq \mathcal{X}$, $A(\mathcal{X} \oplus \mathcal{Y}) = A\mathcal{X} \oplus A\mathcal{Y}$, and $(\mathcal{X} \cap \mathcal{Y}) \oplus \mathcal{Z} \subseteq (\mathcal{X} \oplus \mathcal{Z}) \cap (\mathcal{Y} \oplus \mathcal{Z})$. ■

The proofs for these statements can be found in [28] and [31].

III. Problem Setup

We consider linear discrete-time systems of the form

$$x_{t+1} = Ax_t + Bu_t + w_t \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state and $u_t \in \mathbb{R}^m$ is the control input at time $t \in \mathbb{N}$. The matrix pair (A, B) is assumed to be stabilizable. The disturbance w_t is assumed to be time-varying, unknown, and to satisfy $w_t \in \mathcal{W} \subseteq \mathbb{R}^n$, $t \in \mathbb{N}$, where $\mathcal{W}$ is a known C-set. Furthermore, the probability distribution of the disturbance w_t is assumed to be known. In particular, we assume w_t to be independently and identically distributed for all $t \in \mathbb{N}$ according to the bounded probability density function $\mathbf{p}_w : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with a support that is bounded by $\mathcal{W}$. Furthermore, hard constraints, $x_t \in \mathcal{X}$, $u_t \in \mathcal{U}$, $t \in \mathbb{N}$, on the input and state are given, where $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are C-sets. We assume that the state x_t is available to the event-triggered controller (at the sensor side) as a measurement at any time step $t \in \mathbb{N}$, which is consistent with a setup where the controller is directly integrated in the sensor of the plant. The communication network for which we would like to reduce the number of transmissions is situated between the controller and the actuator, as illustrated in Fig. 1. In particular, we assume that the actuator does not have direct access to the system state and has to receive this information via this communication network.

In order to save communication, the input u_t will be determined by an event-triggered controller of the form

$$u_t = \kappa(x_{t_j}, t - t_j), \quad t \in \mathbb{N}_{[t_j, t_{j+1}-1]} \tag{2a}$$

$$t_{j+1} = \inf\{t \in \mathbb{N}_{\geq t_j+1} \mid x_t \notin \mathcal{E}(x_{t_j}, t - t_j)\} \tag{2b}$$

where $j \in \mathbb{N}$ and $t_0 = 0$. That is, the control values are only updated at the *event instances* t_j based on the state x_{t_j}. The event instances are determined based on the *event conditions* $x_t \notin \mathcal{E}(x_{t_j}, t - t_j)$. At the time instances between t_j and t_{j+1} the input u_t is open loop, that is, not depending explicitly on the current state x_t. This makes it possible to transmit a whole sequence $u_{t_j}, u_{t_j+1}, \ldots, u_{t_j+N-1}$ for an $N \in \mathbb{N}_{\geq 1}$ to the actuator in one packet at time t_j. Of this sequence, only the first $t_{j+1} - t_j$ inputs are applied to the system, as the next packet arrives at time t_{j+1}. This setup makes it necessary to guarantee $t_{j+1} - t_j \leq N$ for all $j \in \mathbb{N}$.

Our goal is to design the controller $\kappa : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m$ and the set-valued function $\mathcal{E} : \mathbb{R}^n \times \mathbb{N} \to 2^{\mathbb{R}^n}$ for the closed-loop system consisting of (1) and (2) such that

1) the constraints $x_t \in \mathcal{X}$, $u_t \in \mathcal{U}$, $t \in \mathbb{N}$, are robustly satisfied;
2) a C-set $\mathcal{Y} \subseteq \mathbb{R}^n$ is robustly asymptotically stable; and
3) the expected value of the interevent times satisfies $\mathbb{E}[t_{j+1} - t_j] = \bar{\Delta}$ for a given $\bar{\Delta} \geq 1$.

We expect a tradeoff between $\bar{\Delta}$ and the size of the set $\mathcal{Y}$, with the tradeoff depending on the probability distribution $\mathbf{p}_w$.

Remark 1: In many applications, it is more natural to have an event-triggered element in the communication channel between the sensor and the actuator, or having the controller cosituated with the actuator, in contrast to the setup presented here, where the event-triggered element is placed between the controller and the actuator. However, if a packet-based communication scheme is employed and the controller is contained in the actuator, it is in principle possible to send a predicted sequence of system states from the actuator back to the sensor at each time the control actions are updated due to an event at the sensor side (which causes the current system state to be transmitted from the sensor to the actuator). Such an alternative setup is depicted in Fig. 2, where it is still assumed that the actuator has to receive state information via the communication network. The results in this paper can be applied to the alternative setup without restrictions, at the price of more communications in the system due to the required back-channel from the actuator to the sensors. Compare [4], where just such a setup was proposed. ■

Remark 2: The assumption of the disturbance being independently and identically distributed may not be satisfied for applications where the source of the disturbance is known. In

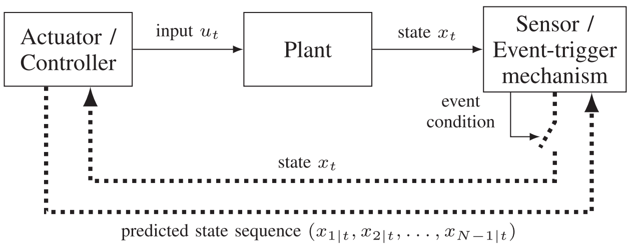

Fig. 2. Control structure with event triggering in the sensor-to-controller path.

this case, it would be more appropriate to assume the disturbance to be generated by an exosystem and to include an estimator of the state of this exosystem in the control scheme. The setup considered in [12] goes in this direction. Such an approach requires output-feedback methods to be applied, which is beyond the scope of this paper. ■

IV. Event-Triggered Tube MPC

We propose a solution to the problem stated in Section III based on Tube MPC. That is, the functions κ and $\mathcal{E}$ are determined by the solution of a finite horizon optimal control problem which is to be solved online at the event instants t_j, $j \in \mathbb{N}$. The constraints in the optimization problem are tightened in order to guarantee robust constraint satisfaction. In particular, we employ the method proposed in [19] to compute the tightened constraint sets. If necessary, we artificially increase the assumed bound on the disturbances in the computations in order to take into account the additional uncertainty introduced by the event-triggered implementation of the controller.

The control scheme is based on an auxiliary feedback law defined by the matrix $K \in \mathbb{R}^{m\times n}$, which is assumed to be the desired feedback for the plant if constraints are ignored. The following assumption is required to hold.

Assumption 1: The matrix $A + BK$ is Schur stable. ■

A. Setup of the MPC Scheme

The finite horizon optimal control problem is defined as follows for an $x_t \in \mathbb{R}^n$ with $t \in \mathbb{N}$. The decision variable of the optimization problem is

$$\mathbf{d} = ((x_{0|t}, \ldots, x_{N|t}), (u_{0|t}, \ldots, u_{N-1|t})) \in \mathbb{D}_N \tag{3}$$

where $\mathbb{D}_N = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$ and $N \in \mathbb{N}_{\geq 1}$ is the prediction horizon. The constraints

$$x_{0|t} = x_t \tag{4a}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}, \quad x_{i+1|t} = Ax_{i|t} + Bu_{i|t} \tag{4b}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}, \quad x_{i|t} \in \mathcal{X}_i \tag{4c}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}, \quad u_{i|t} \in \mathcal{U}_i \tag{4d}$$

$$x_{N|t} \in \mathcal{X}_\mathrm{f} \tag{4e}$$

are imposed on $\mathbf{d}$, where the variables $x_{i|t}$ represent a predicted trajectory for the undisturbed system generated by the inputs $u_{i|t}$ according to (4b). The sets $\mathcal{X}_i$ and $\mathcal{U}_i$, $i \in \mathbb{N}_{[0,N-1]}$, are tightened constraint sets, depending on the step i in the prediction. The set $\mathcal{X}_\mathrm{f}$ is a terminal set. Define the set of all feasible decision variables for a given point $x_t \in \mathbb{R}^n$ by

$$\mathcal{D}_N(x_t) = \{\mathbf{d} \in \mathbb{D}_N \mid (4a) \text{ to } (4e)\}. \tag{5}$$

The tightened constraint sets $\mathcal{X}_i$ and $\mathcal{U}_i$ are defined by

$$\mathcal{X}_i := \mathcal{X} \ominus \mathcal{F}_i,\ i \in \mathbb{N}_{[0,N-1]} \tag{6a}$$

$$\mathcal{U}_i := \mathcal{U} \ominus K\mathcal{F}_i,\ i \in \mathbb{N}_{[0,N-1]} \tag{6b}$$

where the sets $\mathcal{F}_i \subseteq \mathbb{R}^n$, $i \in \mathbb{N}$, are chosen in order to capture the worst case uncertainty in the prediction, taking into account that feedback is only present if an event occurs. The terminal set $\mathcal{X}_\mathrm{f}$, as well as the sets $\mathcal{F}_i$, $i \in \mathbb{N}$, will be defined in Section IV-B.

The cost function for the finite horizon optimal control problem is based on the desired feedback $u = Kx$ and penalizes the difference between the computed $u_{i|t}$ and the desired value $Kx_{i|t}$. In particular, it is defined for all $t \in \mathbb{N}$ and all $\mathbf{d} \in \mathbb{D}_N$ by

$$J_N(\mathbf{d}) = \sum_{i=0}^{N-1} \ell(u_{i|t} - Kx_{i|t}) \tag{7}$$

for a stage cost function $\ell : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$.

The finite horizon optimal control problem to be solved in order to obtain κ and $\mathcal{E}$ in (2) is defined for all $t \in \mathbb{N}$ and all $x_t \in \mathbb{R}^n$ by

$$J_N^0(x_t) = \min_{\mathbf{d}\in\mathcal{D}_N(x_t)} J_N(\mathbf{d}) \tag{8a}$$

$$\mathbf{d}^\star(x_t) = \arg\min_{\mathbf{d}\in\mathcal{D}_N(x_t)} J_N(\mathbf{d}) \tag{8b}$$

Remark 3: In the case of nonunique minimizers, it is assumed that $\mathbf{d}^\star(x_t)$ is an arbitrary minimizer to the optimization problem. ■

The set where the optimization problem in (8) is feasible is defined by $\hat{\mathcal{X}}_N := \{x \in \mathbb{R}^n \mid \mathcal{D}_N(x) \neq \emptyset\}$. Given any $\mathbf{d}^\star(x_{t_j}) = ((x^\star_{0|t_j}, \ldots, x^\star_{N|t_j}), (u^\star_{0|t_j}, \ldots, u^\star_{N-1|t_j}))$, where t_j is assumed to be an event instant, the event conditions are defined by $\mathcal{E}(x_{t_j}, t - t_j) := x^\star_{t-t_j|t_j} \oplus \mathcal{T}_{t-t_j}$, for given closed sets $\mathcal{T}_i \subseteq \mathbb{R}^n$, $i \in \mathbb{N}_{[1,N]}$, and $t \in \mathbb{N}_{[t_j+1,t_j+N]}$. That is, an event is triggered if the actual trajectory deviates too much from the predicted trajectory of the undisturbed system. For simplicity, we define $\mathcal{T}_N := \emptyset$, such that an event is guaranteed to be triggered within the prediction horizon. Furthermore, we define $\mathcal{T}_0 = \{0\}$. The control law is defined by $\kappa(x_{t_j}, t - t_j) = u^\star_{t-t_j|t_j}$ for $t \in \mathbb{N}_{[t_j,t_j+N-1]}$. That is, the finite horizon optimal input is applied in an open-loop fashion to the plant until the next event occurs and the next optimal control problem is solved. The closed-loop system under the event-triggered controller is given by

$$x_{t+1} = Ax_t + Bu^\star_{t-t_j|t_j} + w_t,\ t \in \mathbb{N}_{[t_j,t_{j+1}-1]} \tag{9a}$$

$$t_{j+1} = \min\{t \in \mathbb{N}_{\geq t_j+1} \mid x_t \notin x^\star_{t-t_j|t_j} \oplus \mathcal{T}_{t-t_j}\} \tag{9b}$$

with $w_t \in \mathcal{W}$, $j, t \in \mathbb{N}$, $t_0 = 0$, and $x_0 \in \mathbb{R}^n$. Note that the closed-loop system in (9) matches the dynamical system considered in Definition 1 with $z_t = (x_{t_i}^\top, t - t_i)^\top$ and $g : x_0 \mapsto (x_0^\top, 0)^\top$.

B. Assumptions on the constraints

In the following, assumptions on the sets involved in the definition of the triggering conditions and the optimal control problem will be given that ensure robust constraint satisfaction and robust stability properties for the closed-loop system (9). The sets $\mathcal{F}_i$, $i \in \mathbb{N}$, used to describe the uncertainty in the prediction are defined by

$$\mathcal{F}_i := \bigoplus_{j=0}^{i-1} (A+BK)^j \bar{\mathcal{W}} \tag{10}$$

where the set $\bar{\mathcal{W}} \subseteq \mathbb{R}^n$ is an artificial overapproximation of the set $\mathcal{W}$ of disturbances acting on the system, that is, $\mathcal{W} \subseteq \bar{\mathcal{W}}$, chosen in a way such that the event-triggered behavior of the closed-loop system is taken into account. Compare [19], where the sets $\mathcal{F}_i$ are defined with $\bar{\mathcal{W}} = \mathcal{W}$. In particular, the following assumption is made on the sets $\mathcal{T}_i$ and $\bar{\mathcal{W}}$. Different methods for choosing these sets will be discussed in Section VI.

Assumption 2: Let $\mathcal{H}_0 := \{0\}$ and

$$\mathcal{H}_{i+1} := A(\mathcal{H}_i \cap \mathcal{T}_i) \oplus \mathcal{W} \tag{11}$$

for $i \in \mathbb{N}_{[0,N]}$. It holds that

$$\mathcal{H}_i \subseteq \mathcal{F}_i, \; i \in \mathbb{N}_{[0,N]}. \tag{12}$$

■

Remark 4:

1) Assumption 2 is not restrictive: if there exists an upper bound on the threshold sets containing the origin, that is $(\{0\} \cup \mathcal{T}_i) \subseteq \bar{\mathcal{T}}$ for all $i \in \mathbb{N}_{[0,N]}$, then it holds that $\mathcal{H}_i \subseteq A\bar{\mathcal{T}} \oplus \mathcal{W}$ for all $i \in \mathbb{N}_{[0,N]}$. Hence, using $0 \in \mathcal{W}$, the inclusion in (12)—and therefore Assumption 2—is satisfied with $\bar{\mathcal{W}} = A\bar{\mathcal{T}} \oplus \mathcal{W}$. In general, however, this choice of $\bar{\mathcal{W}}$ may be conservative.
2) From (10), it follows that $\mathcal{F}_0 = \{0\}$. With $\mathcal{T}_0 = \{0\}$, we have that $\mathcal{H}_1 = \mathcal{W}$, such that with (12) for $i = 1$ it follows that $\mathcal{W} \subseteq \bar{\mathcal{W}}$, implying that $0 \in \mathcal{F}_i$ for $i \in \mathbb{N}$. Furthermore, it holds that
$$(A+BK)^j \mathcal{F}_i \oplus \mathcal{F}_j = \mathcal{F}_{i+j} \tag{13}$$
for $i, j \in \mathbb{N}$, see also [27]. ■
3) Note that we use a single sequence of sets $\mathcal{F}_i$, $i \in \mathbb{N}_{[0,N]}$, to capture all possible future states under event-triggered feedback here. In order to obtain a tighter bound on the uncertainty in the prediction, a different sequence of sets could be used for every different assumed sequence of interevent times within the prediction horizon. As the sequence of future interevent times is unknown, the constraints in the MPC problem would have to be enforced for every possible one of these sequences, leading to an exponential growth of complexity in the prediction horizon N. ■

The following assumption on the terminal set $\mathcal{X}_f \subseteq \mathbb{R}^n$, equivalent to the choice of the terminal set in [19], is required to hold.

Assumption 3: It holds that

$$\mathcal{X}_\mathrm{f} \subseteq \mathcal{X} \ominus \mathcal{F}_N \tag{14a}$$

$$K\mathcal{X}_\mathrm{f} \subseteq \mathcal{U} \ominus K\mathcal{F}_N \tag{14b}$$

$$(A+BK)\mathcal{X}_\mathrm{f} \oplus (A+BK)^N \bar{\mathcal{W}} \subseteq \mathcal{X}_\mathrm{f}. \tag{14c}$$

■

Lemma 2: It holds that

$$(A+BK)^k \mathcal{X}_\mathrm{f} \subseteq \mathcal{X} \ominus \mathcal{F}_{N+k} \tag{15a}$$

$$K(A+BK)^k \mathcal{X}_\mathrm{f} \subseteq \mathcal{U} \ominus K\mathcal{F}_{N+k} \tag{15b}$$

$$(A+BK)^k \mathcal{X}_\mathrm{f} \oplus (A+BK)^N \mathcal{F}_k \subseteq \mathcal{X}_\mathrm{f} \tag{15c}$$

for all $k \in \mathbb{N}$. ■

The proof is given in the appendix.

V. Main Properties of the MPC Scheme

In this section, the most important properties of the proposed event-triggered MPC scheme are presented, that is, well-definedness of the controller, robust constraint satisfaction, and asymptotic stability of a compact set for the closed-loop system.

A. Recursive Feasibility and Robust Constraint Satisfaction

The following lemma ensures that system (9) is well defined in the sense that if the optimization problem in (8), defining the controller and the event conditions, is feasible at initialization, then it remains feasible for all event instants (recursive feasibility).

Lemma 3: Let any $t \in \mathbb{N}$, any $x_t \in \mathbb{R}^n$, and any $\mathbf{d} = ((x_{0|t}, \ldots, x_{N|t}), (u_{0|t}, \ldots, u_{N-1|t})) \in \mathcal{D}_N(x_t)$ be given. Let further $x_{t+s+1} = Ax_{t+s} + Bu_{s|t} + w_{t+s}$ with $w_{t+s} \in \mathcal{W}$ for all $s \in \mathbb{N}_{[0,i-1]}$, where

$$t + i = \min\{j \in \mathbb{N}_{\geq t+1} \mid x_j \notin x_{j-t|t} \oplus \mathcal{T}_j\}. \tag{16}$$

Then there exists a $\tilde{\mathbf{d}} \in \mathcal{D}_N(x_{t+i})$. ■

Proof: Let $t + i$ satisfy (16) for an $i \in \mathbb{N}_{[1,N]}$. Then, it holds that $x_{t+j+1} - x_{j+1|t} = A(x_{t+j} - x_{j|t}) + w_{t+j}$ for all $j \in \mathbb{N}_{[0,i-1]}$. As $\mathcal{T}_0 = \{0\}$ and no event was triggered at the time points $t + j$ for $j \in \mathbb{N}_{[1,i-1]}$, it holds that $x_{t+j} - x_{j|t} \in \mathcal{T}_j$ for all $j \in \mathbb{N}_{[0,i-1]}$. By (11) and (12), using induction, it follows that

$$x_{t+j} - x_{j|t} \in \mathcal{H}_j \subseteq \mathcal{F}_j \tag{17}$$

for all $j \in \mathbb{N}_{[0,i]}$. The remainder of the proof can be obtained by extending the results in [19], as sketched in the following. Consider the decision variable $\tilde{\mathbf{d}} := ((x_{0|t+i}, \ldots, x_{N|t+i}), (u_{0|t+i}, \ldots, u_{N-1|t+i}))$, where

$$x_{j|t+i} := (A+BK)^j (x_{t+i} - x_{i|t}) + x_{j+i|t} \quad j \in \mathbb{N}_{[0,N]} \tag{18a}$$

$$u_{j|t+i} := K(A+BK)^j (x_{t+i} - x_{i|t}) + u_{j+i|t} \quad j \in \mathbb{N}_{[0,N-1]} \tag{18b}$$

$$x_{j+i|t} := (A+BK)^{j+i-N} x_{N|t}, \; j \in \mathbb{N}_{[N-i+1,N]} \tag{18c}$$

$$u_{j+i|t} := K(A+BK)^{j+i-N} x_{N|t}, \; j \in \mathbb{N}_{[N-i,N-1]} \tag{18d}$$

which is obtained by applying a feedback on the error between the actual state x_{t+i} and the predicted state $x_{i|t}$ [see (18a) and (18b)], and by extending the predicted trajectory at time t by a linear feedback [see (18c) and (18d)].

Using (13), (17), and Lemma 2, it readily follows that $\tilde{\mathbf{d}} \in \mathcal{D}_N(x_{t+i})$. ■

The following theorem guarantees the satisfaction of the constraints in the closed-loop system.

Theorem 1: For all $x_0 \in \hat{\mathcal{X}}_N$ and any realization of the disturbances $w_t \in \mathcal{W}$, $t \in \mathbb{N}$, it holds that $x_t \in \mathcal{X}$ and $\kappa(x_{t_j}, t - t_j) \in \mathcal{U}$ for all $t \in \mathbb{N}_{[t_j, t_{j+1}-1]}$, $j \in \mathbb{N}$, for the closed-loop system (9). ■

Proof: By Lemma 3, the closed-loop system is well defined. This implies that for all $t \in \mathbb{N}_{[t_j, t_{j+1}-1]}$, $j \in \mathbb{N}$, it holds that $\kappa(x_{t_j}, t - t_j) = u_{t-t_j|t}$ for a $u_{t-t_j|t} \in \mathcal{U} \ominus K\mathcal{F}_{t-t_j} \subseteq \mathcal{U}$, see (4d) and (6b), where the latter inclusion follows from the fact that $0 \in \mathcal{F}_i$, see also Remark 4 2). Hence, the input constraints are satisfied for all times in the closed-loop system. Furthermore, by (17), it holds that $x_t \in x^\star_{t-t_j|t_j} \oplus \mathcal{F}_{t-t_j}$ for all $t \in \mathbb{N}_{[t_j, t_{j+1}-1]}$, $j \in \mathbb{N}$, with $x^\star_{t-t_j|t_j} \oplus \mathcal{F}_{t-t_j} \subseteq \mathcal{X}$, see (4c) and (6a), such that $x_t \in \mathcal{X}$ for all $t \in \mathbb{N}$. Hence, also the state constraint is satisfied for all times in the closed-loop system. ■

B. Closed-Loop Stability Guarantees

In addition to the stage cost function ℓ used in (7), the stability proof presented below relies on functions $V_\mathrm{f} : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^n \to \mathbb{R}$ satisfying the following assumptions.

Assumption 4: The function ℓ is continuous and positive semidefinite and the functions V_f and q are continuous, positive definite, and radially unbounded. Furthermore, for all $x \in \mathbb{R}^n$ and all $v \in \mathbb{R}^m$, it holds that

$$V_\mathrm{f}((A+BK)x + Bv) \leq V_\mathrm{f}(x) - q(x) + \ell(v). \tag{19}$$

■

From Assumption 4, it follows that there exist $\mathcal{K}_\infty$-functions α_1 and α_2, such that for all $x \in \mathbb{R}^n$ and all $v \in \mathbb{R}^m$ it holds that

$$q(x) \geq \alpha_1(|x|) \tag{20}$$

$$V_\mathrm{f}(x) \leq \alpha_2(|x|) \tag{21}$$

and

$$\ell(v) \geq \alpha_1(|Bv|). \tag{22}$$

Remark 5: Assumption 4 requires V_f to be an ISS-Lyapunov function for the system described by $x_{t+1} = (A+BK)x_t + Bv_t$, where $v_t \in \mathbb{R}^m$ is the input at time t. See also [32], where an ISS-control Lyapunov function is used as a terminal cost. ■

In Appendix IX, we show how Assumption 4 can be satisfied in the case of quadratic and piecewise linear functions.

Lemma 4: For all $i \in \mathbb{N}$, all $x \in \mathbb{R}^n$, and all $v_j \in \mathbb{R}^m$, $j \in \mathbb{N}_{[0,i-1]}$, it holds that

$$V_\mathrm{f}\left((A+BK)^i x + \sum_{j=0}^{i-1}(A+BK)^{i-1-j}Bv_j\right) \leq V_\mathrm{f}(x) - \sum_{j=0}^{i-1} q\left((A+BK)^j x + \sum_{k=0}^{j-1}(A+BK)^{j-1-k}Bv_k\right) + \sum_{j=0}^{i-1}\ell(v_j). \tag{23}$$

■

Proof: It follows from (19) in Assumption 4 by induction.

Next, several properties of the optimal cost function will be stated, which ensure that it can be used in the construction of a Lyapunov function for the closed-loop system. The following lemma ensures a decrease of the optimal cost function along trajectories of the closed-loop system (9).

Lemma 5: Let any $t \in \mathbb{N}$, any $x_t \in \mathbb{R}^n$, and any $\mathbf{d} = ((x_{0|t}, \ldots, x_{N|t}), (u_{0|t}, \ldots, u_{N-1|t})) \in \mathcal{D}_N(x_t)$ be given. Let further $x_{t+s+1} = Ax_{t+s} + Bu_{s|t} + w_{t+s}$ with $w_{t+s} \in \mathcal{W}$ for all $s \in \mathbb{N}_{[0,i-1]}$, where

$$t + i = \min\{j \in \mathbb{N}_{\geq t+1} \mid x_j \notin x_{j-t|t} \oplus \mathcal{T}_j\}. \tag{24}$$

Then, there exists a $\tilde{\mathbf{d}} \in \mathcal{D}_N(x_{t+i})$ with

$$J_N(\tilde{\mathbf{d}}) \leq J_N(\mathbf{d}) - \sum_{j=0}^{i-1}\ell(u_{j|t} - Kx_{j|t}). \tag{25}$$

■

Proof: It follows readily that $\tilde{\mathbf{d}}$ as defined in the proof of Lemma 3 satisfies the requirements. ■

Let $\mathcal{Y} \subseteq \mathbb{R}^n$ denote the minimal robust positively invariant set for the dynamics $x_{t+1} = (A+BK)x_t + w_t$ with $w_t \in \bar{\mathcal{W}}$, $t \in \mathbb{N}$. In the following, it will be shown that the set $\mathcal{Y}$ is robustly asymptotically stable for the closed-loop system (9). Define the set

$$\bar{\mathcal{X}}_N := \{x \in \mathbb{R}^n \mid (A+BK)^N x \in \mathcal{X}_\mathrm{f}, (A+BK)^j x \in \mathcal{X}_j \; K(A+BK)^j x \in \mathcal{U}_j, j \in \mathbb{N}_{[0,N-1]}\} \tag{26}$$

which is the set of all states for which the optimization problem in (8) admits a feasible solution resulting from the application of the linear feedback $u = Kx$ at each predicted time step. Define further for any $x \in \hat{\mathcal{X}}_N$

$$V_N^0(x) := J_N^0(x) + \min_{y \in \mathcal{Y}} V_\mathrm{f}(x - y). \tag{27}$$

Remark 6: The function V_f is only used for the analysis of the closed-loop system but is not a part of the MPC optimization problem (where $J_N(\mathbf{d})$ is minimized). Note that adding V_f to this optimization problem in (8) does not change the set of minimizers, as the additional term does not depend on $\mathbf{d}$. ■

The following lemmas provide bounds on the function V_N^0.

Lemma 6: There exists a $\Gamma \in \mathbb{R}_{>0}$ such that for all $x \in \hat{\mathcal{X}}_N$, it holds that $V_N^0(x) \leq \Gamma$. Furthermore, if there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $\mathcal{Y} \oplus \epsilon\mathcal{B} \subseteq \bar{\mathcal{X}}_N$, then there exists a $\mathcal{K}_\infty$-function α_3 such that $V_N^0(x) \leq \alpha_3(|x|_\mathcal{Y})$ for all $x \in \hat{\mathcal{X}}_N$. ■

Proof: The first part of the statement follows from the continuity of the functions ℓ and V_f, and the boundedness of $\mathcal{X}$ and $\mathcal{U}$. The second part can be shown by modifying the proof of [33, Th. III.2]. In particular, note that $J_N^0(x) = 0$ for all $x \in \bar{\mathcal{X}}_N$. It follows that

$$V_N^0(x) = \min_{y \in \mathcal{Y}} V_\mathrm{f}(x - y) \leq \min_{y \in \mathcal{Y}} \alpha_2(|x - y|) = \alpha_2\left(\min_{y \in \mathcal{Y}} |x - y|\right) = \alpha_2(|x|_\mathcal{Y}) \tag{28}$$

for all $x \in \bar{\mathcal{X}}_N$. Hence, if there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $\mathcal{Y} \oplus \epsilon\mathcal{B} \subseteq \bar{\mathcal{X}}_N$, then $|x|_\mathcal{Y} \leq \epsilon$ implies $x \in \bar{\mathcal{X}}_N$ and therefore $V_N^0(x) \leq \alpha_2(|x|_\mathcal{Y})$. Together with the first part of the statement, this implies $V_N^0(x) \leq \alpha_3(|x|_\mathcal{Y})$ for all $x \in \hat{\mathcal{X}}_N$, where $\alpha_3(s) := \max\{1, \Gamma/\alpha_2(\epsilon)\}\alpha_2(s)$ for all $s \in \mathbb{R}_{\geq 0}$. ■

Lemma 7: For all $x_0 \in \hat{\mathcal{X}}_N$ and any realization of the disturbances $w_t \in \mathcal{W}$, $t \in \mathbb{N}$, it holds that

$$V_N^0(x_{t_{j+1}}) \leq V_N^0(x_{t_j}) - \sum_{t=t_j}^{t_{j+1}-1} \alpha_1(|x_t|_\mathcal{Y}) \tag{29}$$

for all $j \in \mathbb{N}$ for the closed-loop system (9). ■

The proof is given in the appendix.

We are now ready to state our main stability theorem.

Theorem 2: For all $x_0 \in \hat{\mathcal{X}}_N$ and any realization of the disturbances $w_t \in \mathcal{W}$, $t \in \mathbb{N}$, it holds that $\lim_{t\to\infty} |x_t|_\mathcal{Y} = 0$ for the closed-loop system (9). Furthermore, if there exists an $\epsilon > 0$ such that $\mathcal{Y} \oplus \epsilon\mathcal{B} \subseteq \bar{\mathcal{X}}_N$, then the set $\mathcal{Y}$ is robustly asymptotically stable for the closed-loop system (9). ■

Proof: For any $x_0 \in \hat{\mathcal{X}}_N$ and any $t \in \mathbb{N}$, it holds that

$$\begin{aligned}\sum_{i=0}^{t-1} \alpha_1(|x_i|_\mathcal{Y}) &\leq \sum_{j=0}^{t-1} \sum_{i=t_j}^{t_{j+1}-1} \alpha_1(|x_i|_\mathcal{Y}) \\ &\overset{\text{Lemma 7}}{\leq} \sum_{j=0}^{t-1} V_N^0(x_{t_j}) - V_N^0(x_{t_{j+1}}) \\ &= V_N^0(x_0) - V_N^0(x_{t_t}) \leq V_N^0(x_0).\end{aligned} \tag{30}$$

By Lemma 6, it holds that $V_N^0(x_0)$ is bounded such that, as t was arbitrary in the derivation above, it holds that $\sum_{i=0}^{\infty} \alpha_1(|x_i|_\mathcal{Y}) \leq V_N^0(x_0)$. As α_1 is a $\mathcal{K}_\infty$-function, it follows that $\lim_{t\to\infty} |x_t|_\mathcal{Y} = 0$. Furthermore, it follows that $\alpha_1(|x_t|_\mathcal{Y}) \leq V_N^0(x_0)$, $t \in \mathbb{N}$, such that if $\mathcal{Y} \oplus \epsilon\mathcal{B} \subseteq \bar{\mathcal{X}}_N$ for an $\epsilon \in \mathbb{R}_{>0}$, by Lemma 6 it holds that $|x_t|_\mathcal{Y} \leq \alpha_1^{-1}(\alpha_3(|x_0|_\mathcal{Y}))$, $t \in \mathbb{N}$, implying that the set $\mathcal{Y}$ is robustly asymptotically stable. ■

C. Event-Triggered Implementation of Standard Tube MPC

In this section, we describe how the parameters of the scheme may be chosen if the main objective is a large region of attraction $\hat{\mathcal{X}}_N$ and a small asymptotic bound $\mathcal{Y}$ for the closed-loop system.

In particular, let the threshold sets be defined by[1]

$$\mathcal{T}_i = A^{-1}(\mathcal{F}_{i+1} \ominus \mathcal{W}) \tag{31}$$

for $i \in \mathbb{N}_{[1,N-1]}$.

Lemma 8: With $\mathcal{T}_i$ as defined in (31) for $i \in \mathbb{N}_{[1,N-1]}$, Assumption 2 is satisfied. ■

Proof: For $i = 0$, the condition in (12) is trivially satisfied. Assume now that $\mathcal{H}_i$ is defined as in Assumption 2 for all $i \in \mathbb{N}_{[0,N]}$. Let further $\mathcal{T}_i = A^{-1}(\mathcal{F}_{i+1} \ominus \mathcal{W})$ for all $i \in \mathbb{N}_{[1,N-1]}$ and note that $\mathcal{W} \subseteq \bar{\mathcal{W}}$. It follows that

$$\begin{aligned}\mathcal{H}_i &= A(\mathcal{H}_{i-1} \cap \mathcal{T}_{i-1}) \oplus \mathcal{W} \\ &\subseteq A\mathcal{T}_{i-1} \oplus \mathcal{W} \\ &= AA^{-1}(\mathcal{F}_i \ominus \mathcal{W}) \oplus \mathcal{W} \\ &\subseteq (\mathcal{F}_i \ominus \mathcal{W}) \oplus \mathcal{W} \subseteq \mathcal{F}_i\end{aligned} \tag{32}$$

for $i \in \mathbb{N}_{[1,N]}$. ■

[1]Note that there is no need to assume that A is nonsingular here.

Under these restrictions, the asymptotic bound is minimized for the choice $\bar{\mathcal{W}} = \mathcal{W}$. In this way, the resulting event-triggered MPC scheme requires the same tightening of constraints, and guarantees the same worst case asymptotic bound, as the all-time triggered scheme in [19]. This also implies that any periodically triggered Tube MPC scheme (not necessarily updating the inputs at every time point t) can be improved in terms of the average required communication in this way by using an appropriate event-triggered MPC scheme in its place. In the event-triggered scheme, whenever the periodically updated (all-time) scheme would normally schedule an update, an event condition along the lines of (31) would be checked beforehand. The actual amount of reduction in communication depends on the particular probability density function $\mathbf{p}_w$.

VI. Probabilistic Guarantees

In this section, we propose techniques to choose the parameters $\bar{\mathcal{W}}$ and $\mathcal{T}_i$ based on knowledge about the probability density function $\mathbf{p}_w$.

A. Event-Triggered Implementation With Assigned Probability Distribution of the Interevent Times

In this section, we provide a means to design the sets $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1}$ and the set $\bar{\mathcal{W}}$, such that the probabilities for the time between event instants attain desired values in the closed-loop system and Assumption 2 are satisfied. Consider the probability that an event is triggered at time point $t + i$ given that the last event occurred at time point t

$$\mathbb{P}(x_{t+i} \notin x_{i|t} \oplus \mathcal{T}_i, x_{t+j} \in x_{j|t} \oplus \mathcal{T}_j, j \in \mathbb{N}_{[1,i-1]}). \tag{33}$$

It holds that $x_{t+j} - x_{j|t} = \sum_{s=0}^{j-1} A^s w_{t+j-1-s}$ for $j \in \mathbb{N}_{[1,i]}$. Furthermore, the disturbances are assumed to be distributed identically and independently, such that the probability in (33) is independent of x_t and t and is given by

$$P_i := \mathbb{P}\left(\sum_{s=0}^{i-1} A^s w_{i-1-s} \notin \mathcal{T}_i, \sum_{s=0}^{j-1} A^s w_{j-1-s} \in \mathcal{T}_j, j \in \mathbb{N}_{[1,i-1]}\right) \tag{34}$$

where the disturbances w_j, $j \in \mathbb{N}_{[0,i-1]}$, are generated according to the probability density function $\mathbf{p}_w$. As $\mathcal{T}_N = \emptyset$, and hence an event is guaranteed to occur for an $i \in \mathbb{N}_{[1,N]}$, it holds that

$$\sum_{i=1}^{N} P_i = 1, \quad P_i \geq 0,\, i \in \mathbb{N}_{[1,N]}. \tag{35}$$

As $\mathbf{p}_w$ was assumed to be bounded, it is possible to assign any values to P_i in (34) that satisfy (35), by appropriately choosing the sets $\mathcal{T}_i$. In turn, choosing the probabilities P_i allows the assignment of the average interevent time. It is also possible to guarantee a minimum interevent time $i_{\min} \in \mathbb{N}_{[1,N]}$ by choosing $P_i = 0$ for $i \in \mathbb{N}_{[1,i_{\min}-1]}$.

We propose the following simple method of choosing the sets $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1}$. Let $\mathcal{T} \subseteq \mathbb{R}^n$ be any PC-set and define $\mathcal{T}_i = \rho_i \mathcal{T}$ for $\rho_i \in \mathbb{R}_{\geq 0}$ and $i \in \mathbb{N}_{[1,N-1]}$. With this definition, it holds that $\rho_r \leq \rho_s \Leftrightarrow \mathcal{T}_r \subseteq \mathcal{T}_s$ for all $r, s \in \mathbb{N}_{[1,N-1]}$.

Lemma 9: Let $\mathcal{T} \subseteq \mathbb{R}^n$ be a PC-set and let

$$P_i(\rho_1, \ldots, \rho_{i-1}, \rho_i) := \mathbb{P}\left(\sum_{s=0}^{i-1} A^s w_{i-1-s} \notin \rho_i \mathcal{T}, \sum_{s=0}^{j-1} A^s w_{j-1-s} \in \rho_j \mathcal{T}, j \in \mathbb{N}_{[1,i-1]}\right) \tag{36}$$

for all $\rho_j \in \mathbb{R}_{\geq 0}$ for $j \in \mathbb{N}_{[1,i]}$, $i \in \mathbb{N}_{[1,N-1]}$. For $\rho_j, j \in \mathbb{N}_{[1,i-1]}$ fixed, it holds that P_i is a continuous and monotonically nonincreasing function of ρ_i. Furthermore, it holds that $P_i(\rho_1, \ldots, \rho_{i-1}, 0) = 1 - \sum_{j=1}^{i-1} P_i$ and $P_i(\rho_1, \ldots, \rho_{i-1}, \rho_i) = 0$ for ρ_i sufficiently large. ■

The proof is given in the appendix.

Lemma 9 ensures that for any given desired P_i, an appropriate selection of $(\rho_1, \rho_2, \ldots, \rho_i)$ exists. Note that if the scalars ρ_j are given for $j \in \mathbb{N}_{[1,i-1]}$, then P_i is a function of ρ_i only. Hence, for desired values of P_i and given set $\mathcal{T}$, the values of ρ_i may be computed sequentially, starting with ρ_1. The expected value of the interevent time is given by $\mathbb{E}[t_{j+1} - t_j] = \sum_{i=1}^{N} iP_i$ for any given $j \in \mathbb{N}$. Hence, by choosing P_i, (and, in turn, $\mathcal{T}_i$) accordingly, any desired value $\mathbb{E}[t_{j+1} - t_j] = \bar{\Delta}$ for $\bar{\Delta} \in [1, N]$ may be achieved.

Remark 7: By [34, Th. 1], it follows that the average event frequency converges to $1/\bar{\Delta}$ for the closed-loop system as time increases in the sense that $\lim_{t\to\infty} \frac{1}{t}\mathbb{E}(\max\{j \in \mathbb{N} \mid t_j \leq t\}) = 1/\bar{\Delta}$. ■

If the sets $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1}$ are known, the enlarged disturbance set $\bar{\mathcal{W}}$ may be determined by defining $\bar{\mathcal{W}} := \bar{\rho}\mathcal{W}$ for a $\bar{\rho} \in \mathbb{R}_{\geq 1}$ such that Assumption 2 is satisfied. In this case, it holds that $\mathcal{Y} = \bar{\rho}\mathcal{Y}^\star$, where $\mathcal{Y}^\star$ is the minimal robust positively invariant set for the case $\bar{\mathcal{W}} = \mathcal{W}$, which follows immediately from [30, eq. (3)]. Similarly, with this choice, the uncertainty sets $\mathcal{F}_i$ appearing in the MPC scheme result from the uncertainty sets appearing in standard Tube MPC by a scaling with $\bar{\rho}$. Hence, this approach shares some similarities with the periodically triggered robust MPC approach in [22].

B. Event-Triggered Implementation With Assigned Expected Value of the Interevent Times

If only the expected value of the interevent $\mathbb{E}[t_{j+1} - t_j]$, but not the specific probabilities P_i are of interest, the following method of choosing $\mathcal{T}_i$ allows a better tradeoff between the average sampling rate and the size of the asymptotic bound. For this, we parameterize the assumed disturbance bound by $\hat{\mathcal{W}} = \hat{\rho}\mathcal{W}$, $\hat{\rho} \in \mathbb{R}_{\geq 0}$, and choose $\mathcal{T}_i$ analogously to (31), that is

$$\mathcal{T}_i(\hat{\rho}) := A^{-1}\left(\left(\bigoplus_{j=0}^{i}(A+BK)^j \hat{\rho}\mathcal{W}\right) \ominus \mathcal{W}\right) \tag{37}$$

for $i \in \mathbb{N}_{[1,N-1]}$. The following statement establishes conditions under which $\mathbb{E}[t_{j+1} - t_j]$ is a continuous and monotonically nonincreasing function of $\hat{\rho}$, which ensures that an appropriate $\hat{\rho}$ exists guaranteeing $\mathbb{E}[t_{j+1} - t_j] = \bar{\Delta}$ for a given $\bar{\Delta} \in \mathbb{N}_{[1,N]}$.

Lemma 10: Let $\hat{\Delta} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denote the function mapping $\hat{\rho}$ in (37) to the expected value $\mathbb{E}[t_{j+1} - t_j]$ of the time between events. Assume that $\mathcal{W}$ is a polyhedron and that A is nonsingular. Then, it holds that $\hat{\Delta}$ is monotonically nondecreasing and continuous. Furthermore, it holds that $\hat{\Delta}(0) = 1$ and $\hat{\Delta}(\hat{\rho}) = N$ for $\hat{\rho}$ sufficiently large. ■

The proof is given in the appendix.

Finally, in order to satisfy Assumption 2, we choose $\bar{\mathcal{W}} = \bar{\rho}\mathcal{W}$ with $\bar{\rho} = \max\{1, \hat{\rho}\}$ such that $\mathcal{Y} = \bar{\rho}\mathcal{Y}^\star$ as in Section VI-A.

Remark 8: Note that if the system description in (1) is obtained by discretizing a continuous-time linear system by a sample-and-hold method, then A will always be nonsingular. ■

TABLE I
DISTRIBUTION OF INTEREVENT TIMES FOR AN EVENT-TRIGGERED IMPLEMENTATION OF A TUBE MPC SCHEME

Interevent time	Frequency
1	77.96%
2	12.82%
3	4.63%
4	1.95%
5	1.12%
6	0.68%
7	0.34%
8	0.22%
9	0.11%
10	0.19%

VII. NUMERICAL EXAMPLES

In this section, we provide three examples showing the reduction of necessary communication with the proposed scheme.

A. Event-Triggered Implementation of a Given Tube MPC Scheme

Let the system be given by

$$x_{t+1} = \begin{bmatrix} 1.1 & 0.2 \\ 0 & 1.2 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t + w_t \tag{38}$$

where w_t is independently uniformly distributed on $\mathcal{W} = [-1, 1]^2$ for $t \in \mathbb{N}$. The feedback matrix K has been chosen LQ-optimal with the weighting matrices $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $R = 1$. In the first example, we investigated the closed-loop behavior for sets $\mathcal{T}_i$, $i \in \mathbb{N}_{[1,N-1]}$, constructed as proposed in Section V-C. Consequently, we defined $\bar{\mathcal{W}} = \mathcal{W}$.

1) Asymptotic Behavior: In a first step, we only investigated the sampling frequency in the control scheme. Note that the triggering behavior only depends on the realization of the disturbance sequence, such that neither the initial condition x_0, nor the constraints, nor the stage cost function ℓ have an influence on the number of communications in the closed-loop system. A simulation of $T_{\text{sim}} = 10^5$ steps yielded the distribution of time steps between events displayed in Table I. The average time between events was 1.42, which amounts to a 29% reduction in communication compared to a control scheme updated at every point in time. Note that the region of attraction and worst-case asymptotic bound on the system state are exactly the same for the all-time scheme in [19] and the event-triggered scheme presented here. The reason for this is that the uncertainty induced by not using the actual state information to update the input at every point in time is *not* modeled as an additional disturbance bounded by the thresholds sets (as, for example, done in [5]). Instead, we exploit the fact that this uncertainty is already included in the uncertainty description used in standard Tube MPC, by virtue of the particular choice of thresholds in Section V-C. Hence, we do not require additional constraint tightening.

Consider further the performance index $J_{\mathrm{perf}} := 1/T_{\mathrm{sim}} \sum_{t=0}^{T_{\mathrm{sim}}-1} x_t^\top Q x_t + u_t^\top R u_t$. For this example, where we initialized the system state at $x_0 = (0,0)^\top$, the performance index for the closed-loop system with an MPC update at every time step was $J_{\mathrm{perf}}^{\mathrm{pt}} = 5.69$. The performance index for the event-triggered scheme was $J_{\mathrm{perf}}^{\mathrm{et}} = 5.79$, which amounts to a 1.73% increase when compared to the scheme with updates at every point in time. We explain the worsening of the performance by noting that although the worst case bound on the uncertainty is the same as in an all-time scheme, its probability distribution is deformed such that it becomes more likely that the system state is localized closer to the boundary of the asymptotic bound in the event-triggered case. The reason for this is the existence of time spans of (unstable, in this example) open-loop control. The sequence of disturbances in the simulation of both control schemes was chosen to be identical.

2) Transient Behavior: In order to investigate the transient behavior of the closed-loop system, we consider now the constraints $\mathcal{X} = [-30, 30]^2$ and $\mathcal{U} = [-10, 10]$. The terminal set $\mathcal{X}_{\mathrm{f}}$ was computed as the maximal set satisfying Assumption 3, using the algorithm in [27]. As proposed in [19], the stage function ℓ was chosen as $\ell : v \mapsto v^\top L v$ with $L = R + B^\top P B$, where P is the stabilizing solution of the discrete-time algebraic Riccati equation with the weighting matrices R and Q.

We simulated the resulting closed-loop system for 300 random initial conditions in $[-30, 30]^2$, where initial conditions in $\mathcal{X}_{\mathrm{f}}$ and initial conditions for which the MPC problem was infeasible were rejected and resampled. For each initial condition, we simulated the closed-loop system both with the periodically triggered controller and the event-triggered controller, for 10 random realizations of the disturbance sequence and $T_{\mathrm{sim}} = 10$ steps. These disturbance realizations were resampled for each initial condition, but kept the same for the simulation with the periodically triggered and the event-triggered controller. The performance index for the periodically triggered controller, averaged over all simulations, was $J_{\mathrm{perf}}^{\mathrm{pt}} = 364.12$. The average performance index for the event-triggered scheme was $J_{\mathrm{perf}}^{\mathrm{et}} = 366.04$, which amounts to a 0.5% increase when compared to the scheme with updates at every point in time. The average time between events was 1.31. In Fig. 3, exemplary state trajectories of the closed-loop systems are depicted together with the terminal set and the sampled initial conditions. The associated input trajectories are displayed in Fig. 4.

B. Event-Triggered Tube MPC With Assigned Distribution of Interevent Times

In the second example, we implemented the scheme as proposed in Section VI-A. Consider the same setup as in the first example. We chose $\mathcal{T} = [-1, 1]^2$ and used a stochastic approximation approach as proposed in [35] to determine iteratively, starting with $i = 1$, appropriate values for ρ_i such that the probabilities P_i attained their desired values. These values were chosen to $P_1 = P_5 = 0.1$, $P_2 = P_4 = 0.2$, $P_3 = 0.4$, and $P_6 = P_7 = \cdots = P_{10} = 0$, which imply a desired average of 3 time steps between event instants and, thus, an average reduction in communication by 66.7% compared to a control scheme updated at every point in time. The application of stochastic approximation resulted in $\rho_1 = 0.9485$, $\rho_2 = 1.3766$, $\rho_3 = 0.9774$, and $\rho_4 = 0.7736$. The remaining ρ_i for $i \in \mathbb{N}_{[5,10]}$ were set to 0. In order to validate the results, we evaluated the probabilities associated with these ρ_i by a Monte-Carlo simulation, resulting in the 95% confidence intervals $P_1 \in [0.0979, 0.1016]$, $P_2 \in [0.1970, 0.2019]$, $P_3 \in [0.3961, 0.4022]$, $P_4 \in [0.1850, 0.1898]$, and $P_5 \in [0.1077, 0.1116]$. The value of $\bar{\rho}$ was computed to $\bar{\rho} = 2.0318$, which is at the same time the factor describing the increase in the guaranteed asymptotic bound on the system state and the increase in necessary constraint tightening (resulting also in a change of the terminal set $\mathcal{X}_{\mathrm{f}}$). The region of attraction depends on the constraints on the state and input, which were not considered in these examples. Note that a tightening of constraints does not necessarily lead to a reduction of the region of attraction. A simulation of 10^5 steps yielded the distribution of time steps between events displayed in Table II.

The average time between events was 3.0149, which amounts to a 66.8% reduction in communication. For this example, where we again initialized the system state at $x_0 = (0,0)^\top$, we obtained $J_{\mathrm{perf}}^{\mathrm{pt}} = 5.75$ and $J_{\mathrm{perf}}^{\mathrm{et}} = 7.83$, which amounts to a 36% increase of the performance index for the event-triggered scheme when compared to a scheme with updates at every point in time.

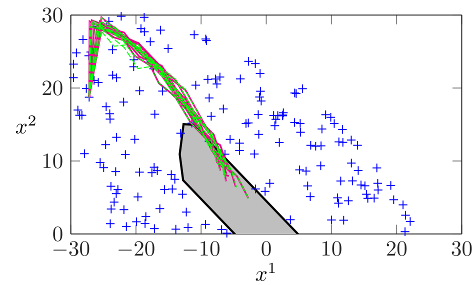

Fig. 3. Random initial conditions used for the simulations (blue crosses) and resulting state trajectories for $x_0 = (-27.19, 18, 74)^\top$, (magenta and solid: periodically triggered scheme, green and dashed: event-triggered scheme). The terminal set is depicted in gray. As the setup is symmetric, only the top half of the state constraint set is depicted.

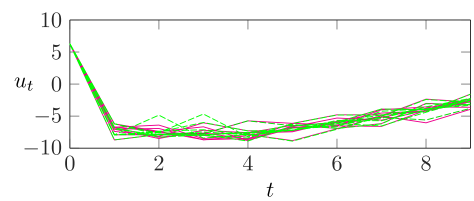

Fig. 4. Resulting input trajectories for $x_0 = (-27.19, 18, 74)^\top$, (magenta and solid: periodically triggered scheme, green and dashed: event-triggered scheme).

TABLE II
DISTRIBUTION OF INTEREVENT TIMES FOR AN EVENT-TRIGGERED TUBE MPC SCHEME WITH ASSIGNED TRIGGER PROBABILITIES

Interevent time	Frequency
1	10.04%
2	19.82%
3	39.92%
4	19.01%
5	11.20%

TABLE III
DISTRIBUTION OF INTEREVENT TIMES FOR AN EVENT-TRIGGERED TUBE MPC SCHEME WITH ASSIGNED AVERAGE SAMPLING RATE

Interevent time	Frequency
1	41.18%
2	17.62%
3	10.91%
4	7.86%
5	6.18%
6	4.32%
7	3.15%
8	2.51%
9	1.80%
10	4.45%

C. Event-Triggered Tube MPC With Assigned Average Interevent Time

In the third example, we implemented the scheme as proposed in Section VI-B. Consider again the setup as in the first example. Similarly to the second example, we used a stochastic approximation approach to find a $\hat{\rho}$ corresponding to the definition of the threshold sets in (37), such that the average intersampling time is approximately 3. This resulted in $\bar{\rho} = \hat{\rho} = 1.3655$, which is significantly smaller than the value of $\bar{\rho}$ in the previous example (2.0318). An *a posteriori* Monte-Carlo simulation yielded a 95% confidence interval for the average time between events of $[2.9761, 3.0077]$.

A simulation of 10^5 steps yielded the distribution of time steps between events displayed in Table III. The average time between events was 3.0032. For this example, where we again initialized the system state at $x_0 = (0, 0)^\top$, we obtained $J_{\mathrm{perf}}^{\mathrm{pt}} = 5.71$ and $J_{\mathrm{perf}}^{\mathrm{et}} = 6.92$, which amounts to a 21% increase of the performance index for the event-triggered scheme when compared to a scheme with updates at every point in time, again a significant improvement over the results in the second example (where the increase in the performance index was 36%).

D. Discussion

Similar to other event-triggered schemes (see, e.g., [13]), the reduction in communication for the scheme presented here is expected to be especially large for sporadically occurring disturbances, that is, for probability density functions $\mathbf{p}_w$, which are concentrated around the origin and imply a large difference between the average and the worst case disturbance magnitude. However, as shown in the examples, even for uniform disturbances a significant reduction can be achieved. As feedback is present at fewer time instances in the event-triggered scheme than in an all-time scheme, the threshold design proposed in Section V-C trades a reduction in communication for worsening of the closed-loop performance, as seen in the first example.

Note that the closed-loop behavior of the event-triggered scheme depends very much on the *assumed* bound on the disturbances. If, in an all-time scheme, the actual disturbance magnitude is lower than expected, this is reflected in the asymptotic bound on the closed-loop system state being also lower. In an event-triggered scheme on the other hand, the asymptotic bound might still be the one for the assumed worst case disturbance magnitude, even if the actual disturbances are smaller. As an interpretation, while an all-time triggered scheme exploits less-than-worst-case disturbances by bringing the state closer to the origin, the event-triggered scheme exploits such disturbances by reducing the communication rate.

VIII. CONCLUSIONS AND OUTLOOK

We have presented a robust event-triggered MPC scheme based on Tube MPC methods. It was shown that the required amount of communication in the control system can be reduced without sacrificing the guarantees offered by a periodically updated Tube MPC scheme. Further reduction of the required communication and assignment of a desired expected value of the time between events is possible by allowing a larger guaranteed asymptotic bound on the system state and tightening the constraints in the prediction. In this way, the event-triggered scheme can be tuned to get a desired tradeoff between the number of communications and the closed-loop control properties.

The results in this paper rely on the fact that the disturbances are independent and that the expected value of the time between events only depends on the disturbances occurring in exactly this time span. These assumptions are not necessarily satisfied in the case of output feedback or disturbances generated by a (randomly disturbed) exosystem, both of which are subject to future research. Another point requiring further investigation is input-to-state stability for event-triggered schemes of the kind presented in this paper. Furthermore, properties of the underlying communication network, such as quantization, delay, and packet loss, should be taken into account in future considerations. Finally, a future research direction will be the application of Stochastic MPC techniques (see [36] for a recent overview of the subject) to the problem of event-triggered stabilization; these methods allow knowledge of the stochastic properties of the disturbance to be taken into account in order to relax robust constraints toward chance constraints and to consider the expected value of the infinite horizon performance index as a cost function.

APPENDIX A
CONSTRUCTION OF THE FUNCTION V_{f}

Assume first that the function ℓ is quadratic, that is, given by $\ell(v) = v^\top L v$ with $L = L^\top \geq 0$. Furthermore, considering

(22), assume that there exists a matrix $M = M^\top > 0$ such that $L \geq B^\top M B$. By Assumption 1, there exist matrices $S = S^\top > 0$ and $Q = Q^\top > 0$ such that $(A+BK)^\top S(A+BK) = S - Q$. It follows that for all $x \in \mathbb{R}^n$ and all $v \in \mathbb{R}^m$, it holds that

$$\begin{aligned}
&\eta\left((A+BK)x+Bv\right)^\top S\left((A+BK)x+Bv\right)\\
&= \eta x^\top (A+BK)^\top S(A+BK)x\\
&\quad + \eta 2x^\top (A+BK)^\top SBv + \eta v^\top B^\top SBv\\
&= \eta x^\top Sx - \eta x^\top Qx\\
&\quad - \eta\left(\frac{Q}{\sqrt{2}}x - \sqrt{2}(A+BK)^\top SBv\right)^\top Q^{-1}\\
&\quad \times \left(\frac{Q}{\sqrt{2}}x - \sqrt{2}(A+BK)^\top SBv\right) + \frac{\eta}{2}x^\top Qx\\
&\quad + \eta v^\top\big(2B^\top S(A+BK)Q^{-1}(A+BK)^\top SB + B^\top SB\big)v\\
&\quad \leq \eta x^\top Sx - \frac{\eta}{2}x^\top Qx + v^\top Lv
\end{aligned} \tag{39}$$

if $\eta \in \mathbb{R}_{>0}$ and $L - \eta(2B^\top S(A+BK)Q^{-1}(A+BK)^\top SB + B^\top SB) \geq 0$. The latter inequality holds if $M - \eta(2S(A+BK)Q^{-1}(A+BK)^\top S + S) \geq 0$, such that, as $M > 0$, it is always possible to find an $\eta \in \mathbb{R}_{>0}$ such that these inequalities are satisfied. Hence, Assumption 4 is satisfied with $V_{\mathrm{f}}(x) = \eta x^\top Sx$, and $q(x) = \frac{\eta}{2}x^\top Qx$, where η, S, and Q satisfy the requirements above.

Second, assume that ℓ is piecewise linear and given by $\ell(v) = \|Lv\|_\infty$ for a matrix $L \in \mathbb{R}^{p\times n}$ with some $p \in \mathbb{N}$, and that there exists an $a_\ell \in \mathbb{R}_{>0}$ such that $\|Lv\|_\infty \geq a_\ell\|Bv\|_\infty$ for all $v \in \mathbb{R}^m$. By Assumption 1, there exists a matrix $S \in \mathbb{R}^{r\times n}$ such that $\|S(A+BK)x\|_\infty \leq \|Sx\|_\infty - \|x\|_\infty$ for $x \in \mathbb{R}^n$ (see, for example, [37] and [38]). It follows that for all $x \in \mathbb{R}^n$ and all $v \in \mathbb{R}^m$, it holds that

$$\begin{aligned}
&\eta\|S((A+BK)x+Bv)\|_\infty\\
&\qquad \leq \eta\|S(A+BK)x\|_\infty + \eta\|SBv\|_\infty\\
&\qquad \leq \eta\|Sx\|_\infty - \eta\|x\|_\infty + \eta\|SBv\|_\infty\\
&\qquad \leq \eta\|Sx\|_\infty - \eta\|x\|_\infty + \eta\|S\|_\infty\|Bv\|_\infty\\
&\qquad \leq \eta\|Sx\|_\infty - \eta\|x\|_\infty + \|Lv\|_\infty
\end{aligned} \tag{40}$$

if $\eta \in \mathbb{R}_{>0}$ and $\eta\|S\|_\infty \leq a_\ell$. Hence, Assumption 4 is satisfied with $V_{\mathrm{f}}(x) = \eta\|Sx\|_\infty$, and $q(x) = \eta\|x\|_\infty$, where η and S satisfy the requirements above.

Appendix B
Proof of Lemma 2

Proof: The statement follows by induction using Lemma 1, (13), and Assumption 3. The induction base is provided by Assumption 3 and $\mathcal{F}_0 = \{0\}$. Assume now that (15c) holds for some $k \in \mathbb{N}$. It follows that

$$\begin{aligned}
&(A+BK)^{k+1}\mathcal{X}_{\mathrm{f}} \oplus (A+BK)^N\mathcal{F}_{k+1}\\
&\overset{(13)}{=} (A+BK)^{k+1}\mathcal{X}_{\mathrm{f}} \oplus (A+BK)^N\left((A+BK)\mathcal{F}_k \oplus \bar{\mathcal{W}}\right)\\
&= (A+BK)\Bigg(\underbrace{(A+BK)^k\mathcal{X}_{\mathrm{f}} \oplus (A+BK)^N\mathcal{F}_k}_{\subseteq\mathcal{X}_{\mathrm{f}}\text{ by induction assumption }(15c)}\Bigg)\\
&\qquad \oplus (A+BK)^N\bar{\mathcal{W}}
\end{aligned} \tag{41}$$

$$\overset{\text{Assumption 3}}{\subseteq} \mathcal{X}_{\mathrm{f}}$$

thereby completing the induction step proving (15c) for all $k \in \mathbb{N}$. Furthermore, using this result, it holds that

$$\begin{aligned}
&(A+BK)^k\mathcal{X}_f \oplus \mathcal{F}_{N+k}\\
&\quad \overset{(13)}{=} (A+BK)^k\mathcal{X}_f \oplus (A+BK)^N\mathcal{F}_k \oplus \mathcal{F}_N\\
&\quad \overset{(15c)}{\subseteq} \mathcal{X}_{\mathrm{f}} \oplus \mathcal{F}_N \overset{(14a)}{\subseteq} \mathcal{X}
\end{aligned}$$

and

$$\begin{aligned}
&K(A+BK)^k\mathcal{X}_f \oplus K\mathcal{F}_{N+k}\\
&\quad \overset{(13)}{=} K(A+BK)^k\mathcal{X}_f \oplus K(A+BK)^N\mathcal{F}_k \oplus K\mathcal{F}_N\\
&\quad \overset{(15c)}{\subseteq} K\mathcal{X}_{\mathrm{f}} \oplus K\mathcal{F}_N \overset{(14b)}{\subseteq} \mathcal{U}.
\end{aligned}$$

Appendix C
Proof of Lemma 7

Proof: Let any $x_0 \in \hat{\mathcal{X}}_N$, any realization of the disturbances $w_t \in \mathcal{W}$, $t \in \mathbb{N}$, and any $j \in \mathbb{N}$ be given. Let further $\mathbf{d}^\star_{t_j}(x_{t_j}) = ((x^\star_{0|t_j}, \ldots, x^\star_{N|t_j}), (u^\star_{0|t_j}, \ldots, u^\star_{N-1|t_j}))$. Then, by (17), it holds that $x_t = x^\star_{t-t_j|t_j} + f_t$ for some $f_t \in \mathcal{F}_{t-t_j}$ for all $t \in \mathbb{N}_{[t_j, t_{j+1}]}$. With $v_{t-t_j} := u^\star_{t-t_j|t_j} - Kx^\star_{t-t_j|t_j}$, $t \in \mathbb{N}_{[t_j, t_{j+1}-1]}$, it holds that

$$\begin{aligned}
V_N^0(x_{t_{j+1}}) &\overset{\text{Lemma 5}}{\leq} J_N^0(x_{t_j}) - \sum_{i=0}^{t_{j+1}-t_j-1} \ell(v_i)\\
&\qquad + \min_{y\in\mathcal{Y}} V_{\mathrm{f}}(x^\star_{t_{j+1}-t_j|t_j} + f_{t_{j+1}} - y)\\
&\leq J_N^0(x_{t_j}) - \sum_{i=0}^{t_{j+1}-t_j-1} \ell(v_i) + V_{\mathrm{f}}(x^\star_{t_{j+1}-t_j|t_j} + f_{t_{j+1}} - \tilde{y}_{t_{j+1}})
\end{aligned} \tag{42}$$

where $\tilde{y}_t := (A+BK)^{t-t_j}y_{t_j} + f_t$, $t \in \mathbb{N}_{[t_j, t_{j+1}]}$, and $y_{t_j} \in \mathcal{Y}$ such that $V_{\mathrm{f}}(x_{t_j} - y_{t_j}) = \min_{y\in\mathcal{Y}} V_{\mathrm{f}}(x_{t_j} - y)$. By (10) and the assumption of $\mathcal{Y}$ being robust positively invariant, it holds that $\tilde{y}_t \in \mathcal{Y}$, $t \in \mathbb{N}_{[t_j, t_{j+1}]}$, which is used to obtain the last inequality in (42). Furthermore, noting that $x^\star_{t-t_j|t_j} = (A+BK)^{t-t_j}x_{t_j} + \sum_{j=0}^{t-t_j-1}(A+BK)^{t-t_j-1-j}Bv_j$,

$t \in \mathbb{N}_{[t_j, t_{j+1}]}$, it follows that

$$\begin{aligned} V_N^0(x_{t_{j+1}}) &\leq J_N^0(x_{t_j}) + V_{\mathrm{f}}\Bigg((A+BK)^{t_{j+1}-t_j}(x_{t_j} - y_{t_j}) \\ &\quad + \sum_{j=0}^{t_{j+1}-t_j-1} (A+BK)^{t_{j+1}-t_j-1-j} B v_j\Bigg) - \sum_{i=0}^{t_{j+1}-t_j-1} \ell(v_i) \\ &\overset{\text{Lemma 4}}{\leq} J_N^0(x_{t_j}) - \sum_{i=0}^{t_{j+1}-t_j-1} q\Bigg((A+BK)^i(x_{t_j} - y_{t_j}) \\ &\quad + \sum_{k=0}^{i-1}(A+BK)^{i-1-k} B v_k\Bigg) + V_{\mathrm{f}}(x_{t_j} - y_{t_j}) \\ &= V_N^0(x_{t_j}) - \sum_{t=t_j}^{t_{j+1}-1} q(x^\star_{t-t_j|t_j} - (A+BK)^{t-t_j} y_{t_j}) \\ &= V_N^0(x_{t_j}) - \sum_{t=t_j}^{t_{j+1}-1} q(x_t - \tilde{y}_t) \\ &\leq V_N^0(x_{t_j}) - \sum_{t=t_j}^{t_{j+1}-1} \min_{\bar{y}_t \in \mathcal{Y}} q(x_t - \bar{y}_t) \\ &\overset{(20)}{\leq} V_N^0(x_{t_j}) - \sum_{t=t_j}^{t_{j+1}-1} \alpha_1(|x_t|_{\mathcal{Y}}). \end{aligned} \tag{43}$$

Appendix D
Proof of Lemma 9

Proof: Define $\bar{P}_0 := 1$ and

$$\begin{aligned} &\bar{P}_i(\rho_1, \ldots, \rho_{i-1}, \rho_i) \\ &:= \mathbb{P}\left(\sum_{s=0}^{i-1} A^s w_{i-1-s} \in \rho_i \mathcal{T}, \sum_{s=0}^{j-1} A^s w_{j-1-s} \in \rho_j \mathcal{T}, j \in \mathbb{N}_{[1,i-1]}\right) \end{aligned} \tag{44}$$

for all $\rho_j \in \mathbb{R}_{\geq 0}$, $j \in \mathbb{N}_{[1,i]}$, and $i \in \mathbb{N}_{[1,N-1]}$. Define $\bar{P}_N(\rho_1, \ldots, \rho_N) = 0$ for all $\rho_j \in \mathbb{R}_{\geq 0}$, $j \in \mathbb{N}_{[1,N]}$, where the unused variable ρ_N has been added for simplicity of exposition. Note that it holds that $P_i(\rho_1, \ldots, \rho_{i-1}, \rho_i) = \bar{P}_{i-1}(\rho_1, \ldots, \rho_{i-1}) - \bar{P}_i(\rho_1, \ldots, \rho_i), \rho_j \in \mathbb{R}_{\geq 0}, j \in \mathbb{N}_{[1,i]}$, and $i \in \mathbb{N}_{[1,N]}$. Here, we make use of the fact that $\mathcal{T}_N = \emptyset$, such that an event is triggered N time steps after the last event if no event has been triggered for $N-1$ time steps. Define the (invertible) change of variables $v_{j-1} := \sum_{s=0}^{j-1} A^s w_{j-1-s}$ for $j \in \mathbb{N}_{[1,i]}$. In fact, there exists a nonsingular matrix $T \in \mathbb{R}^{in \times in}$ such that $\boldsymbol{v} = T\boldsymbol{w}$ where $\boldsymbol{w} := (w_0^\top, \ldots, w_{i-1}^\top)^\top$ and $\boldsymbol{v} := (v_0^\top, \ldots, v_{i-1}^\top)^\top$. Let the (joint) probability density function of $\boldsymbol{w}$ be given by $\mathbf{p}_{\mathrm{w}}$. Then, the (joint) probability density function of $\boldsymbol{v}$ is given by $\mathbf{p}_{\mathrm{v}}$ with $\mathbf{p}_{\mathrm{v}}(\boldsymbol{v}) = |\det(T^{-1})| \mathbf{p}_{\mathrm{w}}(T^{-1}\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{R}^{in}$ (see, for example, [39]).

It follows that

$$\begin{aligned} &\bar{P}_i(\rho_1, \ldots, \rho_{i-1}, \rho_i) \\ &= \int_{\{\boldsymbol{w} \in \mathbb{R}^{in} \,|\, \sum_{s=0}^{i-1} A^s w_{i-1-s} \in \rho_i \mathcal{T}, \sum_{s=0}^{j-1} A^s w_{j-1-s} \in \rho_j \mathcal{T}, j \in \mathbb{N}_{[1,i-1]}\}} \mathbf{p}_{\mathrm{w}}(\boldsymbol{w})\, \mathrm{d}\boldsymbol{w} \\ &= \int_{\{\boldsymbol{v} \in \mathbb{R}^{in} \,|\, v_{i-1} \in \rho_i \mathcal{T}, v_j \in \rho_j \mathcal{T}, j \in \mathbb{N}_{[1,i-2]}\}} \mathbf{p}_{\mathrm{v}}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v} \\ &= \int_{\rho_i \mathcal{T} \times \rho_{i-1} \mathcal{T} \times \cdots \times \rho_1 \mathcal{T}} \mathbf{p}_{\mathrm{v}}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v}, \end{aligned} \tag{45}$$

where we use $\mathrm{d}\boldsymbol{w}$ and $\mathrm{d}\boldsymbol{v}$ to denote in-dimensional volume differentials. In the following, let $\rho_{i,1}, \rho_{i,2} \in \mathbb{R}_{\geq 0}$ be arbitrary with $\rho_{i,1} \leq \rho_{i,2}$. It holds that $\rho_{i,1}\mathcal{T} \subseteq \rho_{i,2}\mathcal{T}$, such that $\bar{P}_i(\rho_1, \ldots, \rho_{i-1}, \rho_{i,1}) \leq \bar{P}_i(\rho_1, \ldots, \rho_{i-1}, \rho_{i,2})$, which proves the monotonicity of P_i in ρ_i. Furthermore, it holds that

$$\begin{aligned} &\bar{P}_i(\rho_1, \ldots, \rho_{i-1}, \rho_{i,2}) \\ &= \int_{\rho_{i,1} \mathcal{T} \times \rho_{i-1} \mathcal{T} \times \cdots \times \rho_1 \mathcal{T}} \mathbf{p}_{\mathrm{v}}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v} \\ &\quad + \int_{(\rho_{i,2}\mathcal{T} \setminus \rho_{i,1}\mathcal{T}) \times \rho_{i-1} \mathcal{T} \times \cdots \times \rho_1 \mathcal{T}} \mathbf{p}_{\mathrm{v}}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v} \\ &\leq \int_{\rho_{i,1} \mathcal{T} \times \rho_{i-1} \mathcal{T} \times \cdots \times \rho_1 \mathcal{T}} \mathbf{p}_{\mathrm{v}}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v} \\ &\quad + \sup_{\boldsymbol{v} \in \mathbb{R}^{in}} \mathbf{p}_{\mathrm{v}}(\boldsymbol{v}) \operatorname{vol}((\rho_{2,i}\mathcal{T} \setminus \rho_{i,1}\mathcal{T}) \times \rho_{i-1}\mathcal{T} \times \cdots \times \rho_1\mathcal{T}). \end{aligned} \tag{46}$$

It follows that

$$\begin{aligned} &|\bar{P}_i(\rho_1, \ldots, \rho_{i-1}, \rho_{i,2}) - \bar{P}_i(\rho_1, \ldots, \rho_{i-1}, \rho_{i,1})| \\ &\leq \sup_{\boldsymbol{v} \in \mathbb{R}^{in}} \mathbf{p}_{\mathrm{v}}(\boldsymbol{v}) \operatorname{vol}((\rho_{i,2}\mathcal{T} \setminus \rho_{i,1}\mathcal{T}) \times \rho_{i-1}\mathcal{T} \times \cdots \times \rho_1\mathcal{T}). \end{aligned} \tag{47}$$

Furthermore, it holds that

$$\begin{aligned} &\operatorname{vol}((\rho_{i,2}\mathcal{T} \setminus \rho_{i,1}\mathcal{T}) \times \rho_{i-1}\mathcal{T} \times \cdots \times \rho_1\mathcal{T}) \\ &\quad = \operatorname{vol}((\rho_{i,2}\mathcal{T} \setminus \rho_{i,1}\mathcal{T}) \operatorname{vol}(\rho_{i-1}\mathcal{T}) \cdots \operatorname{vol}(\rho_1\mathcal{T}) \\ &\quad = (\operatorname{vol}(\rho_{i,2}\mathcal{T}) - \operatorname{vol}(\rho_{i,1}\mathcal{T})) \operatorname{vol}(\rho_{i-1}\mathcal{T}) \cdots \operatorname{vol}(\rho_1\mathcal{T}) \\ &\quad = (\rho_{i,2}^n - \rho_{i,1}^n) \operatorname{vol}(\mathcal{T})^i \prod_{j=1}^{i-1} \rho_j^n. \end{aligned} \tag{48}$$

We now investigate the continuity of $\bar{P}_i$ in the last argument at an arbitrary point $(\rho_1, \ldots, \rho_i) \in \mathbb{R}^i_{\geq 0}$. We consider two cases. In the first case, let $\rho_{i,1} = \rho_i$ be fixed and let $\rho_i \leq \rho_{i,2}$. From (47) and (48), it follows that for every $\delta > 0$, there exists an $\epsilon > 0$ such that $\rho_i \leq \rho_{i,2} < \rho_i + \epsilon$ implies that $|\bar{P}_i(\rho_1, \ldots, \rho_{i-1}, \rho_{i,2}) - \bar{P}_i(\rho_1, \ldots, \rho_{i-1}, \rho_i)| < \delta$.

In the second case, let $\rho_{i,2} = \rho_i$ be fixed and let $0 \leq \rho_{i,1} \leq \rho_i$. From (47) and (48), it follows that for every $\delta > 0$, there exists an $\epsilon > 0$ such that $\max\{0, \rho_i - \epsilon\} \leq \rho_{i,1} \leq \rho_i$ implies that $|\bar{P}_i(\rho_1, \ldots, \rho_{i-1}, \rho_i) - \bar{P}_i(\rho_1, \ldots, \rho_{i-1}, \rho_{i,1})| < \delta$. Hence, $\bar{P}_i$, and thus P_i, are continuous in their last argument at $(\rho_1, \ldots, \rho_i)$, and, as this point was arbitrary, the functions are continuous on their domains in the last argument. Finally, as $\mathbf{p}_w$ is a bounded function, it follows from (44) that $\bar{P}_i(\rho_1,\ldots,\rho_{i-1},0)=0$, implying $P_i(\rho_1,\ldots,\rho_{i-1},0)=\bar{P}_{i-1}(\rho_1,\ldots,\rho_{i-1})=1-\sum_{j=1}^{i-1}P_j(\rho_1,\ldots,\rho_{j-1})$, which follows from the definition of $\bar{P}_{i-1}$; furthermore, as $\mathbf{p}_w$ has a bounded support, there exists a ρ_i, sufficiently large, such that $\bar{P}_i(\rho_1,\ldots,\rho_{i-1},\rho_i)=\bar{P}_{i-1}(\rho_1,\ldots,\rho_{i-1})$ in (44), implying $P_i(\rho_1,\ldots,\rho_{i-1},\rho_i)=0$. ■

Appendix E
Proof of Lemma 10

Before we prove Lemma 10, we need to establish several auxiliary results.

Lemma 11: Let $\mathcal{F}\subseteq\mathbb{R}^n$ be a bounded polyhedron containing the origin defined by $\mathcal{F}:=\{x\in\mathbb{R}^n\,|\,H_ix\le h_i, i\in\mathbb{N}_{[1,r_F]}\}$ with $H_i\in\mathbb{R}^{1\times n}$ and $h_i\in\mathbb{R}$ for $i\in\mathbb{N}_{[1,r_F]}$. Assume that the interior of $\mathcal{F}$ is nonempty. Let $\mathcal{W}\subseteq\mathbb{R}^n$ be a polyhedron defined by $\mathcal{W}:=\mathrm{co}\{w_1,\ldots,w_{r_W}\}$ with $w_j\in\mathbb{R}^n$, $j\in\mathbb{N}_{[1,r_W]}$. Let $A\in\mathbb{R}^{n\times n}$ be nonsingular. Define the set-valued function $\mathcal{T}:\mathbb{R}_{\ge0}\to2^{\mathbb{R}^n}$ with $\mathcal{T}(\rho):=A^{-1}(\rho\mathcal{F}\ominus\mathcal{W})$ for all $\rho\in\mathbb{R}_{\ge0}$. Define finally the function $f:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$ by $f(\rho):=\mathrm{vol}(\mathcal{T}(\rho))$ for all $\rho\in\mathbb{R}_{\ge0}$. It holds that f is continuous and monotonically nondecreasing. ■

Proof: Using [28, Th. 2.2] (with reference to [40]), it holds that

$$\mathcal{T}(\rho)=\left\{x\in\mathbb{R}^n\;\middle|\;\begin{array}{r}H_iAx\le\rho h_i-\max_{j\in\mathbb{N}_{[1,r_W]}}H_iw_j,\\ i\in\mathbb{N}_{[1,r_F]}\end{array}\right\}. \quad (49)$$

As the origin was assumed to be contained in $\mathcal{F}$, it follows that $h_i\ge0$ for all $i\in\mathbb{N}_{[1,r_F]}$. Hence, (49) implies that $\mathcal{T}(\rho_1)\subseteq\mathcal{T}(\rho_2)$ for all $\rho_1,\rho_2\in\mathbb{R}_{\ge0}$ with $\rho_1\le\rho_2$. This already proves that f is monotonically nondecreasing. Consider now the case that $\mathcal{T}(\rho)$ is empty for all ρ. Then, it holds that $f(\rho)=0$ for all $\rho\in\mathbb{R}_{\ge0}$ and the proof is complete. In the following, assume that there exists a $\bar\rho\in\mathbb{R}_{\ge0}$ such that $\mathcal{T}(\bar\rho)$ is nonempty. The graph of $\mathcal{T}$ is given by

$$\begin{aligned}\mathrm{gph}(\mathcal{T})&=\{(x,\rho)\in\mathbb{R}^n\times\mathbb{R}_{\ge0}\,|\,x\in\mathcal{T}(\rho)\}\\&=\Big\{(x,\rho)\in\mathbb{R}^n\times\mathbb{R}_{\ge0}\;\Big|\\&\qquad H_iAx\le\rho h_i-\max_{j\in\mathbb{N}_{[1,r_W]}}H_iw_j,\ i\in\mathbb{N}_{[1,r_F]}\Big\}.\end{aligned} \quad (50)$$

It holds that the graph of $\mathcal{T}$ is closed such that by [41, Th. 5.7], $\mathcal{T}$ is outer semicontinuous on $\mathbb{R}_{\ge0}$. Define

$$\begin{aligned}\rho^\star&=\min\{\rho\in\mathbb{R}_{\ge0}\,|\,\exists x\in\mathcal{T}(\rho)\}\\&=\min\{\rho\in[0,\bar\rho]\,|\,\exists x\in\mathcal{T}(\rho)\}\\&=\min_{\rho\in\mathbb{R},x\in\mathbb{R}^n}(1,0,\ldots,0)\cdot(\rho,x^\top)^\top \quad (51)\\&\quad\text{s.t.}\\&\quad 0\le\rho\le\bar\rho\\&\quad H_iAx\le\rho h_i-\max_{j\in\mathbb{N}_{[1,r_W]}}H_iw_j,\ i\in\mathbb{N}_{[1,r_F]}.\end{aligned}$$

This optimization problem is a linear program, which is feasible by assumption and whose objective function is bounded by construction. Hence, it is ensured that a solution exists and $\rho^\star$ is well defined. Furthermore, the graph of $\mathcal{T}$ is convex, such that by [41, Th. 5.9] $\mathcal{T}$ is inner semicontinuous on $\mathbb{R}_{>\rho^\star}$. Hence, $\mathcal{T}$ is continuous on $\mathbb{R}_{>\rho^\star}$. As $\mathcal{T}(\rho)=\emptyset$ for $\rho\in[0,\rho^\star)$, it follows that $f(\rho)=0$ for $\rho\in[0,\rho^\star)$. Assume now that $f(\rho^\star)>0$. It follows that the interior of $\mathcal{T}(\rho^\star)$ is nonempty and there exist an $x\in\mathbb{R}^n$ and an $\eta\in\mathbb{R}_{>0}$ such that $\{x\}\oplus\eta\mathcal{B}\subseteq\mathcal{T}(\rho^\star)$. Let further $\mu\in\mathbb{R}_{>0}$ such that $\mu\mathcal{W}\subseteq A\mathcal{B}$. Such a γ exists as A is nonsingular. Hence,

$$\begin{aligned}x&\in A^{-1}(\rho^\star\mathcal{F}\ominus\mathcal{W})\ominus\eta\mathcal{B}\\&=A^{-1}(\rho^\star\mathcal{F}\ominus(\mathcal{W}\oplus\eta A\mathcal{B}))\\&\subseteq A^{-1}(\rho^\star\mathcal{F}\ominus(\mathcal{W}\oplus\eta\mu\mathcal{W}))\\&=A^{-1}(\rho^\star\mathcal{F}\ominus(1+\eta\mu)\mathcal{W}).\end{aligned} \quad (52)$$

It follows that

$$\begin{aligned}\frac{1}{1+\eta\mu}x&\in A^{-1}\left(\frac{1}{1+\eta\mu}\rho^\star\mathcal{F}\ominus\mathcal{W}\right)\\&=\mathcal{T}\left(\frac{1}{1+\eta\mu}\rho^\star\right)\end{aligned} \quad (53)$$

which implies that $\mathcal{T}(\rho)\ne\emptyset$ for a $\rho<\rho^\star$, contradicting the definition of $\rho^\star$ in (51). Hence, it holds that $f(\rho^\star)=0$, and thus $f(\rho)=0$ for all $\rho\in[0,\rho^\star]$.

To prove continuity of f, we will first establish right-continuity, then left-continuity. As $\mathcal{T}$ is outer semicontinuous for all $\rho\in\mathbb{R}_{\ge0}$ and compact-valued, by [41, Proposition 5.12], it holds that for all $\rho'\in\mathbb{R}_{\ge0}$ and all $\bar\epsilon\in\mathbb{R}_{>0}$, there exists a $\bar\delta\in\mathbb{R}_{>0}$ such that for all $\rho\in[\rho',\rho'+\bar\delta)$ it holds that $\mathcal{T}(\rho)\subseteq\mathcal{T}(\rho')\oplus\bar\epsilon\mathcal{B}$. By the Steiner–Minkowski formula (see, for example, [40]), for all $\rho'\in\mathbb{R}_{\ge0}$ there exists a continuous function $p:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$ with $p(0)=0$ such that for all $\bar\epsilon\in\mathbb{R}_{>0}$

$$\begin{aligned}\mathrm{vol}(\mathcal{T}(\rho')\oplus\bar\epsilon\mathcal{B})&=\mathrm{vol}(\mathcal{T}(\rho'))+p(\bar\epsilon)\\&=f(\rho')+p(\bar\epsilon).\end{aligned} \quad (54)$$

Hence, for all $\rho'\in\mathbb{R}_{\ge0}$ and all $\bar\epsilon\in\mathbb{R}_{>0}$, there exists a $\delta\in\mathbb{R}_{>0}$ such that for all $\rho\in[\rho',\rho'+\delta)$ it holds that $f(\rho')\le f(\rho)\le f(\rho')+p(\bar\epsilon)$. As p is continuous, it follows that for all $\rho'\in\mathbb{R}_{\ge0}$ and all $\epsilon\in\mathbb{R}_{>0}$, there exists a $\delta\in\mathbb{R}_{>0}$ such that for all $\rho\in[\rho',\rho'+\delta)$ it holds that $|f(\rho)-f(\rho')|\le\epsilon$, proving that f is right-continuous.

Next, we will show that f is left-continuous. For all $\rho'\in\mathbb{R}_{\ge0}$ where $f(\rho')=0$, it also holds that $f(\rho)=0$ for $\rho\in[0,\rho']$, such that f is left-continuous at ρ'. Assume in the following that $\rho'\in\mathbb{R}_{\ge0}$ and $f(\rho')>0$. This implies that $\rho'>\rho^\star$ as $f(\rho^\star)=0$. Hence, $\mathcal{T}$ is inner semicontinuous at ρ' such that by [41, Proposition 5.12] for all $\bar\epsilon\in\mathbb{R}_{>0}$ there exists a $\bar\delta\in\mathbb{R}_{>0}$ such that for all $\rho\in[\rho'-\bar\delta,\rho')$ it holds that $\mathcal{T}(\rho')\subseteq\mathcal{T}(\rho)\oplus\bar\epsilon\mathcal{B}$. Furthermore, as $f(\rho')>0$, the interior of $\mathcal{T}(\rho')$ is nonempty such that there exist an $x'\in\mathbb{R}^n$ and an $\eta'\in\mathbb{R}_{>0}$ such that $x'\oplus\eta'\mathcal{B}\subseteq\mathcal{T}(\rho')$. Hence, $\mathcal{T}(\rho')\subseteq\mathcal{T}(\rho)\oplus\bar\epsilon\mathcal{B}$ implies

$$\mathcal{T}(\rho')\subseteq\mathcal{T}(\rho)\oplus\frac{\bar\epsilon}{\eta'}(\mathcal{T}(\rho')\oplus\{-x'\}). \quad (55)$$

If $\bar{\epsilon} \in (0, \eta')$, it follows that

$$\left(1 - \frac{\bar{\epsilon}}{\eta'}\right) \mathcal{T}(\rho') \subseteq \mathcal{T}(\rho) \oplus \frac{\bar{\epsilon}}{\eta'}\{-x'\} \tag{56}$$

and, hence, $f(\rho) \geq (1 - \frac{\bar{\epsilon}}{\eta'})^n f(\rho')$. As $(1 - \frac{\bar{\epsilon}}{\eta'})^n$ is continuous in $\bar{\epsilon}$ and becomes 1 for $\bar{\epsilon} = 0$, it follows that for all $\epsilon \in \mathbb{R}_{>0}$ there exists a $\delta \in \mathbb{R}_{>0}$ such that for all $\rho \in [\rho' - \delta, \rho')$ it holds that $|f(\rho) - f(\rho')| \leq \epsilon$, proving that f is left-continuous at ρ'. With the results above, it follows that f is left-continuous at every $\rho' \in \mathbb{R}_{\geq 0}$ and, hence, continuous on $\mathbb{R}_{\geq 0}$.

Lemma 12: It holds that $\hat{\rho}_1 \leq \hat{\rho}_2 \Rightarrow \mathcal{T}_i(\hat{\rho}_1) \subseteq \mathcal{T}_i(\hat{\rho}_2)$, $i \in \mathbb{N}_{[1,N-1]}$ for all $\hat{\rho}_1, \hat{\rho}_2 \in \mathbb{R}_{\geq 0}$. ■

Proof: Let $\hat{\rho}_1, \hat{\rho}_2 \in \mathbb{R}_{\geq 0}$ with $\hat{\rho}_1 \leq \hat{\rho}_2$. It holds that

$$\begin{aligned}\mathcal{T}_i(\hat{\rho}_1) &= A^{-1}\left(\left(\bigoplus_{j=0}^{i}(A+BK)^j \hat{\rho}_1 \mathcal{W}\right) \ominus \mathcal{W}\right) \\ &\overset{0\in\mathcal{W}}{\subseteq} A^{-1}\left(\left(\bigoplus_{j=0}^{i}(A+BK)^j \right.\right. \\ &\qquad \left.\left. \times (\hat{\rho}_1\mathcal{W} \oplus (\hat{\rho}_2 - \hat{\rho}_1)\mathcal{W})\right) \ominus \mathcal{W}\right) \\ &= A^{-1}\left(\left(\bigoplus_{j=0}^{i}(A+BK)^j \hat{\rho}_2 \mathcal{W}\right) \ominus \mathcal{W}\right) \\ &= \mathcal{T}_i(\hat{\rho}_2)\end{aligned} \tag{57}$$

for all $i \in \mathbb{N}_{[1,N-1]}$.

Lemma 13: The functions $\bar{P}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$\begin{aligned}\bar{P}_i(\hat{\rho}) = \mathbb{P}\left(\sum_{s=0}^{i-1} A^s w_{i-1-s} \in \mathcal{T}_i(\hat{\rho}),\right. \\ \left.\sum_{s=0}^{j-1} A^s w_{j-1-s} \in \mathcal{T}_j(\hat{\rho}), j \in \mathbb{N}_{[1,i-1]}\right)\end{aligned} \tag{58}$$

for $i \in \mathbb{N}_{[1,N-1]}$ and all $\hat{\rho} \in \mathbb{R}_{\geq 0}$ are continuous. ■

Proof: First notice that

$$\mathcal{T}_i(\hat{\rho}) = A^{-1}\left(\hat{\rho}\mathcal{F}_{i+1} \ominus \mathcal{W}\right) \tag{59}$$

for $i \in \mathbb{N}_{[1,N-1]}$ and, as $\mathbf{p}_w$ was assumed to be bounded, it holds that the interior of $\mathcal{W}$ and, hence, also the interior of $\mathcal{F}_i$ for $i \in \mathbb{N}$ is nonempty. Define the (invertible) change of variables $v_{j-1} := \sum_{s=0}^{j-1} A^s w_{j-1-s}$ for $j \in \mathbb{N}_{[1,i]}$. Define further $\boldsymbol{w} := (w_0^\top, \ldots, w_{i-1}^\top)^\top$ and $\boldsymbol{v} := (v_0^\top, \ldots, v_{i-1}^\top)^\top$ and let the (joint) probability density function of $\boldsymbol{w}$ be given by $\mathbf{p}_\mathbf{w}$. As before in the proof of Lemma 9, there exists an invertible matrix $T \in \mathbb{R}^{in \times in}$ such that $\boldsymbol{v} = T\boldsymbol{w}$. Hence, the (joint) probability density function of $\boldsymbol{v}$ is given by $\mathbf{p}_\mathbf{v}$ with $\mathbf{p}_\mathbf{v}(\boldsymbol{v}) = |\det(T^{-1})|\mathbf{p}_\mathbf{w}(T^{-1}\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{R}^{in}$. Let $\hat{\rho}_1, \hat{\rho}_2 \in \mathbb{R}_{\geq 0}$ with $\hat{\rho}_1 \leq \hat{\rho}_2$ arbitrary. It holds that

$$\begin{aligned}\bar{P}_i(\hat{\rho}_2) &= \int_{\mathcal{T}_i(\hat{\rho}_2)\times\cdots\times\mathcal{T}_1(\hat{\rho}_2)} \mathbf{p}_\mathbf{v}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \\ &= \int_{\mathcal{T}_i(\hat{\rho}_1)\times\cdots\times\mathcal{T}_1(\hat{\rho}_1)} \mathbf{p}_\mathbf{v}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \\ &\quad + \int_{(\mathcal{T}_i(\hat{\rho}_2)\times\cdots\times\mathcal{T}_1(\hat{\rho}_2))\setminus(\mathcal{T}_i(\hat{\rho}_1)\times\cdots\times\mathcal{T}_1(\hat{\rho}_1))} \mathbf{p}_\mathbf{v}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \\ &= \bar{P}_i(\hat{\rho}_1) + \int_{(\mathcal{T}_i(\hat{\rho}_2)\times\cdots\times\mathcal{T}_1(\hat{\rho}_2))\setminus(\mathcal{T}_i(\hat{\rho}_1)\times\cdots\times\mathcal{T}_1(\hat{\rho}_1))} \mathbf{p}_\mathbf{v}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} \\ &\leq \bar{P}_i(\hat{\rho}_1) + \\ &\quad \sup_{\boldsymbol{v}\in\mathbb{R}^{in}} \mathbf{p}_\mathbf{v}(\boldsymbol{v})\,\mathrm{vol}\left((\mathcal{T}_i(\hat{\rho}_2)\times\cdots\times\mathcal{T}_1(\hat{\rho}_2))\setminus\right. \\ &\qquad \left.(\mathcal{T}_i(\hat{\rho}_1)\times\cdots\times\mathcal{T}_1(\hat{\rho}_1))\right) \\ &= \bar{P}_i(\hat{\rho}_1) + g(\hat{\rho}_1, \hat{\rho}_2)\end{aligned} \tag{60}$$

with $g : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$, $g : (\hat{\rho}_1, \hat{\rho}_2) \mapsto \sup_{\boldsymbol{v}\in\mathbb{R}^{in}} \mathbf{p}_\mathbf{v}(\boldsymbol{v})(\prod_{j=1}^{i} \mathrm{vol}(\mathcal{T}_j(\hat{\rho}_2)) - \prod_{j=1}^{i}\mathrm{vol}(\mathcal{T}_j(\hat{\rho}_1)))$, which is, by Lemma 11 and noting that $\mathcal{W}$ is a bounded polyhedron and A nonsingular, a continuous function. Moreover, it holds that $g(\hat{\rho}, \hat{\rho}) = 0$ for any $\hat{\rho} \in \mathbb{R}_{\geq 0}$. It follows that

$$\bar{P}_i(\hat{\rho}_1) \overset{\text{Lemma 12}}{\leq} \bar{P}_i(\hat{\rho}_2) \leq \bar{P}_i(\hat{\rho}_1) + g(\hat{\rho}_1, \hat{\rho}_2) \tag{61}$$

and thus

$$\bar{P}_i(\hat{\rho}_2) - g(\hat{\rho}_1, \hat{\rho}_2) \leq \bar{P}_i(\hat{\rho}_1) \leq \bar{P}_i(\hat{\rho}_2) \tag{62}$$

for any $\hat{\rho}_1, \hat{\rho}_2 \in \mathbb{R}_{\geq 0}$ with $\hat{\rho}_1 \leq \hat{\rho}_2$. Hence, $\bar{P}_i$ are continuous functions.

Proof: We are now ready to prove Lemma 10. Let $P_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $i \in \mathbb{N}_{[1,N]}$ denote the function mapping $\hat{\rho}$ to the probability that an event occurs at i time steps after the last event, given by

$$\begin{aligned}P_i(\hat{\rho}) = \mathbb{P}\left(\sum_{s=0}^{i-1} A^s w_{i-1-s} \notin \mathcal{T}_i(\hat{\rho}),\right. \\ \left.\sum_{s=0}^{j-1} A^s w_{j-1-s} \in \mathcal{T}_j(\hat{\rho}), j \in \mathbb{N}_{[1,i-1]}\right)\end{aligned} \tag{63}$$

for all $\hat{\rho} \in \mathbb{R}_{\geq 0}$. Recall the definition of $\bar{P}_i$ as in Lemma 13. With $\bar{P}_0(\hat{\rho}) := 1$ and $\bar{P}_N(\hat{\rho}) := 0$ for all $\hat{\rho} \in \mathbb{R}_{\geq 0}$, it holds that $P_i(\hat{\rho}) = \bar{P}_{i-1}(\hat{\rho}) - \bar{P}_i(\hat{\rho})$ for all $i \in \mathbb{N}_{[1,N]}$, where we make use of the fact that $\mathcal{T}_N = \emptyset$, such that an event is triggered N time steps after the last event, if no event has been triggered for $N-1$ time steps. Hence, by Lemma 13, the functions P_i are continuous. Furthermore, it holds that

$$\hat{\Delta}(\hat{\rho}) = \sum_{j=1}^{N} j P_j(\hat{\rho}) \tag{64}$$

for all $\hat{\rho} \in \mathbb{R}_{\geq 0}$, such that $\hat{\Delta}$ is also continuous. Consider further that the functions $\bar{P}_i$ as defined in Lemma 13 and $\bar{P}_N(\hat{\rho}) := 0$, $\hat{\rho} \in \mathbb{R}_{\geq 0}$, denote the functions mapping $\hat{\rho}$ to the probability that no event occurred for i time steps after the last event. It holds that

$$\bar{P}_i(\hat{\rho}) = \mathbb{P}\left(\sum_{s=0}^{j-1} A^s w_{j-1-s} \in \mathcal{T}_j(\hat{\rho}), j \in \mathbb{N}_{[1,i]}\right) \quad (65)$$

for all $i \in \mathbb{N}_{[1,N]}$. Let now $\hat{\rho}_1, \hat{\rho}_2 \in \mathbb{R}_{\geq 0}$ and $\hat{\rho}_1 \leq \hat{\rho}_2$. From Lemma 12 and (65), it follows that $\bar{P}_i(\hat{\rho}_1) \leq \bar{P}_i(\hat{\rho}_2)$, for $i \in \mathbb{N}_{[0,N]}$, such that

$$\begin{aligned}\hat{\Delta}(\hat{\rho}_1) &= \sum_{j=1}^{N} j P_j(\hat{\rho}_j) \\ &= \sum_{j=1}^{N} j(\bar{P}_{j-1}(\hat{\rho}_1) - \bar{P}_j(\hat{\rho}_1)) \\ &= \bar{P}_0 + \sum_{j=1}^{N-1} \bar{P}_j(\hat{\rho}_1) \\ &= 1 + \sum_{j=1}^{N-1} \bar{P}_j(\hat{\rho}_1) \leq 1 + \sum_{j=1}^{N-1} \bar{P}_j(\hat{\rho}_2) = \hat{\Delta}(\hat{\rho}_2) \quad (66)\end{aligned}$$

showing that the function $\hat{\Delta}$ is monotonically nondecreasing. Finally, as the interior of $\mathcal{W}$ was assumed nonempty, it holds that $\mathcal{T}_i(0) = \emptyset$ for all $i \in \mathbb{N}_{[1,N-1]}$ implying $P_1(0) = 1$ in (63) and, hence, $\hat{\Delta}(0) = 1$; furthermore, as the support of $\mathbf{p}_w$ is bounded, it follows that $P_i(\hat{\rho}) = 0$ for $i \in \mathbb{N}_{[1,N-1]}$ and $P_N(\hat{\rho}) = 1$ for sufficiently large $\hat{\rho}$, leading to $\hat{\Delta}(\hat{\rho}) = N$. ■

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