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Robust Event-Triggered MPC for Constrained Linear Discrete-Time Systems with Guaranteed Average Sampling Rate ⋆

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Abstract: We propose a robust event-triggered model predictive control (MPC) scheme for linear time-invariant discrete-time systems subject to bounded additive stochastic disturbances and hard constraints on the input and state. For given probability distributions of the disturbances acting on the system, we design event conditions such that the average frequency of communication between the controller and the actuator in the closed-loop system attains a given value. We employ Tube MPC methods to guarantee robust constraint satisfaction and a robust asymptotic bound on the system state. Moreover, we show that based on a given periodically updated Tube MPC scheme, an appropriate event-triggered MPC scheme can be designed, with the same guarantees on constraints and region of attraction, but with a reduced number of average communications.

Keywords: predictive control, event-triggered control, robustness

1. INTRODUCTION

Networked control systems are distributed control systems in which the communication between sensors, controllers, and actuators takes place over a communication network, see for example Hespanha et al. (2007). If the bandwidth of the network is small compared to the number of users, and/or if communications induce a non-trivial cost in terms of energy, which is especially the case in wireless communication, efforts should be made in reducing the overall amount of communication in the control system. One method that is suited to reduce the required communication is event-triggered control. Instead of updating the control input of the plant at periodic time instances, in event-triggered control new inputs are only transmitted to the actuators if certain well-defined events occur in the plant. Typically, these events are defined in terms of the plant output or state leaving a certain set. For a recent overview of event-triggered control, please refer to Heemels et al. (2012).

In event-triggered model predictive control (MPC), an event is usually triggered if the plant state deviates by a certain amount from the prediction of the state that was computed in the MPC optimization problem at the last event, see for example Bernardini and Bemporad (2012); Lehmann et al. (2013); Eqtami (2013); Li et al. (2014), and the references therein. Alternatively, the MPC cost function may be used to define the event conditions, see for example Varutti et al. (2010). These control schemes, including the scheme proposed in this paper, require a whole sequence of predicted inputs to be transmitted to the actuators at a given event. This setup matches many communication protocols in which the size of communicated packets is fixed, such that it is more efficient to transmit whole sequences of inputs only at infrequent times, than to transmit a single input at each point in time, although the overall amount of transmitted *information* is the same (or even higher), see Georgiev and Tilbury (2004) and Bernardini and Bemporad (2012) with reference to Feeney and Nilsson (2001). See also Quevedo et al. (2011), where an MPC scheme using packetized communication is proposed. This control structure is illustrated in Figure 1, which corresponds to the structure also used in Lehmann et al. (2013).

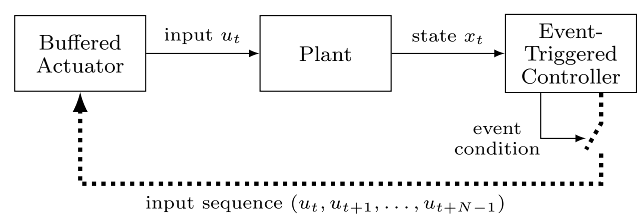

Fig. 1. Control structure with buffered actuator.

In this paper, we propose a robust event-triggered MPC scheme based on the Tube MPC approach presented in Chisci et al. (2001). In Tube MPC, the uncertainty in the prediction of the future plant state due to disturbances is described by a sequence of sets (the so-called "tube"), which are centered around the prediction of a nominal system, see also Langson et al. (2004); Mayne et al. (2005). The main idea in Tube MPC is to assume that feedback is applied to the plant at every time step, which allows to limit the growth of the uncertainty in the prediction, as

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the (future) feedback will counteract the effect of the disturbances. An important observation (motivating our proposed scheme) is that if a disturbance of significantly lower than worst-case magnitude affects the system at a given time step, then the deviation of the plant trajectory from the predicted trajectory will not be greater than what was previously predicted as a worst case, even if *no* feedback is applied at the given time step. We exploit this observation in order to derive a robust event-triggered controller that does *not* update the inputs of the plant in the very event of such less-than-worst-case disturbances, thereby saving communication, and possibly computational power, in the process. Interestingly, our newly proposed design methodology allows any periodically updated Tube MPC scheme to be used for creating an event-triggered MPC scheme retaining the guarantees of its periodically updated counterpart concerning robust constraint satisfaction, region of attraction, and asymptotic bound, with a reduced average amount of communication between the controller and the actuators. Additionally, we present a method of artificially increasing the assumed bound on the disturbances in order to further reduce the communication in the system. In particular, we show how, based on the knowledge of the probability distribution of the disturbances, event conditions can be designed such that the time between events is a random variable with a predefined, arbitrary probability distribution with finite and discrete support.

The remainder of the paper is structured in the following way. Section 2 contains notes on notation and several preliminary results. The formal problem statement is given in Section 3. The robust event-triggered MPC scheme is presented in Section 4 and its relevant properties are described in Section 5. The design of the event conditions is explained in Section 6. Section 7 contains numerical examples illustrating our results and Section 8 concludes the paper with an outlook on open questions.

2. NOTATION AND PRELIMINARIES

$\mathbb{N}$ denotes the set of non-negative integers. For $q, s \in \mathbb{N}$, let $\mathbb{N}_{[q,s]}$ denote the set $\{r \in \mathbb{N} \mid q \leq r \leq s\}$. For a given real number $a \in \mathbb{R}$, we use $\mathbb{R}_{\geq a}$ and $\mathbb{R}_{>a}$ to denote the set of real numbers greater than a, or greater than or equal to a, respectively. For symmetric matrices $S = S^\top \in \mathbb{R}^{n\times n}$, we use $S > 0$ and $S \geq 0$ to denote the fact that S is positive definite and positive semi-definite, respectively. Given sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$, a scalar α, and matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$ we define $\alpha\mathcal{X} := \{\alpha x \mid x \in \mathcal{X}\}$, $A\mathcal{X} := \{Ax \mid x \in \mathcal{X}\}$, and $B^{-1}\mathcal{X} := \{x \mid Bx \in \mathcal{X}\}$. The Minkowski set addition is defined by $\mathcal{X} \oplus \mathcal{Y} := \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$. Given a vector $x \in \mathbb{R}^n$ we define $\mathcal{X} \oplus x := x \oplus \mathcal{X} := \{x\} \oplus \mathcal{X}$. The Pontryagin set difference is defined by $\mathcal{X} \ominus \mathcal{Y} := \{z \in \mathbb{R}^n \mid z \oplus \mathcal{Y} \subseteq \mathcal{X}\}$, see Kolmanovsky and Gilbert (1995, 1998). Given a sequence of sets $\mathcal{X}_i$ for $i \in \mathbb{N}_{[a,b]}$ with $a, b \in \mathbb{N}$, we define $\bigoplus_{i=a}^{b} \mathcal{X}_i = \mathcal{X}_a \oplus \mathcal{X}_{a+1} \oplus \ldots \oplus \mathcal{X}_b$. By convention, the empty sum is equal to $\{0\}$. Similarly, for any vectors $v_i \in \mathbb{R}^n$, $i \in \mathbb{N}$, we define $\sum_{i=a}^{b} v_i = 0$ for any $a, b \in \mathbb{N}$ if $a > b$. We call a compact, convex set containing the origin a C-set. A C-set containing the origin in its (nonempty) interior is called a PC-set. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. If additionally $\alpha(s) \to \infty$ as $s \to \infty$, α is said to belong to class $\mathcal{K}_\infty$. The Euclidean norm of a vector $v \in \mathbb{R}^n$ is denoted by $|v|$. Given any compact set $\mathcal{Y} \subseteq \mathbb{R}^n$, the distance between v and $\mathcal{Y}$ is defined by $|v|_\mathcal{Y} := \min_{s\in\mathcal{Y}} |v - s|$. Define finally the Euclidean unit ball by $\mathcal{B} := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

Definition 1. Given a dynamical system of the form $x_{t+1} = f(t, x_t, w_t)$, $t \in \mathbb{N}$, $f : \mathbb{N} \times \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^n$ with a compact set $\mathcal{W} \subseteq \mathbb{R}^n$, a set $\mathcal{Y} \subseteq \mathbb{R}^n$ is *robustly asymptotically stable with region of attraction* $\hat{\mathcal{X}}$ for this system, if there exists a class $\mathcal{K}$-function α, such that $|x_t|_\mathcal{Y} \leq \alpha(|x_0|_\mathcal{Y})$, $t \in \mathbb{N}$, and $\lim_{t\to\infty} |x_t|_\mathcal{Y} = 0$, for all $x_0 \in \hat{\mathcal{X}}$, $w_t \in \mathcal{W}$, compare Rawlings and Mayne (2009).

Definition 2. Given a dynamical system described by $x_{t+1} = Ax_t + w_t$ with $x_t \in \mathbb{R}^n$, $w_t \in \mathcal{W}$, $t \in \mathbb{N}$, where $\mathcal{W} \subseteq \mathbb{R}^n$ is a C-set and A is a Schur matrix, the *minimal robust positively invariant set* is the nonempty compact set $\mathcal{Y}^\star \subseteq \mathbb{R}^n$ satisfying $A\mathcal{Y}^\star \oplus \mathcal{W} \subseteq \mathcal{Y}^\star$, which is contained in every compact set $\bar{\mathcal{Y}} \subseteq \mathbb{R}^n$ satisfying $A\bar{\mathcal{Y}} \oplus \mathcal{W} \subseteq \bar{\mathcal{Y}}$, see also Kolmanovsky and Gilbert (1998), Raković et al. (2005).

3. PROBLEM SETUP

We consider linear discrete-time systems of the form

$$x_{t+1} = Ax_t + Bu_t + w_t, \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state and $u_t \in \mathbb{R}^m$ is the control input at time $t \in \mathbb{N}$. The disturbance w_t is assumed to be time-varying, unknown, and to satisfy $w_t \in \mathcal{W} \subseteq \mathbb{R}^n$, $t \in \mathbb{N}$, where $\mathcal{W}$ is a known C-set. Furthermore, the probability distribution of the disturbance w_t is assumed to be known. In particular, we assume w_t to be independently and identically distributed for all $t \in \mathbb{N}$ according to the bounded probability density function $\mathbf{p}_w : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with a support that is bounded by $\mathcal{W}$. Further, hard constraints $x_t \in \mathcal{X}$, $u_t \in \mathcal{U}$, $t \in \mathbb{N}$ on the input and state are given, where $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are C-sets. We assume that the state x_t is available to the controller as a measurement at any time step $t \in \mathbb{N}$. The communication network for which we would like to reduce the number of transmissions is situated between the controller and the actuator, as illustrated in Figure 1.

In order to save communication, the input u_t will be determined by an event-triggered controller of the form

$$u_t = \kappa(x_{t_j}, t - t_j), \quad t \in \mathbb{N}_{[t_j, t_{j+1}-1]} \tag{2a}$$

$$t_{j+1} = \inf\{t \in \mathbb{N}_{\geq t_j+1} \mid x_t \notin \mathcal{E}(x_{t_j}, t - t_j)\}, \tag{2b}$$

where $j \in \mathbb{N}$ and $t_0 = 0$. That is, the control values are only updated at the *sampling instants* t_j, which are determined based on the *event conditions* $x_t \notin \mathcal{E}(x_{t_j}, t - t_j)$. At the time instances between t_j and t_{j+1} the input u_t is open-loop, that is, not depending explicitly on the current state x_t. This makes it possible to transmit the whole sequence $u_{t_j}, u_{t_j+1}, \ldots, u_{t_{j+1}-1}$ to the actuator in one packet at time t_j.

Our goal is to design the controller $\kappa : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m$ and the set-valued function $\mathcal{E} : \mathbb{R}^n \times \mathbb{N} \to 2^{\mathbb{R}^n}$ for the closed-loop system consisting of (1) and (2) such that (i) the constraints $x_t \in \mathcal{X}$, $u_t \in \mathcal{U}$, $t \in \mathbb{N}$, are robustly satisfied, (ii) a C-set $\mathcal{Y} \subseteq \mathbb{R}^n$ is robustly asymptotically stable, and (iii) the expected value of the inter-sampling times satisfies $\mathbb{E}[t_{j+1} - t_j] = \bar{\Delta}$ for a given $\bar{\Delta} \geq 1$. We expect a trade-off between $\bar{\Delta}$ and the size of the set $\mathcal{Y}$, with the trade-off depending, amongst others, on the probability distribution $\mathbf{p}_w$.

The control scheme will be based on an auxiliary feedback law defined by the matrix $K \in \mathbb{R}^{m\times n}$, which is assumed to be the desired feedback for the plant if the constraints are ignored. The following assumption is required to hold.

Assumption 1. The matrix $A + BK$ is Schur.

4. EVENT-TRIGGERED TUBE MPC

We propose a solution to the problem stated in Section 3 based on Tube MPC. That is, the functions κ and $\mathcal{E}$ are determined by the solution of a finite horizon optimal control problem which is to be solved online at the sampling instants t_j, $j \in \mathbb{N}$. The constraints in the optimization problem are tightened in order to guarantee robust constraint satisfaction. In particular, we employ the method proposed in Chisci et al. (2001) to compute the tightened constraint sets. We artificially increase the assumed bound on the disturbances in the computations in order to take into account the event-triggered implementation of the controller.

4.1 Setup of the MPC scheme

The finite horizon optimal control problem is defined as follows for an $x_t \in \mathbb{R}^n$ with $t \in \mathbb{N}$. The decision variable of the optimization problem is

$$\mathbf{d}_t = ((x_{0|t}, \ldots, x_{N|t}), (u_{0|t}, \ldots, u_{N-1|t})) \in \mathbb{D}_N, \tag{3}$$

where $\mathbb{D}_N = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$ and $N \in \mathbb{N}_{\geq 1}$ is the prediction horizon. The constraints

$$x_{0|t} = x_t, \tag{4a}$$

$$\forall i \in \mathbb{N}_{[0,N-1]},\ x_{i+1|t} = Ax_{i|t} + Bu_{i|t}, \tag{4b}$$

$$\forall i \in \mathbb{N}_{[0,N-1]},\quad x_{i|t} \in \mathcal{X}_i, \tag{4c}$$

$$\forall i \in \mathbb{N}_{[0,N-1]},\quad u_{i|t} \in \mathcal{U}_i, \tag{4d}$$

$$x_{N|t} \in \mathcal{X}_\mathrm{f} \tag{4e}$$

are imposed on $\mathbf{d}_t$, where the variables $x_{i|t}$ represent the predicted trajectory for the undisturbed system generated by the inputs $u_{i|t}$ according to (4a) and (4b). The sets $\mathcal{X}_i$ and $\mathcal{U}_i$, $i \in \mathbb{N}_{[0,N-1]}$, are tightened constraint sets, depending on the step i in the prediction. The set $\mathcal{X}_\mathrm{f}$ is a terminal set. Define the set of all feasible decision variables for a given point $x_t \in \mathbb{R}^n$ by

$$\mathcal{D}_N(x_t) = \{\mathbf{d}_t \in \mathbb{D}_N \mid \text{(4a) to (4e)}\}. \tag{5}$$

The tightened constraint sets $\mathcal{X}_i$ and $\mathcal{U}_i$ are defined by

$$\mathcal{X}_i := \mathcal{X} \ominus \mathcal{F}_i, \quad i \in \mathbb{N}_{[0,N-1]}, \tag{6a}$$

$$\mathcal{U}_i := \mathcal{U} \ominus K\mathcal{F}_i,\ i \in \mathbb{N}_{[0,N-1]}, \tag{6b}$$

where the sets $\mathcal{F}_i \subseteq \mathbb{R}^n$ are chosen in order to capture the worst-case uncertainty in the prediction, taking into account that feedback is only present if an event occurs. The terminal set $\mathcal{X}_\mathrm{f}$, as well as the sets $\mathcal{F}_i$, $i \in \mathbb{N}$, will be defined in Subsection 4.2.

The cost function for the finite horizon optimal control problem is based on the deviation of the predicted input from the desired feedback $u = Kx$ and is defined for all $t \in \mathbb{N}$ and all $\mathbf{d}_t \in \mathbb{D}_N$ by

$$J_N(\mathbf{d}_t) = \sum_{i=0}^{N-1} \ell(u_{i|t} - Kx_{i|t}) \tag{7}$$

for a stage cost function $\ell : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$.

The finite horizon optimal control problem to be solved in order to obtain κ and $\mathcal{E}$ is defined for all $t \in \mathbb{N}$ and all $x_t \in \mathbb{R}^n$ by

$$J_N^0(x_t) = \min_{\mathbf{d}_t \in \mathcal{D}_N(x_t)} J_N(\mathbf{d}_t) \tag{8a}$$

$$\mathbf{d}_t^\star(x_t) = \operatorname*{arg\,min}_{\mathbf{d}_t \in \mathcal{D}_N(x_t)} J_N(\mathbf{d}_t) \tag{8b}$$

Remark 1. In the case of non-unique minimizers, it is assumed that $\mathbf{d}_t^\star(x_t)$ is any solution to the optimization problem.

The set where the optimization problem in (8) is feasible is defined by $\hat{\mathcal{X}}_N := \{x \in \mathbb{R}^n \mid \mathcal{D}_N(x) \neq \emptyset\}$. Given any $\mathbf{d}_{t_j}^\star(x_{t_j}) = ((x_{0|t_j}^\star, \ldots, x_{N|t_j}^\star), (u_{0|t_j}^\star, \ldots, u_{N-1|t_j}^\star))$, where t_j is assumed to be a sampling instant, the event conditions are defined by $\mathcal{E}(x_{t_j}, t - t_j) := x_{t-t_j|t_j}^\star \oplus \mathcal{T}_{t-t_j}$, for given closed sets $\mathcal{T}_i \subseteq \mathbb{R}^n$, $i \in \mathbb{N}_{[1,N]}$ and $t \in \mathbb{N}_{[t_j+1,t_j+N]}$. That is, an event is triggered if the actual trajectory deviates too much from the predicted trajectory of the undisturbed system. As the actuator runs out of buffered inputs after N steps, an event has to be triggered within the prediction horizon. For this reason we define $\mathcal{T}_N := \emptyset$. Furthermore, we define $\mathcal{T}_0 = \{0\}$. The control law is defined by $\kappa(x_{t_j}, t - t_j) = u_{t-t_j|t_j}^\star$ for $t \in \mathbb{N}_{[t_j,t_j+N-1]}$. That is, the finite horizon optimal input is applied in an open-loop fashion to the plant until the next event occurs and the next optimal control problem is solved. The closed-loop system under the event-triggered controller is given by

$$x_{t+1} = Ax_t + Bu_{t-t_j|t_j}^\star + w_t,\ t \in \mathbb{N}_{[t_j,t_{j+1}-1]} \tag{9a}$$

$$t_{j+1} = \min\{t \in \mathbb{N}_{\geq t_j+1} \mid x_t \notin x_{t-t_j|t_j}^\star \oplus \mathcal{T}_{t-t_j}\} \tag{9b}$$

with $w_t \in \mathcal{W}$, $j, t \in \mathbb{N}$, $t_0 = 0$, and $x_0 \in \mathbb{R}^n$. As will be shown in Section 6.2, this definition of the triggering conditions ensures that the times between sampling instants t_j only depend on the realization of the disturbances w_t, but are independent of the initial condition x_0.

4.2 Assumptions on the constraints

In the following, assumptions on the sets involved in the definition of the triggering conditions and the optimal control problem will be given that ensure robust constraint satisfaction and robust stability properties for the closed-loop system (9). The sets $\mathcal{F}_i$, $i \in \mathbb{N}$, used to describe the uncertainty in the prediction, are defined by

$$\mathcal{F}_i := \bigoplus_{j=0}^{i-1} (A + BK)^j \bar{\mathcal{W}}, \tag{10}$$

where the set $\bar{\mathcal{W}} \subseteq \mathbb{R}^n$ is an artificial over-approximation of the set $\mathcal{W}$ of disturbances acting on the system, chosen in a way such that the event-triggered behavior of the closed-loop system is taken into account. Compare Chisci et al. (2001), where the sets $\mathcal{F}_i$ are defined with $\bar{\mathcal{W}} = \mathcal{W}$. In particular, the following assumption is made on the sets $\mathcal{T}_i$ and $\bar{\mathcal{W}}$. Different methods for choosing these sets will be discussed in Section 6.

Assumption 2. It holds that

$$A\mathcal{T}_i \oplus \mathcal{W} \subseteq \mathcal{F}_{i+1},\ i \in \mathbb{N}_{[0,N-1]}. \tag{11}$$

The reasoning behind this assumption is that if no event is triggered at a given time $t+i$, then the deviation between the actual system state x_{t+i} and its prediction $x_{i|t}^\star$ is contained in the set $\mathcal{T}_i$. Assumption 2 ensures that even without feedback the worst case deviation at the next point in time is bounded by the uncertainty bound $\mathcal{F}_{i+1}$, which is computed under the assumption of feedback.

Remark 2. From (10) it follows that $\mathcal{F}_0 = \{0\}$. Further, by (11) for $i = 0$ and the assumption that $\mathcal{T}_0 = \{0\}$, it follows that $\mathcal{W} \subseteq \mathcal{F}_1 = \bar{\mathcal{W}}$. Moreover, with (10) and $0 \in \mathcal{W}$ we have $0 \in \mathcal{F}_i$ for $i \in \mathbb{N}$. Furthermore, it holds that

$$(A + BK)^j \mathcal{F}_i \oplus \mathcal{F}_j = \mathcal{F}_{i+j} \tag{12}$$

for $i, j \in \mathbb{N}$, see also Kolmanovsky and Gilbert (1995).

Remark 3. Note that using a single sequence of sets $\mathcal{F}_i$, $i \in \mathbb{N}_{[0,N]}$, to capture all possible future states under event-triggered feedback is a simplification. A sharper approximation can be obtained by using a different sequence of sets for every different future sequence of inter-sampling times $t_{j+1} - t_j$. As the number of these possible future sequences grows exponentially in the prediction horizon, the conservative over-approximation above is used in order to limit the computational effort involved with the MPC scheme.

The following assumption on the terminal set $\mathcal{X}_f \subseteq \mathbb{R}^n$, equivalent to the choice of the terminal set in Chisci et al. (2001), is required to hold.

Assumption 3. It holds that

$$\mathcal{X}_\mathrm{f} \subseteq \mathcal{X} \ominus \mathcal{F}_N \tag{13a}$$

$$K\mathcal{X}_\mathrm{f} \subseteq \mathcal{U} \ominus K\mathcal{F}_N \tag{13b}$$

$$(A+BK)\mathcal{X}_\mathrm{f} \oplus (A+BK)^N \bar{\mathcal{W}} \subseteq \mathcal{X}_\mathrm{f}. \tag{13c}$$

5. MAIN PROPERTIES OF THE MPC SCHEME

In this section, the most important properties of the proposed event-triggered MPC scheme are presented, that is, well-definedness of the controller, robust constraint satisfaction, and asymptotic stability of a compact set for the closed-loop system. Due to lack of space, the proofs of the statements have been omitted.

The following lemma ensures that system (9) is well defined in the sense that if the optimization problem in (8), defining the controller and the event conditions, is feasible at initialization, then it remains feasible for all sampling instants (recursive feasibility).

Lemma 1. Let any $t \in \mathbb{N}$, any $x_t \in \mathbb{R}^n$, and any $\mathbf{d}_t = ((x_{0|t}, \dots, x_{N|t}), (u_{0|t}, \dots, u_{N-1|t})) \in \mathcal{D}_N(x_t)$ be given. Let further $x_{t+s+1} = Ax_{t+s} + Bu_{s|t} + w_{t+s}$ with $w_{t+s} \in \mathcal{W}$ for all $s \in \mathbb{N}_{[0,i-1]}$, where

$$t + i = \min\{j \in \mathbb{N}_{\geq t+1} \mid x_j \notin x_{j-t|t} \oplus \mathcal{T}_j\}. \tag{14}$$

Then it holds that $\mathcal{D}_N(x_{t+i}) \neq \emptyset$.

The next theorem guarantees the satisfaction of the constraints in the closed-loop system.

Theorem 1. For all $x_0 \in \hat{\mathcal{X}}_N$ and any realization of the disturbances $w_t \in \mathcal{W}$, $t \in \mathbb{N}$, it holds that $x_t \in \mathcal{X}$ and $\kappa(x_{t_j}, t - t_j) \in \mathcal{U}$ for all $t \in \mathbb{N}_{[t_j, t_{j+1}-1]}$, $j \in \mathbb{N}$, for the closed-loop system (9).

In order to establish stability properties of the closed-loop system, the auxiliary functions $V_\mathrm{s} : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^n \to \mathbb{R}$ are introduced. In particular, the stage cost function ℓ, q, and V_s are required to satisfy the following assumption.

Assumption 4. The functions ℓ, V_s, and q are continuous and positive semi-definite. Further, for all $x \in \mathbb{R}^n$ and all $v \in \mathbb{R}^m$ it holds that

$$V_\mathrm{s}((A+BK)x + Bv) \leq V_\mathrm{s}(x) - q(x) + \ell(v). \tag{15}$$

Finally, there exist $\mathcal{K}_\infty$-functions α_1 and α_2, such that for all $x \in \mathbb{R}^n$ it holds that

$$q(x) \geq \alpha_1(|x|) \tag{16}$$

and

$$V_\mathrm{s}(x) \leq \alpha_2(|x|). \tag{17}$$

Remark 4. Assumption 4 requires V_s to be an ISS-Lyapunov function for the system described by $x_{t+1} = (A+BK)x_t + Bv_t$. See also Copp and Hespanha (2014), where an ISS-control Lyapunov function is used as a terminal cost. The assumption is, for example, satisfied for the quadratic functions $\ell(x) = x^\top Lx$, $V_\mathrm{s}(x) = \eta x^\top Sx$, and $q(x) = \frac{\eta}{2} x^\top Qx$, where $\eta \in \mathbb{R}_{>0}$, $S = S^\top > 0$, $Q = Q^\top > 0$, $L = L^\top > 0$, $(A+BK)^\top S(A+BK) = S - Q$, and $L - \eta(2B^\top S(A+BK)Q^{-1}(A+BK)^\top SB + B^\top SB) \geq 0$. Note that if Assumption 1 is satisfied, it is always possible to find matrices satisfying these inequalities.

Let $\mathcal{Y} \subseteq \mathbb{R}^n$ denote the minimal robust positively invariant set for the dynamics $x_{t+1} = (A+BK)x_t + w_t$ with $w_t \in \bar{\mathcal{W}}$, $t \in \mathbb{N}$. Define for any $x \in \hat{\mathcal{X}}_N$

$$V_N^0(x) := J_N^0(x) + \min_{y \in \mathcal{Y}} V_\mathrm{s}(x - y). \tag{18}$$

The following lemma ensures that this function decreases along trajectories of the closed-loop system in a certain sense.

Lemma 2. For all $x_0 \in \hat{\mathcal{X}}_N$ and any realization of the disturbances $w_t \in \mathcal{W}$, $t \in \mathbb{N}$, it holds that

$$V_N^0(x_{t_{j+1}}) \leq V_N^0(x_{t_j}) - \sum_{t=t_j}^{t_{j+1}-1} \alpha_1(|x_t|_\mathcal{Y}) \tag{19}$$

for all $j \in \mathbb{N}$ for the closed-loop system (9).

Define the set

$$\bar{\mathcal{X}}_N := \{x \in \mathbb{R}^n \mid (A+BK)^N x \in \mathcal{X}_\mathrm{f},$$
$$(A+BK)^j x \in \mathcal{X}_j, K(A+BK)^j x \in \mathcal{U}_j, j \in \mathbb{N}_{[0,N-1]}, \}, \tag{20}$$

which is the set of all states for which the optimization problem in (8) admits a feasible solution resulting from the application of the linear feedback $u = Kx$ at each predicted time step.

Theorem 2. For all $x_0 \in \hat{\mathcal{X}}_N$ and any realization of the disturbances $w_t \in \mathcal{W}$, $t \in \mathbb{N}$, it holds that $\lim_{t\to\infty} |x_t|_\mathcal{Y} = 0$ for the closed-loop system (9). Further, if there exists an $\epsilon > 0$ such that $\mathcal{Y} \oplus \epsilon\mathcal{B} \subseteq \bar{\mathcal{X}}_N$, then the set $\mathcal{Y}$ is robustly asymptotically stable for the closed-loop system (9) with region of attraction $\hat{\mathcal{X}}_N$.

6. PROBABILISTIC GUARANTEES AND CHOICE OF PARAMETERS

In this section, we will give techniques to choose the parameters $\bar{\mathcal{W}}$ and $\mathcal{T}_i$ that guarantee the satisfaction of Assumption 2 and allow a quantification of the worst case asymptotic bound on the system state and the probability distribution of the inter-sampling time.

6.1 Event-triggered implementation of Tube MPC

If the main objective is a large region of attraction $\hat{\mathcal{X}}_N$ and a small asymptotic bound $\mathcal{Y}$ for the closed-loop system, one may choose $\bar{\mathcal{W}} = \mathcal{W}$ and

$$\mathcal{T}_i = A^{-1}(\mathcal{F}_{i+1} \ominus \mathcal{W}) = A^{-1}\left(\bigoplus_{j=1}^{i} (A+BK)^j \mathcal{W}\right), \tag{21}$$

for $i \in \mathbb{N}_{[1,N-1]}$, in order to satisfy Assumption 2. In this way, the resulting event-triggered MPC scheme requires the same tightening of constraints, and guarantees the same worst-case asymptotic bound, as the all-time triggered scheme in Chisci et al. (2001). The amount of reduction in communication depends on the particular probability density function $\mathbf{p}_w$.

Remark 5. This result implies that any periodically triggered Tube MPC scheme (not necessarily updating the inputs at every time point t) can be improved in terms of the average required communication by using an appropriate event-triggered MPC scheme in its place. In the event-triggered scheme, whenever the periodically updated scheme would normally schedule an update, an event condition along the lines of (21) would be checked beforehand.

6.2 Event-triggered implementation with probabilistic guarantees

In this subsection, we provide a means to design the sets $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1}$ and the set $\bar{\mathcal{W}}$, such that a desired average sampling rate is achieved in the closed-loop system and Assumption 2 is satisfied. In particular, consider the probability that an event is triggered at time point $t+i$ given that the last event occurred at time point t,

$$\mathbb{P}(x_{t+i} \notin x^\star_{i|t} \oplus \mathcal{T}_i, x_{t+j} \in x^\star_{j|t} \oplus \mathcal{T}_j, j \in \mathbb{N}_{[1,i-1]}). \quad (22)$$

It holds that $x_{t+j} - x^\star_{j|t} = \sum_{s=0}^{j-1} A^s w_{t+j-1-s}$. Furthermore, the disturbances are assumed to be distributed identically and independently, such that the probability in (22) is independent of x_t and t and is given by

$$P_i := \mathbb{P}\left(\sum_{s=0}^{i-1} A^s w_{i-1-s} \notin \mathcal{T}_i, \sum_{s=0}^{j-1} A^s w_{j-1-s} \in \mathcal{T}_j, j \in \mathbb{N}_{[1,i-1]}\right), \quad (23)$$

where the disturbances w_j, $j \in \mathbb{N}_{[0,i-1]}$, are generated according to the probability density function $\mathbf{p}_w$.

Remark 6. From (23) it can be seen that the triggering behavior of the closed-loop system only depends on the realization of the disturbance sequence, such that neither the initial condition x_0, nor the constraints, nor the stage cost function ℓ have an influence on the number of communications in the closed-loop system.

As $\mathcal{T}_N = \emptyset$, and hence an event is guaranteed to occur for an $i \in \mathbb{N}_{[1,N]}$, it holds that

$$\sum_{i=1}^{N} P_i = 1, \quad P_i \geq 0, \ i \in \mathbb{N}_{[1,N]}. \quad (24)$$

As $\mathbf{p}_w$ was assumed to be bounded, it is possible to assign any values to P_i, $i \in \mathbb{N}_{[1,N]}$ in (23) which satisfy (24), by choosing the sets $\mathcal{T}_i$ accordingly.

We propose the following simple method of choosing the sets $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1}$. Let $\mathcal{T} \subseteq \mathbb{R}^n$ be any PC-set and define $\mathcal{T}_i = \rho_i \mathcal{T}$ for $\rho_i \in \mathbb{R}_{\geq 0}$ and $i \in \mathbb{N}_{[1,N-1]}$. With this definition it holds that $\rho_r \leq \rho_s \Leftrightarrow \mathcal{T}_r \subseteq \mathcal{T}_s$ for all $r, s \in \mathbb{N}_{[1,N-1]}$, such that the probabilities P_i, $i \in \mathbb{N}_{[1,N-1]}$, as functions of $\rho_1, \rho_2, \ldots, \rho_i$, are continuous and monotonically non-increasing in ρ_i for $\rho_1, \rho_2, \ldots, \rho_{i-1}$ fixed. This allows P_i to be approximated (to arbitrary precision) by a bisection approach. Note that if the scalars ρ_j are fixed for $j \in \mathbb{N}_{[1,i-1]}$, then P_i is a function of ρ_i only. Hence, for desired values of P_i and given set $\mathcal{T}$, the values of ρ_i may be computed sequentially, starting with ρ_1. The expected value of the inter-sampling time is given by $\mathbb{E}[t_{j+1} - t_j] = \sum_{i=1}^{N} i P_i$ for any given $j \in \mathbb{N}$. Hence, by choosing P_i, (and, in turn, $\mathcal{T}_i$) accordingly, any desired value $\mathbb{E}[t_{j+1} - t_j] = \bar{\Delta}$ for $\bar{\Delta} \in [1, N]$ can be achieved.

Remark 7. By Theorem 1 in Doob (1948), it follows that the average sampling frequency converges to $1/\bar{\Delta}$ for the closed-loop system as time increases in the sense that $\lim_{t\to\infty} \frac{1}{t}\mathbb{E}(\max\{j \in \mathbb{N} \mid t_j \leq t\}) = 1/\bar{\Delta}$.

If the sets $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{N-1}$ are known, the enlarged disturbance set $\bar{\mathcal{W}}$ may be determined by defining $\bar{\mathcal{W}} := \bar{\rho}\mathcal{W}$ for a $\bar{\rho} \in \mathbb{R}_{\geq 1}$ such that Assumption 2 is satisfied. In this case, it holds that $\mathcal{Y} = \bar{\rho}\mathcal{Y}^\star$, where $\mathcal{Y}^\star$ is the minimal robust positively invariant set for the case $\bar{\mathcal{W}} = \mathcal{W}$, which follows immediately from equation (3) in Raković et al. (2005).

7. NUMERICAL EXAMPLES

In this section, we provide two examples[1] showing the reduction of communication with the proposed scheme.

7.1 Event-triggered implementation of a given Tube MPC scheme

Let the system be given by

$$x_{t+1} = \begin{bmatrix} 1.1 & 0.2 \\ 0 & 1.2 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t + w_t, \quad (25)$$

where w_t is independently uniformly distributed on $\mathcal{W} = [-1, 1]^2$ for $t \in \mathbb{N}$. The feedback matrix K has been chosen LQ-optimal with the weighting matrices $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $R = 1$. In the first example, we investigated the closed-loop behavior for sets $\mathcal{T}_i$, $i \in \mathbb{N}_{[1,N-1]}$, constructed as proposed in Section 6.1. We defined $\bar{\mathcal{W}} = \mathcal{W}$ and implemented the resulting event-triggered MPC scheme with a prediction horizon of $N = 10$. A simulation of $T_{\text{sim}} = 10^5$ steps yielded the distribution of time steps between events displayed in Table 1. The average time between events was 1.42, which amounts to a 29% reduction in communication. Note that the region of attraction and worst-case asymptotic bound on the system state are exactly the same for the scheme in Chisci et al. (2001) and the event-triggered scheme presented here. Consider further the performance index $J_{\text{perf}} := 1/T_{\text{sim}} \sum_{t=0}^{T_{\text{sim}}-1} (x_t^\top Q x_t + u_t^\top R u_t)$. For this example the performance index for the closed-loop system with an MPC update at every time step was $J^{\text{pt}}_{\text{perf}} = 5.69$. The performance index for the event-triggered scheme was $J^{\text{et}}_{\text{perf}} = 5.79$, which amounts to a 1.73% increase when compared to the scheme with updates at every point in time. The sequence of disturbances in the simulation of both control schemes was chosen to be identical.

Table 1. Distribution of inter-sampling times for an event-triggered implementation of a Tube MPC scheme.

inter-sampling time	frequency	inter-sampling time	frequency
1	77.96%	6	0.68%
2	12.82%	7	0.34%
3	4.63%	8	0.22%
4	1.95%	9	0.11%
5	1.12%	10	0.19%

7.2 Event-triggered Tube MPC with assigned distribution of inter-sampling times

In the second example, we implemented the scheme as proposed in Section 6.2. Consider the same setup as in the first example. We chose $\mathcal{T} = [-1, 1]^2$ and used a Monte-Carlo-approach to evaluate the probabilities P_i for given ρ_i with a 95% confidence interval of $[-0.005, 0.005]$. The values for ρ_i were computed with a bisection iteration with an error tolerance of $[-0.005, 0.005]$, using the aforementioned Monte-Carlo-approach in each step. Hence, the probabilities P_i are contained in an interval of $[-0.01, 0.01]$ around their desired values (chosen to be $P_1 = P_5 = 0.1$, $P_2 = P_4 = 0.2$, $P_3 = 0.4$, and $P_6 = P_7 = \ldots = P_{10} = 0$) with

[1] YALMIP (Löfberg (2004)), the Multi-Parametric Toolbox 3.0 (Herceg et al. (2013)) and IBM ILOG CPLEX Optimization Studio (IBM (2014)) were used in the simulations.

95% confidence. The values chosen for the probabilities P_i imply a desired average of 3 time steps between sampling instants and thus an average reduction in communication by 66.7%. The resulting values of ρ_i for $i \in \mathbb{N}_{[1,4]}$ were computed to $\rho_1 = 0.9492$ $\rho_2 = 1.3750$, $\rho_3 = 0.9688$, and $\rho_4 = 0.7188$. The remaining ρ_i for $i \in \mathbb{N}_{[5,10]}$ were set to 0. The value of $\bar{\rho}$ was computed to $\bar{\rho} = 2.3435$, which is at the same time the factor describing the increase in the guaranteed asymptotic bound on the system state and the increase in necessary constraint tightening. The region of attraction depends on the constraints on the state and input, which were not considered in these examples. Note that a tightening of constraints does not necessarily lead to a reduction of the region of attraction. A simulation of 10^5 steps yielded the distribution of time steps between events displayed in Table 2. All frequencies are within an interval of $[-0.01, 0.01]$ around the assigned probabilities P_i as guaranteed with 95% confidence by the combined Monte-Carlo-bisection iteration above. The average time between events was 2.9989, which amounts to a 66.55% reduction in communication. For this example we obtained $J_{\text{perf}}^{\text{pt}} = 5.77$ and $J_{\text{perf}}^{\text{et}} = 7.85$ which amounts to a 36% increase of the performance index for the event-triggered scheme when compared to a scheme with updates at every point in time.

Table 2. Distribution of inter-sampling times for an event-triggered Tube MPC scheme with assigned probabilities.

inter-sampling time	frequency	inter-sampling time	frequency
1	9.88%	4	19.77%
2	20.19%	5	9.60%
3	40.56%		

Remark 8. The disturbances were assumed to be uniformly distributed in the examples. Much more favorable results (greater reduction in communication with smaller performance loss) can be expected for sporadically occurring disturbances, that is for probability distributions which are mostly concentrated around the origin.

8. CONCLUSIONS AND OUTLOOK

We have presented a robust event-triggered MPC scheme based on Tube MPC methods. It was shown that the required amount of communication in the control system can be reduced without sacrificing the guarantees offered by a periodically updated Tube MPC scheme. Further reduction of the required communication and assignment of a desired expected value of the time between events is possible by allowing a larger asymptotic bound on the system state and tightening the constraints in the prediction, while introducing a trade-off between the closed-loop performance and the average required communication.

The results in this paper rely heavily on the fact that the disturbances are independent and that the expected value of the time between events only depends on the disturbances occurring in exactly this time span. These assumptions are not necessarily satisfied in the case of output feedback or disturbances generated by a (randomly disturbed) exosystem, both of which are subject to future research.

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