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Dynamic Thresholds in Robust Event-Triggered Control for Discrete-Time Linear Systems

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Abstract— **We propose a model-based event-triggered control scheme where state measurements are transmitted to the controller only if the deviation between the real system state and the state of a nominal closed-loop model crosses a certain threshold. In contrast to earlier results, the size of the thresholds is dynamically adapted to the unknown magnitude of the disturbances acting on the system. The closed-loop system is shown to be input-to-state stable. For a particular choice of the parameters, the guaranteed asymptotic bound on the system state is the same as for a periodically triggered scheme while still allowing a reduction in communication.**

I. INTRODUCTION

In many networked control systems, where sensors, controllers, and actuators are not necessarily located in one place, the cost of communicating measurements from the sensors to the controller has to be taken into account when designing the control scheme. Especially in wireless networks, communication costs may not be negligible. It has been found that aperiodic sampling schemes can achieve a better closed-loop system performance than periodic sampling schemes with the same average sampling rate, hence achieving a better trade-off between controller performance and network usage, see, for example, [1], [2], where this was shown quantitatively. One way of achieving aperiodic sampling is based on using event-triggered control: the state is measured continuously or periodically, but communicated to the controller only if the measured state violates certain thresholds. For recent overviews of event-triggered control, please refer to [3], [4].

In this paper, we focus on *model-based* event-triggered control. Here, both the sensor and the actuator contain a model of the undisturbed closed-loop system. An event is triggered if the deviation between the real system and the nominal model crosses a threshold. Then, the measured system state is communicated to the actuator, and the nominal system models (both at the sensor and the actuator) are synchronized with the real system. This approach has the advantage that events, and communication, only occur at initialization and in response to disturbances. Hence, for vanishing disturbances also the necessity for communication disappears, while the control objective can still be fulfilled. Model-based event-triggered control schemes were, for example, proposed in [5], [6], [7]. See also [8] for a an event-triggered model predictive controller using similar concepts. In the recent paper [9] it was shown that in the case of bounded disturbances, choosing the thresholds for a model-based event-triggered model predictive controller in a certain way, leads to the same worst-case asymptotic bound on the system state as any periodically triggered control scheme, but with a reduced average sampling rate. However, for event thresholds that are independent of the system state, if there does not exist an a priori upper bound on the time between events[1], the closed-loop system may cease to be input-to-state stable with respect to additive disturbances if the open-loop system is unstable. The reason for this is that updates are only scheduled if the norm of the state becomes sufficiently large, regardless of the actual bound on the disturbances. The closed-loop system remains input-to-state *practically* stable, however, see [10]. In the same work, input-to-state stability for open-loop asymptotically stable systems under event-triggered control was shown.

We propose an event-triggered scheme where the size of the thresholds is dynamically adapted to the magnitude of the disturbances. In particular, an additional state is included in the controller which serves as an estimate of the bound on the disturbances. The thresholds employed in the event-triggering mechanism are multiplied with this estimate. It is shown that the closed-loop system is input-to state stable and that the state of the system converges to the origin if the disturbances acting on the system converge to zero. If the disturbances acting on the system are zero, the size of the thresholds converges exponentially to zero, which is similar to the event-triggered schemes proposed in [11], [12] for undisturbed systems. Other control schemes proposing dynamic thresholds similar to the ones in the present paper were proposed in [13], [14] for undisturbed nonlinear systems and in [15] for continuous-time systems subject to disturbances. Please also refer to the references therein. Note that no estimate of the disturbance magnitude is performed in [15]. Another event-triggered scheme where the event conditions are adapted online can for example be found in [16], where event-triggered control is used to solve a distributed scheduling problem.

The remainder of the paper is structured as follows. Section II contains the notation used in the paper and several preliminary results. The problem setup is stated in Section III. The proposed method and the main results of the paper are presented in Section IV. Remarks on the choice of parameters and possible extensions of the control scheme are given in Section V. Section VI contains a numerical example illustrating the results and Section VII concludes the paper.

II. PRELIMINARIES

Notation: We use $\mathbb{N}$ to denote the non-negative integers and $\mathbb{R}$ to denote the real numbers. Further, for $c \in \mathbb{R}$ we use $\mathbb{N}_{\geq c}$ ($\mathbb{R}_{\geq c}$) to denote the set of integers (real numbers) greater to or equal than c. We analogously define $\mathbb{R}_{>c}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and zero at zero. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function if it is a $\mathcal{K}$-function

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[1]Such a bound may be artificially enforced.

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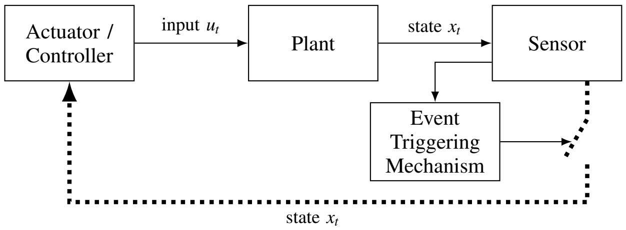

Fig. 1. Event-triggered control structure. Communication takes place between the sensor and the actuator/controller.

in its first argument for every value of its second argument and is strictly decreasing and convergent to zero in its second argument for every value of its first argument. A PC-set is a convex and compact set containing the origin in its interior. For any matrix $M \in \mathbb{R}^{n\times m}$ with entries M_{ij}, $i \in \mathbb{N}_{[1,n]}$, $j \in \mathbb{N}_{[1,m]}$, we use $|M|$ to denote the maximum absolute row sum, that is, $|M| := \max_i \sum_{j=1}^m |M_{ij}|$. For a vector $v \in \mathbb{R}^n$ and a PC-set $\mathcal{F} \subseteq \mathbb{R}^n$, the infinity norm of v is denoted by $|v|$, the distance of v to $\mathcal{F}$ is given by $|v|_{\mathcal{F}} := \min_{f\in\mathcal{F}} |v-f|$, and the size of $\mathcal{F}$ is defined by $|\mathcal{F}| := \max_{f\in\mathcal{F}} |f|$. For any PC set $\mathcal{F} \subseteq \mathbb{R}^n$, the Minkowski gauge function $\Phi_{\mathcal{F}} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is given by $\Phi_{\mathcal{F}} : x \mapsto \min\{s \in \mathbb{R}_{\geq 0} \mid x \in s\mathcal{F}\}$. For any set $\mathcal{A} \subseteq \mathbb{R}^n$ and any matrices $L \in \mathbb{R}^{m\times n}, M \in \mathbb{R}^{n\times m}$ we define $L\mathcal{A} := \{Lx \mid x \in \mathcal{A}\}$ and $M^{-1}\mathcal{A} := \{x \in \mathbb{R}^m \mid Mx \in \mathcal{A}\}$. Given two sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$, the Minkowski set addition is defined by $\mathcal{A} \oplus \mathcal{B} := \{a+b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ and the Pontryagin set difference is defined by $\mathcal{A} \ominus \mathcal{B} := \{x \in \mathbb{R}^n \mid \{x\} \oplus \mathcal{B} \subseteq \mathcal{A}\}$. Given a sequence of sets $\mathcal{X}_i \subseteq \mathbb{R}^n$ for $i \in \mathbb{N}_{[a,b]}$ with $a,b \in \mathbb{N} \cup \{\infty\}$, we define $\bigoplus_{i=a}^b \mathcal{X}_i = \{x \mid \exists x_i \in \mathcal{X}_i, i \in \mathbb{N}_{[a,b]}, x = \sum_{i\in\mathbb{N}_{[a,b]}} x_i\}$. By convention, the empty sum is equal to $\{0\}$.

The following lemmas will be used in the paper.

Lemma 1: Let $\mathcal{A} \subseteq \mathbb{R}^n$, $\mathcal{B} \subseteq \mathbb{R}^m$ be PC-sets. Then for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ it holds that $\left|\binom{x}{y}\right|_{\mathcal{A}\times\mathcal{B}} \leq |x|_{\mathcal{A}} + |y|_{\mathcal{B}} \leq 2\left|\binom{x}{y}\right|_{\mathcal{A}\times\mathcal{B}}$. The proof is given in the appendix.

Lemma 2: For all PC-sets sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$ and all $x, y \in \mathbb{R}^n$ it holds that $|x+y|_{\mathcal{A}\oplus\mathcal{B}} \leq |x|_{\mathcal{A}} + |y|_{\mathcal{B}}$.

We use the following definition of input-to-state stability for a general formulation of a dynamical system using a transition function Ψ. In particular, let W denote the set of infinite sequences in $\mathbb{R}^{n_w}$ and define $\Psi : \mathbb{R}^{n_z} \times W \times \mathbb{N} \to \mathbb{R}^{n_z}$, $(z_0, t, \underline{w}) \mapsto \Psi(z_0, \underline{w}, t)$. Here, $\Psi(z_0, \underline{w}, t)$ is the state at time t for the discrete-time dynamical system with initial state z_0 at time $t = 0$ and input sequence $\underline{w}$.

Definition 1 (cf. [17]): The dynamical system given by Ψ is *input-to-state stable* with respect to w, if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function α such that

$$|\Psi(z_0, \underline{w}, t)| \leq \beta(|z_0|, t) + \alpha\left(\sup_{t\in\mathbb{N}} |w_t|\right) \tag{1}$$

for all $z_0 \in \mathbb{R}^{n_z}$, all $\underline{w} \in W$, with $\underline{w} = \{w_\tau,\ \tau \in \mathbb{N}\}$ and all $t \in \mathbb{N}$.

In order to show input-to-state stability, we will establish that a certain set is asymptotically stable for the closed-loop system.

Definition 2 (cf. [18]): A set $\Omega \subseteq \mathbb{R}^{n_z}$ is *asymptotically stable* for the dynamical system given by Ψ if there exists a $\mathcal{KL}$-function $\bar{\beta}$, such that

$$|\Psi(z_0, \underline{w}, t)|_{\Omega} \leq \bar{\beta}(|z_0|_{\Omega}, t) \tag{2}$$

for all $z_0 \in \mathbb{R}^{n_z}$, all $\underline{w} \in W$, with $\underline{w} = \{w_\tau,\ \tau \in \mathbb{N}\}$ and all $t \in \mathbb{N}$. It is *exponentially stable* if $\bar{\beta} : (s,t) \to c\rho^t s$ for some $c \in \mathbb{R}_{\geq 0}$ and some $\rho \in [0, 1)$.

III. PROBLEM SETUP

Consider the linear time-invariant discrete-time system given by

$$x_{t+1} = Ax_t + Bu_t + w_t, \tag{3}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the input, and $w_t \in \mathbb{R}^n$ is the disturbance at time $t \in \mathbb{N}$. We assume that the state x_t is available as a measurement at any time $t \in \mathbb{N}$. The disturbance w_t is assumed to be unknown but bounded by $w_t \in \gamma\mathcal{W}$, where $\gamma \in \mathbb{R}_{\geq 0}$ and $\mathcal{W} \subseteq \mathbb{R}^n$ is a PC-set. It is assumed that $\mathcal{W}$ is known but γ is not. We assume the control structure shown in Figure 1, where the controller is given by the dynamical system

$$\hat{x}_t = \begin{cases} x_t & t \in \mathcal{E} \\ (A+BK)\hat{x}_{t-1} & t \in \mathbb{N} \setminus \mathcal{E} \end{cases}$$
$$u_t = K\hat{x}_t \tag{4}$$

where $\mathcal{E} \subseteq \mathbb{N} \cup \{\infty\}$ with $0 \in \mathcal{E}$ is the set of event times. Here, the matrix $K \in \mathbb{R}^{m\times n}$ is chosen such that $A+BK$ is Schur stable. The event-triggering mechanism defining the set $\mathcal{E}$ as $\mathcal{E} := \{t_i,\ i \in \mathbb{N}\}$ contains a copy of (4) and is given by

$$t_0 = 0,$$
$$t_{i+1} = \inf\{t \in \mathbb{N},\ t > t_i,\ x_t \notin \{(A+BK)\hat{x}_{t-1}\} \oplus \mathcal{T}_t\} \tag{5}$$

for threshold sets $\mathcal{T}_t \subseteq \mathbb{R}^n$, $t \in \mathbb{N}$. That is, an event is triggered if the system state x_t deviates too far from the controller state $\hat{x}_t$. This setup is similar to the control schemes proposed in [5], [19], [6], [20]. Note that the thresholds are allowed to be time-varying and, in fact, allowed to depend on the evolution of the system state and, hence, on the realization of the disturbance sequence. Define

$$\mathcal{Y} := \bigoplus_{i=0}^{\infty} (A+BK)^i \mathcal{W}, \tag{6}$$

which is the minimal robust positively invariant set under the control law $u_t = Kx_t$ if the worst case bound on the disturbances is given by $w_t \in \mathcal{W}$, $t \in \mathbb{N}$. Note that $\mathcal{Y}$ is a PC-set [21].

The goal is to design a dynamical system generating the sets $\mathcal{T}_t$ such that the following requirements are fulfilled by the resulting closed-loop system.

- If the disturbances are bounded by the (unknown) bound $\sup_{t\in\mathbb{N}} \Phi_{\mathcal{W}}(w_t) = \gamma$, the set $\gamma\mathcal{Y}$ is asymptotically stabilized for the closed-loop system,
- the closed-loop system is input-to-state stable with respect to the disturbances w, and
- if the disturbances converge to zero, then also the state of the closed-loop system converges to zero.

Further, the sets $\mathcal{T}_t$ should be chosen as large as possible in order to facilitate a reduction in the average communication rate.

Assumption 1: Let $\Phi_{\mathcal{W}}(w) \leq d|w|$ for all $w \in \mathbb{R}^n$ and a $d \in \mathbb{R}_{\geq 0}$.

Remark 1: The existence of d in Assumption 1 is guaranteed with Lemma 1 in [22] by noting that $\Phi_{\{x\in\mathbb{R}^n \mid |x|\leq 1\}}(w) = |w|$ for all $w \in \mathbb{R}^n$.

IV. DYNAMIC THRESHOLD DESIGN

We propose the following solution to the problem stated in Section III. While the disturbance w_t is not known to the controller at time t, it can be computed at time $t+1$ with the formula

$$w_t = x_{t+1} - Ax_t - Bu_t \tag{7}$$

as the right-hand side of (7) is completely known at time $t+1$. Define the dynamical system

$$\eta_{t+1} = \lambda\eta_t + (1-\lambda)\Phi_{\mathcal{W}}(w_t), \tag{8}$$

with $\lambda \in [0,1)$ and $\eta_0 \in \mathbb{R}_{\geq 0}$. Define further

$$\mathcal{T}^i := A^{-1}\left(\bigoplus_{j=0}^{i}(A+BK)\mathcal{W} \ominus \mathcal{W}\right) = A^{-1}\bigoplus_{j=1}^{i}(A+BK)\mathcal{W} \tag{9}$$

for $i \in \mathbb{N}$, which are the thresholds proposed in [9]. Let finally

$$\mathcal{T}_t = \eta_t \mathcal{T}^{t-t_i} \tag{10}$$

for all $t \in \mathbb{N}_{[t_i+1,t_{i+1}]}$ and all $i \in \mathbb{N}$. Then the complete closed-loop system is given by

$$\begin{aligned} x_{t+1} &= Ax_t + BK\hat{x}_t + w_t \\ \hat{x}_{t+1} &= \begin{cases} x_{t+1}, & \text{if } t+1 = t_i \text{ for some } i \in \mathbb{N} \\ (A+BK)\hat{x}_t, & \text{if } t+1 \neq t_i \text{ for all } i \in \mathbb{N} \end{cases} \\ t_{i+1} &= \inf\{t \in \mathbb{N},\ t > t_i,\ x_t \notin \{(A+BK)\hat{x}_{t-1}\} \oplus \eta_t \mathcal{T}^{t-t_i}\} \\ \eta_{t+1} &= \lambda \eta_t + (1-\lambda)\Phi_{\mathcal{W}}(w_t), \end{aligned} \tag{11}$$

with $t_0 = 0$ and $\hat{x}_0 = x_0$. The initial value η_0 can be chosen arbitrarily. However, if $\eta_0 = 0$, additional properties can be guaranteed, see Corollary 1. Note that (11) is a dynamical system that can be described by a map Ψ as in Definition 1 with state $z_t = (x_t^\top, \eta_t)^\top$ and input w_t.

The following results establish the input-to-state stability properties of the closed-loop system in (11).

Lemma 3: For the closed-loop system (11), the set $\Omega := \gamma\mathcal{Y} \times [0,\gamma]$ is globally exponentially stable in the sense that there exist $c \in \mathbb{R}_{\geq 0}$, $\rho \in [0,1)$ such that $|(x_t^\top, \eta_t)^\top|_\Omega \leq c\rho^t |(x_0^\top, \eta_0)^\top|_\Omega$, $t \in \mathbb{N}$, for all $x_0 \in \mathbb{R}^n$, $\eta_0 \in \mathbb{R}_{\geq 0}$, and any realization of the sequence of disturbances which satisfies $w_t \in \gamma\mathcal{W}$ for all $t \in \mathbb{N}$ and a $\gamma \in \mathbb{R}_{\geq 0}$.

Theorem 1: The closed-loop system (11) is input-to-state stable with respect to w with state $z_t = (x_t^\top, \eta_t)^\top$. Furthermore, if $\lim_{t\to\infty}|w_t| = 0$, it holds that $\lim_{t\to\infty}|x_t| = 0$.

Corollary 1: If $\eta_0 = 0$, then it holds that $|x_t| \leq \beta_2(|x_0|,t) + \alpha_2(\sup_{t\in\mathbb{N}}|w_t|)$ for a $\mathcal{KL}$-function β_2 and a $\mathcal{K}$-function α_2, for all $x_0 \in \mathbb{R}^n$, and any realization of the sequence of disturbances $w_t \in \mathbb{R}^n$, $t \in \mathbb{N}$.

The proofs for these statements are given in the appendix.

V. DISCUSSION AND REMARKS

A. Simpler Thresholds

In general, the complexity of the thresholds as defined in (9) grows with i, the time since the last event. Simpler thresholds can still be employed in the same manner, as long as they satisfy

$$A\mathcal{T}^i \oplus \mathcal{W} \subseteq \bigoplus_{j=0}^{i}(A+BK)\mathcal{W} \tag{12}$$

for all $i \in \mathbb{N}$, compare [9]. In fact, the thresholds as defined in (9) are the largest sets satisfying (12).

B. Trade-off between asymptotic bound and average sampling rate

The thresholds as defined in (9) guarantee that the asymptotic bound on the system state, that is $\gamma\mathcal{Y}$, is not worse than for a controller $u_t = Kx_t$, which is updated at every time step. A relaxation of the asymptotic bound in favor of a reduced average sampling rate may be employed by replacing the thresholds in (10) with

$$\mathcal{T}_t = \delta\eta_t\mathcal{T}^{t-t_i}, \tag{13}$$

where $\delta \in \mathbb{R}_{\geq 1}$ is an additional tuning parameter. The only change in the results is that Ω in Lemma 3 has to be replaced by $\Omega_\delta = \delta\gamma\mathcal{Y} \times [0,\gamma]$, compare equation (3) in [23]. Thus, a larger δ leads to larger thresholds, with the consequence of a larger asymptotic bound, and, as we conjecture, a reduced average sampling rate.

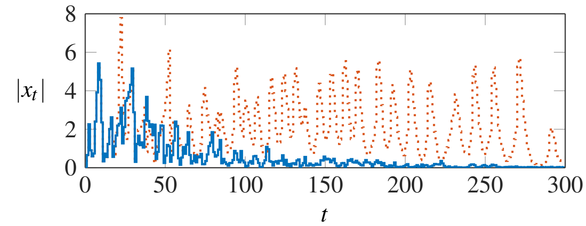

Fig. 2. Evolution of $|x_t|$ for $\lambda = 0.9$ (solid) and $\lambda = 1$ (dotted).

C. Extension to Model Predictive Control

If constraints of the form $(x_t^\top, u_t^\top)^\top \in \mathcal{X} \times \mathcal{U}$, $t \in \mathbb{N}$, are present, a linear controller may have an unsuitably small operating range or may have to be detuned to unsatisfactory performance. An alternative is robust model predictive control, where the input u_t is defined as the first part of a finite-horizon-optimal control sequence, where the optimal control problem is parameterized by the state $\hat{x}_t$. If an upper bound $\bar{\gamma}$ on the disturbance bound γ is known, appropriate constraint tightening (based on $\bar{\gamma}$) guarantees robust constraint satisfaction. The event conditions can be defined as in this paper. Note that an event-triggered predictive control scheme along these lines requires identical optimal control problems to be solved both in the actuator and in the sensor, or the optimal input sequence to be transmitted at every triggering instant. Further, it is required that $\eta_0 \leq \bar{\gamma}$. For details on a particular robust event-triggered predictive control scheme compatible with this paper, see [9].

VI. NUMERICAL EXAMPLE

Consider the system given by

$$x_{t+1} = \begin{bmatrix} 1.4 & 0.3 \\ 0 & 1.3 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t + w_t \tag{14}$$

with $\mathcal{W} = [-1,1]^2$ and $x_0 = (0,0)^\top$. We choose K as the LQ-optimal feedback gain for the weighting matrices $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $R = 1$. We computed the sets $\mathcal{T}_i$ according to (9) for $i \in \mathbb{N}_{[1,19]}$ and set $\mathcal{T}_i = \emptyset$ for $i \in \mathbb{N}_{[\geq 20]}$. The thresholds were computed according to (13) with $\delta = 3$. We set $\eta_0 = 1$.

Remark 2: Note that the threshold sets for $i \in \mathbb{N}_{[\geq 20]}$ satisfy (12) and that their usage is equivalent to forcing an event after 20 steps if no threshold violation occurred for 19 consecutive steps. As remarked in the introduction, this implies input-to-state stability of the closed-loop system even without the threshold adaptation. However, as shown in the first example below, the convergence behavior of the system is improved greatly by employing dynamic thresholds. In the case of using the thresholds as defined in (9) for all $i \in \mathbb{N}$, we expect the improvement to be even more pronounced.

A. Converging disturbances

In the first example, we consider a disturbance sequence generated by drawing w_t independently from a uniform distribution on $\gamma_t\mathcal{W}$ with $\gamma_t = 0.985^t$. In Figure 2, $|x_t|$ is plotted over time for the two cases $\lambda = 0.9$ and $\lambda = 1$, where the latter case leads to an event generation similar to the one proposed in [9], that is, the size of the thresholds is not adapted to the magnitude of the disturbances. Clearly, in the case of $\lambda = 0.9$, the state converges faster to the origin. The evolution of η_t is given in Figure 3. The triggering behavior is shown in Figure 4.

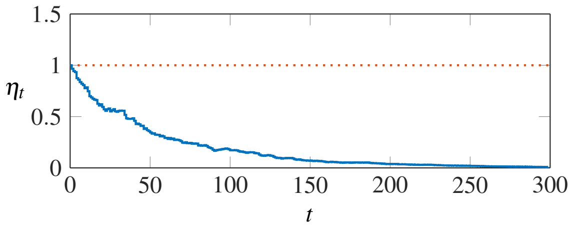

Fig. 3. Evolution of η_t for $\lambda = 0.9$ (solid) and $\lambda = 1$ (dotted).

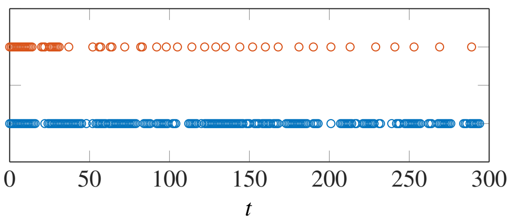

Fig. 4. Events (circles) for for $\lambda = 1$ (upper plot) and $\lambda = 0.9$ (lower plot).

B. Step change in disturbance magnitude

In the second example we consider a disturbance sequence generated by drawing w_t independently from a uniform distribution on $\gamma_t \mathcal{W}$ with

$$\gamma_t = \begin{cases} 10 & t \in \mathbb{N}_{[0,24999]} \\ 0.1 & t \in \mathbb{N}_{[25000,49999]}. \end{cases} \tag{15}$$

In Figure 5, η_t is plotted over time. For the time span $\mathbb{N}_{[0,24999]}$, we reported an average of 1.3717 steps between events and for $\mathbb{N}_{[25000,49999]}$, we reported an average of 1.3764 steps between events. This shows the ability of the scheme to adapt to a change in the magnitude of the disturbances while keeping the average sampling rate constant. Note that the average sampling rate depends on the system dynamics and the probability distribution from which the disturbances are drawn. For sparse disturbance sequences (where large disturbances occur with a far smaller probability than small disturbances), a much lower average sampling rate can be achieved, depending on the choice of parameters δ and λ.

For an example of the triggering behavior under sparse disturbances, consider the same setup, but with w_t now being uniformly distributed on $\gamma_t \mathcal{W}$ with a probability of only 0.1 for a given $t \in \mathbb{N}$, and being distributed on $0.01\gamma_t \mathcal{W}$ with probability 0.9 for any given $t \in \mathbb{N}$ (probabilities being independent for all $t \in \mathbb{N}$). The average number of time steps between events increased to 4.3644. Additionally setting $\lambda = 0.99$ (from initially $\lambda = 0.9$) increased this average to 6.0549.

VII. CONCLUSIONS AND OUTLOOK

We have presented an event-triggered controller where the size of the thresholds is dynamically adapted to the magnitude of the disturbance input. It is shown that the closed-loop system is input-to-state stable with respect to the disturbances. The numerical example demonstrates the ability of the scheme to adapt to the unknown bounds on the disturbances. Future research will focus on stochastic disturbances, where we conjecture that the average sampling rate of the adaptive scheme is invariant under linear scaling of the disturbances. Further, the influence of the tuning parameters δ and λ on the average sampling rate will be studied. Finally, we also want to consider the output-feedback case, in which it is no longer possible to directly measure the disturbance w_t.

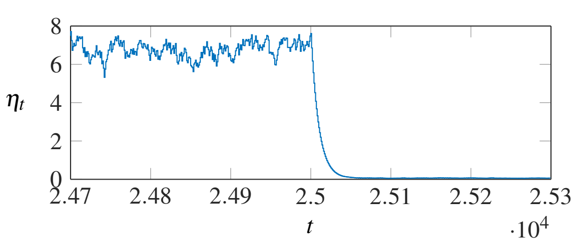

Fig. 5. Evolution of η_t for a step change in the disturbance magnitude.

APPENDIX

The subsequent proofs rely on the following fact.

Lemma 4 (cf. [24], [25]): For all $\tau \in \mathbb{N}$ it holds that $(A+BK)^\tau \mathcal{Y} \oplus \bigoplus_{j=0}^{\tau-1} (A+BK)\mathcal{W} = \mathcal{Y}$.

A. Proof of Lemma 1

It holds that

$$\begin{aligned}
&\left|\begin{pmatrix} x \\ y \end{pmatrix}\right|_{\mathcal{A}\times\mathcal{B}} = \min_{\binom{a}{b}\in\mathcal{A}\times\mathcal{B}} \left|\begin{pmatrix} x-a \\ y-b \end{pmatrix}\right| \\
&\leq \min_{\binom{a}{b}\in\mathcal{A}\times\mathcal{B}} \left|\begin{pmatrix} x-a \\ 0 \end{pmatrix}\right| + \left|\begin{pmatrix} 0 \\ y-b \end{pmatrix}\right| = \min_{\binom{a}{b}\in\mathcal{A}\times\mathcal{B}} |x-a| + |y-b| \\
&= \min_{a\in\mathcal{A}} |x-a| + \min_{b\in\mathcal{B}} |y-b| = |x|_{\mathcal{A}} + |y|_{\mathcal{B}} \leq 2\max\{|x|_{\mathcal{A}}, |y|_{\mathcal{B}}\} \\
&= 2\max\left\{\min_{a\in\mathcal{A}} |x-a|, \min_{b\in\mathcal{B}} |y-b|\right\} \\
&\leq 2 \min_{\binom{a}{b}\in\mathcal{A}\times\mathcal{B}} \max\{|x-a|, |y-b|\} = 2 \min_{\binom{a}{b}\in\mathcal{A}\times\mathcal{B}} \left|\begin{pmatrix} x-a \\ y-b \end{pmatrix}\right| \\
&= 2\left|\begin{pmatrix} x \\ y \end{pmatrix}\right|_{\mathcal{A}\times\mathcal{B}},
\end{aligned} \tag{16}$$

thereby completing the proof. ■

B. Proof of Lemma 3

The stability proof has the following structure. We will construct, for every possible trajectory of the closed-loop system (11) (generated by an initial condition $x_0 \in \mathbb{R}^n$, $\eta_0 \in \mathbb{R}_{\geq 0}$ and a disturbance realization with $w_t \in \gamma\mathcal{W}, t \in \mathbb{N}$) a function $\bar{V}: \mathbb{N} \to \mathbb{R}_{\geq 0}$ with the following properties.

$$\text{(i)} \qquad \bar{V}(0) \leq a_2 \left|\begin{pmatrix} x_0 \\ \eta_0 \end{pmatrix}\right|_{\Omega},$$

$$\text{(ii)} \qquad a_1 \left|\begin{pmatrix} x_t \\ \eta_t \end{pmatrix}\right|_{\Omega} \leq \bar{V}(t), \qquad \text{and}$$

$$\text{(iii)} \qquad \bar{V}(t+1) \leq \rho \bar{V}(t)$$

for all $t \in \mathbb{N}$, where $a_1, a_2 \in \mathbb{R}_{>0}$ and $\rho \in [0,1)$. This will allow us to use $\bar{V}$ as a Lyapunov-like function proving global exponential stability of Ω.

In order to construct $\bar{V}$, we first define a function $V_\gamma: \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, parameterized by the disturbance bound γ. As $A+BK$ is Schur stable, there exist a matrix $P \in \mathbb{R}^{p\times n}$ of rank n and a scalar $\mu \in [0,1)$, such that $|P(A+BK)x| \leq \mu |Px|$ for all $x \in \mathbb{R}^n$, see for example [26], [27]. Define V_γ by

$$\begin{aligned}
V_\gamma(\tau, x, \hat{x}, \eta) :=& \min_{\bar{y}\in\gamma(A+BK)^\tau\mathcal{Y}} |P(\hat{x}-\bar{y})| \\
&+ \min_{\bar{z}\in\gamma\bigoplus_{j=0}^{\tau-1}(A+BK)\mathcal{W}} |P(x-\hat{x}-\bar{z})| + \frac{2|P||(A+BK)\mathcal{Y}|}{1-\lambda} |\eta|_{[0,\gamma]}
\end{aligned} \tag{17}$$

for all $\tau \in \mathbb{N}$, all $x \in \mathbb{R}^n$, all $\hat{x} \in \mathbb{R}^n$, and all $\eta \in \mathbb{R}_{\geq 0}$. By Corollary II.8 in [27] there exist a $a_{p1} \in \mathbb{R}_{>0}$ such that $a_{p1}|x| \leq |Px| \leq |P||x|$ for all $x \in \mathbb{R}^n$. Hence, it holds that

$$\begin{aligned}\left|\begin{pmatrix} x \\ \eta \end{pmatrix}\right|_{\Omega} &\overset{\text{Lemma 1}}{\leq} |x|_{\gamma\mathcal{Y}} + |\eta|_{[0,\gamma]} \\ &\overset{\text{Lemma 2, Lemma 4}}{\leq} |\hat{x}|_{(A+BK)^\tau\gamma\mathcal{Y}} + |x-\hat{x}|_{\gamma\bigoplus_{j=0}^{\tau-1}(A+BK)^j\mathcal{W}} + |\eta|_{[0,\gamma]} \\ &\leq \frac{1}{a_1}V_\gamma(\tau,x,\hat{x},\eta) \end{aligned} \tag{18}$$

for some $a_1 \in \mathbb{R}_{>0}$, all $\tau \in \mathbb{N}$, all $x \in \mathbb{R}^n$, all $\hat{x} \in \mathbb{R}^n$, and all $\eta \in \mathbb{R}_{\geq 0}$. Further, using again Lemma 1, there also exists an $a_2 \in \mathbb{R}_{>0}$ with

$$V_\gamma(0,x,x,\eta) \leq a_2 \left|\begin{pmatrix} x \\ \eta \end{pmatrix}\right|_{\Omega} \tag{19}$$

for all $x \in \mathbb{R}^n$, $\eta \in \mathbb{R}_{\geq 0}$.

Given now a certain closed-loop trajectory, we define $\bar{V}(t) := V_\gamma(t-t_i, x_t, \hat{x}_t, \eta_t)$ for $t \in \mathbb{N}_{[t_i,t_{i+1}-1]}$, $i \in \mathbb{N}$. With (18) and (19), properties (i) and (ii) of $\bar{V}$ are immediately established, considering the initial conditions of the closed-loop system.

The remainder of the proof is concerned with property (iii). Let any $t \in \mathbb{N}_{[t_i,t_{i+1}-1]}$, any $i \in \mathbb{N}$, and any $x_t \in \mathbb{R}^n$, $\hat{x}_t \in \mathbb{R}^n$, $\eta_t \in \mathbb{R}_{\geq 0}$, consistent with the closed-loop system (11) be given for some $w_t \in \gamma\mathcal{W}$, $t \in \mathbb{N}$, some $x_0 \in \mathbb{R}^n$, and some $\eta_0 \in \mathbb{R}_{\geq 0}$. It holds that $|\eta|_{[0,\gamma]} = \min_{\bar{\eta}\in[0,\gamma]}|\eta - \bar{\eta}| = \max\{0, \eta-\gamma\}$ for $\gamma, \eta \in \mathbb{R}_{\geq 0}$ and further that $|\eta_{t+1}|_{[0,\gamma]} = \max\{0,\eta_{t+1}-\gamma\} \leq \lambda \max\{0,\eta_t - \gamma\} = \lambda|\eta_t|_{[0,\gamma]}$, $t \in \mathbb{N}$, for $\gamma \in \mathbb{R}_{\geq 0}$, which follows from the dynamics of η_t and from $\Phi_{\mathcal{W}}(w_t) \leq \gamma$. By the definition of the triggering conditions and (9) it holds that $x_t - \hat{x}_t = f_t$ with

$$\begin{aligned} f_t \in & \eta_t A^{-1}\bigoplus_{j=1}^{t-t_i}(A+BK)\mathcal{W} \\ \subseteq & (\gamma + \max\{0,\eta_t-\gamma\})A^{-1}\bigoplus_{j=1}^{t-t_i}(A+BK)\mathcal{W} \\ = & \gamma A^{-1}\bigoplus_{j=1}^{t-t_i}(A+BK)\mathcal{W} \oplus |\eta_t|_{[0,\gamma]}A^{-1}\bigoplus_{j=1}^{t-t_i}(A+BK)\mathcal{W} \end{aligned} \tag{20}$$

for $t \in \mathbb{N}_{[t_i,t_{i+1}-1]}$. We consider now first the case that no event occurs at time $t+1$, that is, $t+1 \in \mathbb{N}_{[t_i+1,t_{i-1}-1]}$. With $x_{t+1} - \hat{x}_{t+1} = Af_t + w_t$, compare [8], and $w_t \in \gamma\mathcal{W}$ it follows that

$$\begin{aligned} x_{t+1} - \hat{x}_{t+1} &\in \gamma\bigoplus_{j=0}^{t-t_i}(A+BK)\mathcal{W} \oplus |\eta_t|_{[0,\gamma]}\bigoplus_{j=1}^{t-t_i}(A+BK)\mathcal{W} \\ &\subseteq \gamma\bigoplus_{j=0}^{t-t_i}(A+BK)\mathcal{W} \oplus |\eta_t|_{[0,\gamma]}(A+BK)\mathcal{Y}. \end{aligned} \tag{21}$$

Hence, it holds that

$$\begin{aligned} &\frac{2|P||(A+BK)\mathcal{Y}|}{1-\lambda}|\eta_{t+1}|_{[0,\gamma]} \\ &\quad + \min_{\bar{z}_{t+1}\in\gamma\bigoplus_{j=0}^{t-t_i}(A+BK)\mathcal{W}} |P(x_{t+1}-\hat{x}_{t+1}-\bar{z}_{t+1})| \\ &\leq \lambda\frac{2|P||(A+BK)\mathcal{Y}|}{1-\lambda}|\eta_t|_{[0,\gamma]} + |P||(A+BK)\mathcal{Y}||\eta_t|_{[0,\gamma]} \\ &= \frac{(1+\lambda)|P||(A+BK)\mathcal{Y}|}{1-\lambda}|\eta_t|_{[0,\gamma]}. \end{aligned} \tag{22}$$

Consider further that $\hat{x}_{t+1} = (A+BK)\hat{x}_t$ in this first case, and let $y_t \in \gamma(A+BK)^{t-t_i}\mathcal{Y}$ such that $\min_{\bar{y}_t\in\gamma(A+BK)^{t-t_i}\mathcal{Y}}|P(\hat{x}_t - \bar{y}_t)| = |P(\hat{x}_t - y_t)|$. It follows that $(A+BK)y_t \in \gamma(A+BK)^{t+1-t_i}\mathcal{Y}$ such that

$$\begin{aligned} &\min_{\bar{y}_{t+1}\in\gamma(A+BK)^{t+1-t_i}\mathcal{Y}}|P(\hat{x}_{t+1}-\bar{y}_{t+1})| \\ &= \min_{\bar{y}_{t+1}\in\gamma(A+BK)^{t+1-t_i}\mathcal{Y}}|P((A+BK)\hat{x}_t - \bar{y}_{t+1})| \\ &\leq |P((A+BK)\hat{x}_t - (A+BK)y_t)| = |P(A+BK)(\hat{x}_t - y_t)| \\ &\leq \mu|P(\hat{x}_t - y_t)| = \mu\min_{\bar{y}_t\in\gamma(A+BK)^{t-t_i}\mathcal{Y}}|P(\hat{x}_t-\bar{y}_t)|. \end{aligned} \tag{23}$$

Considering (22) and (23), it holds that

$$\begin{aligned} &V_\gamma(t+1-t_i, x_{t+1}, \hat{x}_{t+1}, \eta_{t+1}) \\ &\qquad \leq \max\left\{\mu, \frac{1}{2}(1+\lambda)\right\}V_\gamma(t-t_i, x_t, \hat{x}_t, \eta_t) \end{aligned} \tag{24}$$

for the case that $t+1 \in \mathbb{N}_{[t_i+1,t_{i+1}-1]}$.

Consider now the case that an event is triggered at time $t+1$, that is $t+1 = t_{i+1}$. In this second case it holds that $x_{t+1} - \hat{x}_{t+1} = 0$ and, hence,

$$\begin{aligned} \hat{x}_{t+1} &= x_{t+1} = Ax_t + BK\hat{x}_t + w_t \\ &= (A+BK)\hat{x}_t + A(x_t - \hat{x}_t) + w_t = (A+BK)\hat{x}_t + Af_t + w_t, \end{aligned} \tag{25}$$

where f_t satisfies (20). It holds that $f_t = g_t + h_t$ for some $g_t, h_t \in \mathbb{R}^n$ with $g_t \in \gamma A^{-1}\bigoplus_{j=1}^{t-t_i}(A+BK)\mathcal{W}$ and $h_t \in |\eta_t|_{[0,\gamma]}A^{-1}\bigoplus_{j=1}^{t-t_i}(A+BK)\mathcal{W}$. Let $y_t \in \gamma(A+BK)^{t-t_i}\mathcal{Y}$ such that $\min_{\bar{y}_t\in\gamma(A+BK)^{t-t_i}\mathcal{Y}}|P(\hat{x}_t - \bar{y}_t)| = |P(\hat{x}_t - y_t)|$ and define $y_{t+1} := (A+BK)y_t + Ag_t + w_t$. It holds that

$$y_{t+1} \in \gamma(A+BK)^{t-t_i+1}\mathcal{Y} \oplus \gamma\bigoplus_{j=1}^{t-t_i}(A+BK)\mathcal{W} \oplus \gamma\mathcal{W} \overset{\text{Lemma 4}}{\subseteq} \gamma\mathcal{Y}. \tag{26}$$

It follows that

$$\begin{aligned} &\min_{\bar{y}_{t+1}\in\gamma\mathcal{Y}}|P(\hat{x}_{t+1}-\bar{y}_{t+1})| \\ &= \min_{\bar{y}_{t+1}\in\gamma\mathcal{Y}}|P((A+BK)\hat{x}_t + Ag_t + Ah_t + w_t - \bar{y}_{t+1})| \\ &\leq |P((A+BK)\hat{x}_t + Ag_t + Ah_t + w_t - y_{t+1})| \\ &= |P((A+BK)\hat{x}_t - (A+BK)y_t + Ah_t)| \\ &\leq |P(A+BK)(\hat{x}_t - y_t)| + |PAh_t| \\ &\leq \mu|P(\hat{x}_t - y_t)| + |P||(A+BK)\mathcal{Y}||\eta_t|_{[0,\gamma]} \\ &= \mu\min_{\bar{y}_t\in\gamma(A+BK)^{t-t_i}\mathcal{Y}}|P(\hat{x}_t-\bar{y}_t)| + |P||(A+BK)\mathcal{Y}||\eta_t|_{[0,\gamma]}. \end{aligned} \tag{27}$$

Hence, using the derivations in (22), inequality (24) also holds in the second case, such that

$$V_\gamma(t-t_i, x_t, \hat{x}_t, \eta_t) \leq \rho^t V_\gamma(0, x_0, x_0, \eta_0) \tag{28}$$

with $\rho = \max\left\{\mu, \frac{1}{2}(1+\lambda)\right\} \in [0,1)$ for the closed-loop system (11), any $t \in \mathbb{N}_{[t_i,t_{i+1}-1]}$, any $i \in \mathbb{N}$, any $\hat{x}_0 = x_0 \in \mathbb{R}^n$, any $\eta_0 \in \mathbb{R}_{\geq 0}$, and any realization of the disturbances $w_\tau \in \gamma\mathcal{W}$, $\tau \in \mathbb{N}$. With (18) and (19) it follows that

$$\left|\begin{pmatrix} x_t \\ \eta_t \end{pmatrix}\right|_{\Omega} \leq c\rho^t \left|\begin{pmatrix} x_0 \\ \eta_0 \end{pmatrix}\right|_{\Omega}, \ t \in \mathbb{N}, \tag{29}$$

with $c = \frac{a_2}{a_1}$, showing that the set Ω is exponentially stable for the closed-loop system and thereby completing the proof. ■

C. Proof of Theorem 1

We have

$$
\begin{aligned}
\left|\begin{pmatrix} x_t \\ \eta_t \end{pmatrix}\right| &\leq \left|\begin{pmatrix} x_t \\ \eta_t \end{pmatrix}\right|_\Omega + |\Omega| \overset{\text{Lemma 3}}{\leq} c\rho^t \left|\begin{pmatrix} x_0 \\ \eta_0 \end{pmatrix}\right|_\Omega + |\gamma\mathcal{Y} \times [0,\gamma]| \\
&\leq c\rho^t \left|\begin{pmatrix} x_0 \\ \eta_0 \end{pmatrix}\right| + \gamma|\mathcal{Y} \times [0,1]| \\
&= c\rho^t \left|\begin{pmatrix} x_0 \\ \eta_0 \end{pmatrix}\right| + |\mathcal{Y} \times [0,1]| \sup_{t\in\mathbb{N}} \Phi_{\mathcal{W}}(w_t) \\
&\overset{\text{Assumption 1}}{\leq} c\rho^t \left|\begin{pmatrix} x_0 \\ \eta_0 \end{pmatrix}\right| + |\mathcal{Y} \times [0,1]|\, d \sup_{t\in\mathbb{N}} |w_t|,
\end{aligned} \tag{30}
$$

showing that the closed-loop system is input-to-state stable with $\beta : (s,t) \mapsto c\rho^t s$ and $\alpha : s \mapsto |\mathcal{Y} \times [0,1]|\, ds$, and thereby completing the proof for the first part of the theorem. The second part of the proof extends the proof of convergence in [8] by including the additional variable η_t. In particular, the closed-loop system can be written as

$$x_{t+1} = (A+BK)x_t - BK\tilde{x}_t + w_t, \tag{31}$$

where $\tilde{x}_t := x_t - \hat{x}_t$. Hence, the convergence of $-BK\tilde{x}_t + w_t$ to zero for t approaching infinity is sufficient for the convergence of x_t to zero. Note that $\tilde{x}_t = 0$ or $\tilde{x}_t = A\tilde{x}_{t-1} + w_{t-1}$ for all $t \in \mathbb{N}_{\geq 1}$. From (11) it follows that $\tilde{x}_{t-1} \in \eta_{t-1}\mathcal{T}^{t-1-t_i}$ if $t-1 \in \mathbb{N}_{[t_i,t_{i+1}-1]}$ for some $i \in \mathbb{N}$. Considering that $\mathcal{T}^j \subseteq A^{-1}(\mathcal{Y} \ominus \mathcal{W}) \subseteq A^{-1}\mathcal{Y}$ for all $j \in \mathbb{N}$, it follows that

$$\tilde{x}_t \in \left(A\left(\eta_{t-1}A^{-1}\mathcal{Y}\right) \oplus \{w_{t-1}\}\right) \cup \{0\} \subseteq \eta_{t-1}\mathcal{Y} \oplus A\{w_{t-1}\} \cup \{0\} \tag{32}$$

for all $t \in \mathbb{N}$. As $\mathcal{Y}$ is bounded, it follows that $\tilde{x}_t$ converges to zero if w_t and η_t converge to zero. From (11) it is evident that η_t converges to zero if w_t converges to zero, thereby completing the proof for the second part of the theorem. ■

D. Proof of Corollary 1

The statement follows directly from equation (30). ■

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