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Robust Self-Triggered MPC for Constrained Linear Systems

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Abstract— **In this paper we propose a robust self-triggered model predictive control algorithm for linear systems with additive bounded disturbances and hard constraints on the inputs and state. In self-triggered control, at every sampling instant the time until the next sampling instant is computed online based on the current state of the system. The goal is to achieve a low average sampling rate, thereby minimizing communication in the control system and possibly reducing the number of control updates as is required in sparse control applications. Naturally, and intentionally, our approach leads to long spans of time in which the plant is controlled in an open-loop fashion. Especially for unstable plants or large disturbances this necessitates taking into account the disturbance characteristics in the design of the control law in order to prevent constraint violation in the closed-loop system. We use constraint tightening methods as proposed in Tube Model Predictive Control to guarantee robust constraint satisfaction. The self-triggered controller is shown to stabilize a robust invariant set in the state space for the closed-loop system.**

I. INTRODUCTION

Control systems that feature aperiodic sampling and/or communication may require a lot less information exchange on average than control systems with periodic sampling while preserving a desirable closed-loop behaviour, see for example [1] and the references therein. This becomes relevant if the communication in the control system takes place over a resource-constrained network, especially a wireless network, where sending and receiving information might be costly, e.g. due to the relatively high amounts of energy required for transmitting data.

Two specific types of aperiodic control have gained particular interest, that is "event-triggered control" and "self-triggered control". Please refer to [2] for an overview of these types of control strategies. In event-triggered control, the system output is usually monitored continuously or periodically. The state estimates and control inputs are updated only if certain events occur. Such an event might be, for example, the deviation between the current system output and the estimated output or set point crossing a certain threshold. In self-triggered control on the other hand, the next sampling instant is computed explicitly by the controller on the basis of the current measurement and possibly other information.

In a recent paper, a self-triggered model predictive controller was presented [3]. See also [4] for a self-triggered LQR control (including stochastic disturbances), which could be perceived as the unconstrained version of [3]. In model predictive control (MPC), an optimal control problem is solved at every sampling instant. The control input applied to the plant is the first piece of the optimizing input sequence. MPC is especially useful for control problems with hard constraints on the inputs or state, as the constraints can be taken into account explicitly in the optimization problem. For an overview of MPC the reader is referred to [5]. In the self-triggered MPC controller in [3], the time until the next sampling instant is part of the optimization problem and is to be maximized. Furthermore, constraints are added to ensure that the input to the plant is constant until the next sampling instant and that certain performance bounds are met. The control scheme requires more computations to be performed at each sampling instant than standard MPC with periodic sampling. Hence, one could say that (less) communication is traded for (more) computation, see also [6]. We are only interested in minimizing the amount of information that is exchanged and assume that the computational load is not an issue.

Besides reducing the overall communication in the control system, the mentioned self-triggered control algorithms also achieve "sparse control" that is, the control inputs do not change at every (periodic) sampling instant, thereby reducing the strain on the actuators. See for example [7] and [8] for MPC schemes achieving low variations in the input sequence through the use of the ℓ_1-norm in the formulation of the cost function. Note that, in some publications, "sparse control" refers to control schemes where the inputs are nonzero at infrequent time instances only. While we do not consider this case here, our scheme is easily adapted to this alternative objective as well (we only focus on the case when the control *increments* are nonzero infrequently).

If sparse control is not an objective, but communication is still expensive, a control strategy might be desirable where a whole sequence of control inputs or states predicted by the MPC controller is transmitted to the actuators or sensors (in one packet) at every (aperiodic) sampling instant. The actuators might then apply this sequence of inputs to the system until the next update from the controller arrives, see for example [9]–[11]. In the remainder of this paper we will refer to control strategies where information is transmitted in larger packages at fewer time instants as "packet-based" control [9]. This method might also be used to robustify a control system against communication losses, see for example [12] and [13]. The main difference between sparse control and packet-based control is that in sparse control the input to the system is constant between two sampling intervals, while in packet-based control the input is allowed to be time-varying. In the context of event-based MPC a packet-based control strategy was for example proposed in [14], where packets of information are being exchanged between the sensor and the controller. While the overall amount of information that is transferred in this kind of setup is not lower than in periodic sampling, it has been remarked in [14] with reference to [15] that in wireless communication networks sending information in few large packages might require a lot less energy than sending the same amount of information in many small packages. The same holds true if the packet size is fixed and is much larger than required for the transmission of a single input value.

Both types of control approaches, that is sparse control and

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packet-based control, will be discussed within the same framework in this paper. In fact, the MPC problems for both schemes are identical except for an additional constraint needed for sparse control that forces the first inputs in the predicted sequence to be identical.

A disadvantage that is inherent to self-triggered control is its inability to react instantly to disturbances, due to the fact that the control is strictly open-loop between sampling instants. This is particularly problematic if the plant is unstable and the sampling intervals are large. Therefore, we explicitly take disturbances into account by using robust model predictive control methods. In particular, we focus on Tube MPC [16], [17], which is a robust MPC technique well-suited to handle additive disturbances. The basic idea is to use only the nominal system for predictions and to tighten the constraints on the state and on the input based on a local controller that keeps the trajectories of the disturbed system in a robust invariant tube around the trajectories of the nominal system.

As in [3] and [4], our controller decides the time until the next sampling instant and the input sequence to the plant by solving multiple MPC problems with different lengths of the initial sampling interval. The solution to the (feasible) optimization problem with the longest sampling interval defines the control input. This setup is different to the one presented in [18], where the length of the sampling interval is directly included in the cost function of the optimization problem. The implementation of the optimization problem in [18] and in this work share the similarity that the problem is separated in an outer optimization over the interval length (which is an integer number) and an inner optimal control problem which does not contain integer variables.

The remainder of this paper is structured as follows. The problem setup is given in Section II. In Section III a Tube MPC scheme with multiple-step open-loop control is described. Section IV contains our main contribution, that is, a self-triggered Tube MPC algorithm. A numerical example is given in Section V that illustrates the approach. In Section VI computational aspects are discussed. Conclusions and an outlook are given in Section VII.

Notation: Let $\mathbb{N}_{\geq q}$ and $\mathbb{N}_{[q,s]}$ denote the sets $\{r \in \mathbb{N} \mid r \geq s\}$ and $\{r \in \mathbb{N} \mid q \leq r \leq s\}$, respectively. The set of non-negative real numbers is denoted by $\mathbb{R}_{\geq 0}$. Given sets $\mathbb{A}, \mathbb{B} \subset \mathbb{R}^n$, a scalar c and a matrix $A \in \mathbb{R}^{m \times n}$, we define $c\mathbb{A} := \{x \in \mathbb{R}^n \mid \exists a \in \mathbb{A} : x = ca\}$ and $A\mathbb{A} := \{x \in \mathbb{R}^m \mid \exists a \in \mathbb{A} : x = Aa\}$. The Minkowski set addition is denoted by $\mathbb{A} \oplus \mathbb{B} := \{x \in \mathbb{R}^n \mid \exists a \in \mathbb{A}, b \in \mathbb{B} : x = a + b\}$, the Pontryagin set difference by $\mathbb{A} \ominus \mathbb{B} := \{x \in \mathbb{R}^n \mid \forall b \in \mathbb{B} : x + b \in \mathbb{A}\}$. Further, given a vector $a \in \mathbb{R}^n$ we define $a \oplus \mathbb{A} := \{a\} \oplus \mathbb{A}$. We call a compact, convex set containing the origin a *C*-set; a *C*-set containing the origin in its (non-empty) interior is called a *proper C*-set or a *PC*-set, compare [19], [20]. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathscr{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. If additionally $\alpha(s) \to \infty$ as $s \to \infty$, α is said to belong to class $\mathscr{K}_\infty$. The euclidean norm of a vector $v \in \mathbb{R}^n$ is denoted by $|v|$. Given a compact set $\mathbb{S} \subseteq \mathbb{R}^n$, the distance between a vector $v \in \mathbb{R}^n$ and $\mathbb{S}$ is defined by $|v|_{\mathbb{S}} := \min_{s \in \mathbb{S}} |v - s|$.

II. PROBLEM SETUP AND PRELIMINARIES

We consider discrete-time, linear time-invariant systems subject to additive disturbances of the form

$$x_{t+1} = Ax_t + Bu_t + w_t \tag{1}$$

with $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$, subject to the constraints

$$\forall t \in \mathbb{N}_{\geq 0} : \; x_t \in \mathbb{X} \subset \mathbb{R}^n, \; u_t \in \mathbb{U} \subset \mathbb{R}^m. \tag{2}$$

The disturbance w_t may change at every sampling instant, is not measurable, but satisfies $w_t \in \mathbb{W} \subset \mathbb{R}^n$ for all $t \in \mathbb{N}_{\geq 0}$. We assume that the whole system state x_t is available as a measurement at any time point $t \in \mathbb{N}_{\geq 0}$.

The following assumptions apply throughout the paper.

Assumption 1: The sets $\mathbb{X}$ and $\mathbb{U}$ are *PC*-sets and $\mathbb{W}$ is a *C*-set.

Assumption 2: There exists a matrix $K \in \mathbb{R}^{m \times n}$ such that the eigenvalues of the matrix $A + BK$ are contained in the interior of the unit disc.

The goal is to asymptotically stabilize a bounded set $\mathbb{S}$ containing the origin of system (1) while satisfying (2) and keeping the communication between the controller and the plant to a minimum.

III. TUBE MPC WITH MULTIPLE-STEP OPEN-LOOP CONTROL

Reducing the frequency of control updates requires the system to run in open-loop for extended periods of time. In order to take the effects of uncertainty into account, we propose a Tube MPC scheme where the first M inputs in the prediction horizon are applied in an open-loop fashion. This requires a notion of robust invariance that is different from the one used in [17]. In particular, it leads to a larger robust invariant set, especially if the matrix A is unstable.

Definition 1: A set $\mathbb{E} \subseteq \mathbb{R}^n$ is M-step $(A, B, K, \mathbb{W})$-invariant, if for all $i \in \mathbb{N}_{[1,M]}$ it holds that

$$(A + BK)^i \mathbb{E} \oplus \bigoplus_{j=0}^{i-1} A^j \mathbb{W} \subseteq \mathbb{E}. \tag{3}$$

Remark 1: In general, we are interested in obtaining robust invariant sets *as small as possible*. The definition of the robust invariant set in [17] merely requires $(A + BK)\mathbb{E} \oplus \mathbb{W} \subseteq \mathbb{E}$ which is a requirement also implied by Definition 1. Hence, the *minimal* set satisfying Definition 1 will be a superset of the minimal set satisfying the requirement in [17].

Lemma 1: If a set is M-step $(A, B, K, \mathbb{W})$-invariant, it is also i-step $(A, B, K, \mathbb{W})$-invariant for all $i \in \mathbb{N}_{[1,M]}$.

Proof: It follows directly from the definition. ■

Lemma 2: If a set $\mathbb{E} \subseteq \mathbb{R}^n$ satisfies

$$(A + BK)\mathbb{E} \oplus \bigoplus_{j=0}^{M-1} A^j \mathbb{W} \subseteq \mathbb{E} \tag{4}$$

for matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $K \in \mathbb{R}^{m \times n}$ and a *C*-set $\mathbb{W}$, then it is M-step $(A, B, K, \mathbb{W})$-invariant.

Proof: Since by assumption $0 \in \mathbb{W}$, relation (4) implies that $(A + BK)\mathbb{E} \subseteq \mathbb{E}$. Hence, for all $i \in \mathbb{N}_{[1,M]}$ (4) implies that

$$(A + BK)^i \mathbb{E} \oplus \bigoplus_{j=0}^{i-1} A^j \mathbb{W} \subseteq (A + BK)\mathbb{E} \oplus \bigoplus_{j=0}^{M-1} A^j \mathbb{W} \subseteq \mathbb{E}, \tag{5}$$

completing the proof. ■

While possibly conservative, Lemma 2 allows the construction of M-step $(A, B, K, \mathbb{W})$-invariant sets based on methods for computing robust invariant sets, see for example [21].

Remark 2: It is possible to calculate much smaller M-step $(A, B, K, \mathbb{W})$-invariant sets based on methods similar to those presented in [22]. This matter is under research and beyond the scope of the present paper.

Define now, for a given set $\mathbb{E} \subseteq \mathbb{R}^n$, the tightened constraint sets

$$\mathbb{Z} := \mathbb{X} \ominus \mathbb{E} \text{ and } \mathbb{V} := \mathbb{U} \ominus K\mathbb{E}. \tag{6}$$

Further, let $\mathbb{Z}_\mathrm{f} \subset \mathbb{R}^n$ be a set satisfying

$$\mathbb{Z}_\mathrm{f} \subseteq \mathbb{Z}, \quad K\mathbb{Z}_\mathrm{f} \subseteq \mathbb{V}, \text{ and } (A + BK)\mathbb{Z}_\mathrm{f} \subseteq \mathbb{Z}_\mathrm{f}. \tag{7}$$

Let $\ell\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ be a convex, continuous positive definite stage cost function and $V_\mathrm{f}: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ a convex, continuous positive definite terminal cost function. The following additional assumptions apply throughout the remainder of this paper.

Assumption 3: The sets $\mathbb{Z}$, $\mathbb{V}$, and $\mathbb{Z}_\mathrm{f}$ are *PC*-sets, the set $\mathbb{E}$ is an M-step $(A,B,K,\mathbb{W})$-invariant *C*-set.

Assumption 4: There exist $\mathscr{K}_\infty$-functions α_1, α_2, α_3, such that for all $z \in \mathbb{R}^n$, $v \in \mathbb{R}^m$ it holds that $\ell(z,v) \geq \alpha_1(|z|)$ and for all $z \in \mathbb{Z}_\mathrm{f}$ it holds that $\ell(z,Kz) \leq \alpha_2(|z|)$ and $V_\mathrm{f}(z) \leq \alpha_3(|z|)$. Furthermore, ℓ and V_f are bounded on $\mathbb{X} \times \mathbb{U}$ and $\mathbb{Z}_\mathrm{f}$, respectively.

Assumption 5: For all $z \in \mathbb{Z}_\mathrm{f}$ it holds that

$$V_\mathrm{f}((A+BK)z) \leq V_\mathrm{f}(z) - \ell(z,Kz). \tag{8}$$

Define the decision variable for the Tube MPC scheme as

$$\mathbf{d}_N^M := ((z_0,\dots,z_N),(y_0,\dots,y_{M-1}),(v_0,\dots,v_{N-1}),\bar{u}) \in \mathbb{R}^{(N+1)n} \times \mathbb{R}^{Mn} \times \mathbb{R}^{Nm} \times \mathbb{R}^m := \mathbb{D}_N^M. \tag{9}$$

Given a state $x \in \mathbb{R}^n$ of system (1), define the following constraints on $\mathbf{d}_N^M$.

$$x \in z_0 \oplus \mathbb{E}, \tag{10a}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}: \ z_{i+1} = Az_i + Bv_i, \tag{10b}$$

$$\forall i \in \mathbb{N}_{[1,N-1]}: \ z_i \in \mathbb{Z}, \tag{10c}$$

$$\forall i \in \mathbb{N}_{[1,N-1]}: \ v_i \in \mathbb{V}, \tag{10d}$$

$$z_N \in \mathbb{Z}_\mathrm{f}, \tag{10e}$$

$$y_0 = x, \tag{10f}$$

$$\forall i \in \mathbb{N}_{[0,M-2]}: \ y_{i+1} = Ay_i + B(v_i + K(y_i - z_i)), \tag{10g}$$

$$v_0 + K(y_0 - z_0) \in \mathbb{U}, \tag{10h}$$

$$\forall i \in \mathbb{N}_{[0,M-1]}: \ v_i + K(y_i - z_i) = \bar{u}. \tag{10i}$$

Remark 3: The variables y_i are included mainly for improved readability. It holds that $y_i - z_i = (A+BK)^i(x - z_0)$ and the only purpose of the variables y_i is the definition of the constraints (10h) and (10i). Hence the expression above may be substituted in the optimization problem and the variables y_i removed from the vector of decision variables.

The two different variants of self-triggered control, that is sparse control and packet-based control, differ only in the property that in sparse control the control input is constant between the aperiodic sampling instants. To reflect this, define the constraint set for sparse control by

$$\bar{\mathscr{D}}_N^M(x) \ := \ \left\{ \mathbf{d}_N^M \ \in \ \mathbb{D}_N^M \ \middle| \ \text{(10a) -- (10i) hold.} \right\} \tag{11}$$

and the constraint set for packet-based control by

$$\check{\mathscr{D}}_N^M(x) \ := \ \left\{ \mathbf{d}_N^M \ \in \ \mathbb{D}_N^M \ \middle| \ \text{(10a) -- (10h) hold.} \right\}. \tag{12}$$

In both cases, the sequence of (future) inputs applied to the plant is given by $(v_0 + K(y_0 - z_0),\dots,v_{M-1} + K(y_{M-1} - z_{M-1}))$, with the difference that this sequence is constant in the sparse control case.

For most of the system-theoretic properties the differences between the two variants are irrelevant. Hence, for the sake of simple exposition, instead of distinguishing between $\bar{\mathscr{D}}_N^M(x)$ and $\check{\mathscr{D}}_N^M(x)$ we will write $\mathscr{D}_N^M(x)$, which may stand for either of the two constraint sets. Differences will be highlighted where needed. In the packet-based control approach, the sets $\check{\mathscr{D}}_N^M(x)$ are effectively identical for different values of M, as the variables y_i are not used in this case.

Remark 4: Note that the input constraints in (10) are not tightened for $i = 0$. The reason for this is that this first input is known exactly. Similarly, there is no explicit state constraint on z_0.

Define the cost function for the Tube MPC scheme as

$$J_N(\mathbf{d}_N^M) = \sum_{i=0}^{N-1} \ell(z_i,v_i) + V_\mathrm{f}(z_N). \tag{13}$$

Finally, for a given state $x \in \mathbb{R}^n$ of system (1) and $M \in \mathbb{N}_{[1,N-1]}$, the Tube MPC problem $\mathscr{P}_N^M(x)$ is defined as

$$V_N^M(x) := \min_{\mathbf{d}_N^M \in \mathscr{D}_N^M(x)} J_N(\mathbf{d}_N^M) \tag{14a}$$

$$\hat{\mathbf{d}}_N^M(x) := \operatorname*{argmin}_{\mathbf{d}_N^M \in \mathscr{D}_N^M(x)} J_N(\mathbf{d}_N^M) \tag{14b}$$

with

$$\hat{\mathbf{d}}_N^M(x) = ((\hat{z}_0(x),\dots,\hat{z}_N(x)),(\hat{y}_0(x),\dots,\hat{y}_{M-1}(x)), (\hat{v}_0(x),\dots,\hat{v}_{N-1}(x)),\hat{u}(x)). \tag{15}$$

The following lemma states that the application of an open-loop input sequence that is feasible for the problem $\mathscr{P}_N^M(x)$ guarantees satisfaction of the state and input constraints for the next M time instances.

Lemma 3: Let any $x_0 \in \mathbb{X}$ for which $\mathscr{D}_N^M(x_0) \neq \emptyset$ be given. Let further $\mathbf{d}_N^M := ((z_0,\dots,z_N),(y_0,\dots,y_{M-1}),\ (v_0,\dots,v_{N-1}),\bar{u}) \in \mathscr{D}_N^M(x_0)$ and let for all $i \in \mathbb{N}_{[0,M-1]}: x_{i+1} = Ax_i + B(v_i + K(y_i - z_i)) + w_i$, where $w_i \in \mathbb{W}$. Then for all $i \in \mathbb{N}_{[0,M]}$ it holds that $x_i \in z_i \oplus \mathbb{E}$ and $x_i \in \mathbb{X}$. Furthermore, for all $i \in \mathbb{N}_{[0,M-1]}$ it holds that $v_i + K(y_i - z_i) \in \mathbb{U}$.

Proof: First, it is proven that $x_i \in z_i \oplus \mathbb{E}$. Define $y_M := Ay_{M-1} + B(v_{M-1} + K(y_{M-1} - z_{M-1}))$ and $f_i := y_i - z_i$ for all $i \in \mathbb{N}_{[0,M]}$. With $y_{i+1} = Ay_i + B(v_i + K(y_i - z_i))$, $z_{i+1} = Az_i + Bv_i$ and $y_0 = x_0 \in z_0 \oplus \mathbb{E}$ it holds that $f_{i+1} = (A+BK)f_i$ and $f_0 \in \mathbb{E}$, such that for all $i \in \mathbb{N}_{[0,M]}$ it holds that $f_i \in (A+BK)^i\mathbb{E}$. We also define $e_i := x_i - y_i$ for all $i \in \mathbb{N}_{[0,M]}$. It holds that $x_0 = y_0$ and $x_{i+1} = Ax_i + B(v_i + K(y_i - z_i)) + w_i$ for all $i \in \mathbb{N}_{[0,M-1]}$. It follows that $e_{i+1} = Ae_i + w_i$ and $e_0 = 0$, such that for all $i \in \mathbb{N}_{[0,M]}$ it holds that $e_i \in \bigoplus_{j=0}^{i-1} A^j\mathbb{W}$.

Consider now that $x_i = z_i + f_i + e_i$ for all $i \in \mathbb{N}_{[0,M]}$. It follows that for all $i \in \mathbb{N}_{[0,M]}$

$$x_i \in z_i \oplus (A+BK)^i\mathbb{E} \oplus \bigoplus_{j=0}^{i-1} A^j\mathbb{W}. \tag{16}$$

As $\mathbb{E}$ is an M-step $(A,B,K,\mathbb{W})$-invariant set, this implies that for all $i \in \mathbb{N}_{[0,M]}$ it holds that $x_i \in z_i \oplus \mathbb{E}$. The statement that $x_i \in \mathbb{X}$ follows from this and the constraint $z_i \in \mathbb{Z} = \mathbb{X} \ominus \mathbb{E}$. Considering the input constraints, for $i = 0$ the statement follows directly from constraint (10h). For $i \in \mathbb{N}_{[1,M-1]}$, observe that $y_i - z_i = f_i \in (A+BK)^i\mathbb{E} \subseteq \mathbb{E}$ and, hence, the tightened constraint $v_i \in \mathbb{V} = \mathbb{U} \ominus K\mathbb{E}$ implies $v_i + K(y_i - z_i) \in \mathbb{U}$, thereby completing the proof. ■

IV. SELF-TRIGGERED TUBE MPC

In this section we define a self-triggered robust model predictive control scheme along the same lines as in [3]. The main idea is to maximize the time M until the next sampling instant while guaranteeing that the cost is not much higher than when updating the input at the very next sampling instant. Given an $M_{\max} \in \mathbb{N}_{[1,N-1]}$, let $\mathbb{E}$ be an $M_{\max}$-step $(A,B,K,\mathbb{W})$-invariant set. Given a state x of system (1), for a $c \geq 0$ define the self-triggered MPC problem $\mathscr{P}_N^{\mathrm{st}}(x)$ as

$$\hat{M}(x) := \max\left\{ M \in \mathbb{N}_{[1,M_{\max}]} \ \middle| \ \mathscr{D}_N^M(x) \neq \emptyset,\ V_N^M(x) \leq V_N^1(x) + c \right\} \tag{17a}$$

$$\hat{\mathbf{d}}_N^{\hat{M}}(x) := \operatorname*{argmin}_{\mathbf{d}_N^{\hat{M}} \in \mathscr{D}_N^{\hat{M}(x)}(x)} J_N(\mathbf{d}_N^{\hat{M}}) \tag{17b}$$

with

$$\hat{\mathbf{d}}_N^{\hat{M}}(x) = ((\hat{z}_0(x),\ldots,\hat{z}_N(x)),(\hat{y}_0(x),\ldots,\hat{y}_{\hat{M}-1}(x)), (\hat{v}_0(x),\ldots,\hat{v}_{N-1}(x)),\hat{u}(x)). \quad (18)$$

Note that $\mathscr{D}_N^M(x)\neq\emptyset$ implies $\mathscr{D}_N^1(x)\neq\emptyset$ for any $M\geq 0$.

Remark 5: The scalar c in $\mathscr{P}_N^{\mathrm{st}}(x)$ is a tuning parameter. In the sparse control case, the set $\mathscr{D}_N^M$ contains additional constraints when compared to $\mathscr{D}_N^1$ if $M>1$. Hence, it is to be expected, that in general $V_N^M(x)>V_N^1(x)$. In order to still achieve $\hat{M}(x)>1$ for some x when aiming for sparse control, the parameter c is included. In the packet-based approach, it is possible to set $c=0$, as here $V_N^M(x)=V_N^1(x)$ for any value of M. This implies that in the packet-based approach it always turns out that $\hat{M}=M_{\max}$. Because c enters additively, it has more influence when the current system state is close to the origin, where the MPC cost function attains smaller values. Compare [23], where similar observations have been made in the context of event-triggered control.

This leads to the following control algorithm.

Algorithm 1 Self-Triggered Tube MPC

1: at time point t, obtain current state x_t of system (1)
2: solve problem $\mathscr{P}_N^{\mathrm{st}}(x_t)$, obtain $\hat{M}(x_t)$ and $\hat{\mathbf{d}}_N^{\hat{M}}(x_t)$
3: for $i\in\mathbb{N}_{[0,\hat{M}(x_t)-1]}$ and time points $t+i$ apply $u_{t+i}=\hat{v}_i(x_t)+K(\hat{y}_i(x_t)-\hat{z}_i(x_t))$ to the system
4: at time point $t+\hat{M}(x_t)$, set $t=t+\hat{M}(x_t)$
5: go to 1.

The set of states where Algorithm 1 is feasible is $\hat{\mathbb{X}}_N := \{x\in\mathbb{X}\mid \mathscr{D}_N^1(x)\neq\emptyset\}$. The closed-loop system resulting from the application of Algorithm 1 is

$$x_{t+1} = Ax_t + B(\hat{v}_{t-t_i}(x_{t_i}) + K(\hat{y}_{t-t_i}(x_{t_i}) - \hat{z}_{t-t_i}(x_{t_i}))) + w_t \quad \text{if } t\in\mathbb{N}_{[t_i,t_{i+1}-1]} \quad (19a)$$

$$t_{i+1} = t_i + \hat{M}(x_{t_i}), \quad (19b)$$

for $i\in\mathbb{N}$ with $t_0=0$ and $x_0\in\hat{\mathbb{X}}_N$.

Remark 6: The input applied to the system in the time-interval $\mathbb{N}_{[t_i,t_{i+1}-1]}$ is a function of x_{t_i} only. Therefore, the whole sequence $(\hat{v}_0(x_{t_i})+K(\hat{y}_0(x_{t_i})-\hat{z}_0(x_{t_i})),\ldots,\hat{v}_{\hat{M}(x_{t_i})-1}(x_{t_i})+K(\hat{y}_{\hat{M}(x_{t_i})-1}(x_{t_i})-\hat{z}_{\hat{M}(x_{t_i})-1}(x_{t_i})))$ may be transmitted to the actuator at time t_i. At the other points in $\mathbb{N}_{[t_i,t_{i+1}-1]}$ no communication and no (re)calculation of the control law take place, such that the communication network and the sensors might be shut off until t_{i+1}. Note that if the sparse control version of the algorithm is implemented, that is, constraint (10i) is included in the optimization problem, the input is constant in the interval $\mathbb{N}_{[t_i,t_{i+1}-1]}$.

In the following, *recursive feasibility* of Algorithm 1 and the satisfaction of constraints (2) by system (19) are established. Recursive feasibility means that if there exists a solution to the MPC problem for a given point in time and a given state in the state space, then it is guaranteed that a solution to the MPC problem exists for all future points in time and all future states resulting from the application of the control law returned by the solution of the MPC problem.

Lemma 4: Let $\hat{M}$ and $\hat{\mathbf{d}}_N^{\hat{M}} = ((\hat{z}_0,\ldots,\hat{z}_N),(\hat{y}_0,\ldots,\hat{y}_{\hat{M}-1}),(\hat{v}_0,\ldots,\hat{v}_{N-1}),\hat{u})$ be a solution of $\mathscr{P}_N^{\mathrm{st}}(x_t)$ and let for all $j\in\mathbb{N}_{[0,\hat{M}-1]}$: $x_{t+j+1} = Ax_{t+j}+B(\hat{v}_j+K(\hat{y}_j-\hat{z}_j))+w_{t+j}$, where $w_{t+j}\in\mathbb{W}$. Then for all $j\in\mathbb{N}_{[1,\hat{M}]}$ it holds that $\mathscr{D}_N^1(x_{t+j})\neq\emptyset$ and $x_{t+j}\in\mathbb{X}$, that is, $x_{t+j}\in\hat{\mathbb{X}}_N$.

Proof: Consider

$$\tilde{\mathbf{d}}_N^{1,j} := \big((\hat{z}_j,\ldots,\hat{z}_N,(A+BK)\hat{z}_N,\ldots,(A+BK)^j\hat{z}_N),(x_{t+j}), (\hat{v}_j,\ldots,\hat{v}_{N-1},K(A+BK)^0\hat{z}_N,\ldots,K(A+BK)^{j-1}\hat{z}_N), (\hat{v}_j+K(x_{t+j}-\hat{z}_j))\big) \quad (20)$$

as a candidate solution to $\mathscr{P}_N^{\mathrm{st}}(x_{t+j})$. From (7) and Lemma 3 it follows that $\tilde{\mathbf{d}}_N^{1,j}\in\mathscr{D}_N^1(x_{t+j})$ and $x_{t+j}\in\mathbb{X}$ for $j\in\mathbb{N}_{[1,\hat{M}]}$, which completes the proof. ■

Theorem 1: If $x_0\in\hat{\mathbb{X}}_N$, then for the closed-loop system (19), all $t\in\mathbb{N}$, and all $j\in\mathbb{N}_{[0,\hat{M}(x_t)-1]}$ it holds that $\hat{v}_j(x_t)+K(\hat{y}_j(x_t)-\hat{z}_j(x_t))\in\mathbb{U}$ and for all $t\in\mathbb{N}$ it holds that $x_t\in\mathbb{X}$.

Proof: By Lemma 4 the closed-loop system (19) is well defined. The statement follows from Lemma 3. ■

Before we state our main stability theorem, we need some auxiliary results.

Lemma 5: There exists a $\mathscr{K}_\infty$-function α_4 such that for all $x\in\hat{\mathbb{X}}_N$ it holds that

$$\alpha_1(|x|_{\mathbb{E}})\leq V_N^1(x)\leq\alpha_4(|x|_{\mathbb{E}}). \quad (21)$$

Furthermore, for the closed-loop system (19) with $x_0\in\hat{\mathbb{X}}_N$, for any $i\in\mathbb{N}$ and any $j\in\mathbb{N}_{[1,\hat{M}(x_{t_i})]}$ it holds that

$$V_N^1(x_{t_i+j})\leq V_N^1(x_{t_i})-\alpha_1(|x_{t_i}|_{\mathbb{E}})+c. \quad (22)$$

Due to lack of space, the proof of this lemma is omitted here. The stability properties of the closed-loop system are summarized in the following theorem.

Theorem 2: For every $c\geq 0$ in (17a), such that the set

$$\mathbb{S}_c := \{x\in\hat{\mathbb{X}}_N \mid V_N^1(x)\leq\alpha_4(\alpha_1^{-1}(c))+c\} \quad (23)$$

is contained in the interior of the feasible set $\hat{\mathbb{X}}_N=\{x\in\mathbb{X}\mid\mathscr{D}_N^1(x)\neq\emptyset\}$, the set $\mathbb{S}_c$ is asymptotically stable for the closed-loop system (19) with a region of attraction $\hat{\mathbb{X}}_N$ under the control resulting from the application of Algorithm 1. Furthermore, it holds that $\mathbb{S}_c\subseteq\{z\in\mathbb{R}^n\mid |z|\leq\alpha_1^{-1}(\alpha_4(\alpha_1^{-1}(c))+c)\}\oplus\mathbb{E}$

Proof: It holds that J_N in (13) is a convex function and the set $\{(x,\mathbf{d}_N^1)\in\mathbb{X}\times\mathbb{D}_N^1\mid\mathbf{d}_N^1\in\mathscr{D}_N^1(x)\}$ is convex. Hence, V_N^1 is convex on $\hat{\mathbb{X}}_N$. By [24] this implies that V_N^1 is uniformly continuous on every compact set contained in the interior of $\hat{\mathbb{X}}_N$. Then, the first part of the theorem can be derived from Lemma 5.

For the second part, consider that $V_N^1(x)\leq\alpha_4(\alpha_1^{-1}(c))+c$ implies by Lemma 5 that $|x|_{\mathbb{E}}\leq\alpha_1^{-1}(\alpha_4(\alpha_1^{-1}(c))+c)$ which leads to the second part of the assertion, thereby completing the proof. ■

Remark 7: It holds that $\mathbb{Z}_{\mathrm{f}}\oplus\mathbb{E}\subseteq\hat{\mathbb{X}}_N$, such that by Assumption 3 the interior of $\hat{\mathbb{X}}_N$ is not empty. Furthermore, it holds that $x\in\mathbb{E}\Leftrightarrow V_N^1(x)=0$, such that there always exist $c\geq 0$, such that $\mathbb{S}_c$ is contained in the interior of $\hat{\mathbb{X}}_N$. In fact, if $c=0$ in (17a), the set $\mathbb{E}$ is stabilized, which is similar to the result obtained by standard Tube MPC [17]. Note, however, that due to the different notions of invariance, the set $\mathbb{E}$ in this paper will in general be larger than the one in standard Tube MPC. There exists an obvious trade-off between the size of the set $\mathbb{S}_c$ that is stabilized and the average length of the sampling intervals. A larger average sampling interval requires a larger $M_{\max}$, leading to a larger set $\mathbb{E}$. Furthermore, if the sparse control version of the MPC scheme is applied, a possibly large value of c is required to achieve $M>1$ in a reasonably large region of the state space, which increases the size of $\{z\in\mathbb{R}^n\mid |z|\leq\alpha_1^{-1}(\alpha_4(\alpha_1^{-1}(c))+c)\}$, see also Remark 5.

V. NUMERICAL EXAMPLE

Consider the disturbed double integrator

$$x_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_t + w_t, \tag{24}$$

subject to the constraints $x_t \in [-20,20] \times [-8,8]$, $u_t \in [-8,8]$, and $w_t \in [-0.25,0.25] \times [0.25,0.25]$. The cost functions have been chosen as

$$\ell(z,v) = z^\mathsf{T} Qz + v^\mathsf{T} Rv, \quad V_\mathrm{f}(z) = z^\mathsf{T} Pz, \tag{25}$$

where $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $R = 0.1$. The matrix P is the stabilizing and optimal solution to the discrete-time algebraic Riccati equation with the weighting matrices Q and R. The resulting optimal feedback matrix is K. The prediction horizon was chosen to $N = 20$, the maximal open-loop horizon to $M_\mathrm{max} = 5$. The set $\mathbb{E}$ has been computed based on Lemma 2 and the algorithm for the computation of robust invariant sets in [21], where the parameter ε has been chosen to $\varepsilon = 2$. The set $\mathbb{E}$ is depicted in Fig. 1. The terminal set has been calculated as the maximal output admissible set [25] for the closed-loop system $x_{t+1} = (A + BK)x_t$.

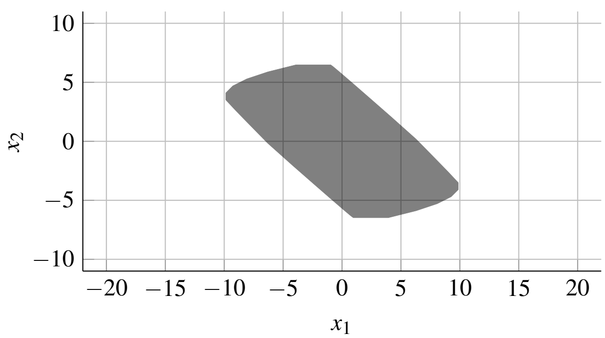

Fig. 1. The M_max-step $(A,B,K,\mathbb{W})$-invariant set $\mathbb{E}$.

The sparse control approach has been chosen for the simulations. For the initial condition $x_0 = (10,6)^\mathsf{T}$, a constant disturbance $w_t = (-0.25,-0.25)^\mathsf{T}$, and $c = 50$ the closed loop has been simulated for 50 steps. For comparison, the same system has been simulated without tightened constraints, that is for $\mathbb{E} = \emptyset$. The terminal set has also been calculated without constraint tightening for the comparison simulation. Trajectories resulting from both simulations are shown in Fig. 2, together with an inner approximation of the feasible region $\hat{\mathbb{X}}_{20}$. The inner approximation has been obtained by calculating $\hat{\mathbb{Z}}_{20} \oplus \mathbb{E}$, where $\hat{\mathbb{Z}}_{20}$ is the set of all $x \in \mathbb{R}^2$ such that $\tilde{\mathscr{D}}^1_{20}(x) \neq \emptyset$, with

$$\tilde{\mathscr{D}}^M_N(x) := \Big\{ \mathbf{d}^M_N \in \mathbb{D}^M_N \;\Big|\; x = z_0,\ \text{(10b)} - \text{(10i) hold}, \\ z_0 \in \mathbb{Z},\ \text{and}\ v_0 \in \mathbb{V} \Big\}. \tag{26}$$

Note that the trajectory resulting from the MPC scheme without constraint tightening violates the state constraints for $t = 2$. Interestingly, the MPC scheme with tightened constraints yielded $\hat{M} = 1$ at $t = 0$, while the MPC scheme without tightened constraints yielded $\hat{M} = 2$. In fact, the MPC scheme with tightened constraints was only feasible for $M = 1$, while the MPC scheme with original constraints was feasible for all $M \in \mathbb{N}_{[1,5]}$. Also note the apparent limit cycle behavior resulting from the constant disturbance acting on the system.

The inputs and the triggering behaviour of the closed-loop systems are shown in Fig. 3. The average time between sampling

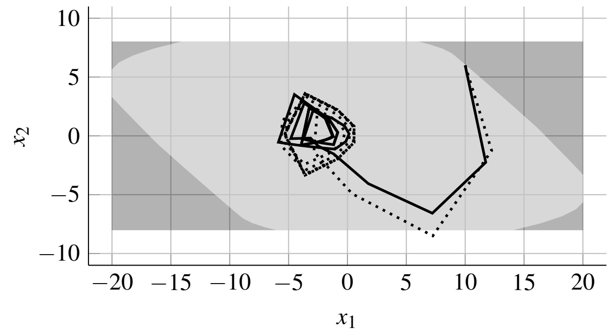

Fig. 2. Trajectories of the closed-loop system for MPC schemes with tightened constraints (solid) and without tightened constraints (dotted). The state constraint set is depicted in dark gray, an inner approximation of the region of attraction $\hat{\mathbb{X}}_{20}$ in light gray.

instants was 2.63 for the MPC scheme with tightened constraints and 3.33 for the scheme without tightened constraints.

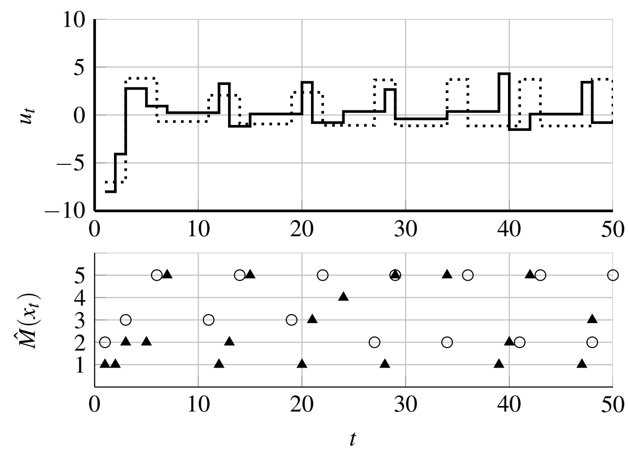

Fig. 3. Inputs u_t and inter-sampling times $\hat{M}$ of the closed-loop system for MPC schemes with tightened constraints (solid, triangles) and without tightened constraints (dotted, circles).

VI. COMPUTATIONAL COMPLEXITY

In the worst case, the algorithm requires the solution of M_max optimization problems every time the inputs to the system are updated. The complexity of the individual optimization problems is roughly the same as for a nominal MPC scheme where uncertainties are not taken into account, the main difference being inequality (10a) whose complexity is determined by the set $\mathbb{E}$. The tightened constraints are, when described by linear inequalities, not more complicated than the original ones, independent of the complexity of $\mathbb{E}$.

Note that the motivation for this paper and the main benefit of the proposed scheme is the reduction of communication in the control system. The price to be paid for this reduction is an increase of necessary computations at each sampling instant.

Usually, it is the worst case computational load that is of interest when implementing the controller, as the hardware must be capable of handling the worst case in a specified amount of time. However, computations on a digital computer require energy and hence the *average* computational load might also be relevant. As the inputs are not updated at every time-step, this average might be approximately the same as when updating the input at every time step, see also the discussion in [3] on this matter. Further, it might be possible to achieve an average computational load that is even lower than when updating at every point in time if $\hat{M}$ in (17a) is for example calculated by bisection.

VII. CONCLUSIONS AND OUTLOOK

In this paper a robust self-triggered MPC scheme was presented. By combining ideas in self-triggered LQR and MPC strategies as in [4] and, respectively, [3], and the methods of Tube MPC in [17], satisfaction of state and input constraints can be guaranteed despite the influence of additive disturbances on the system, while simultaneously achieving a low frequency of control updates and hence communication.

Interestingly, the framework presented in this paper allows the implementation of both sparse control and packet-based control algorithms, thereby showing the application potential of the proposed ideas. In the algorithms, the difference is including additional constraints in the case of sparse control.

Future research will address the approximation of the *minimal M*-step $(A,B,K,\mathbb{W})$-invariant set $\mathbb{E}$. Additionally, robust self-triggered MPC schemes might benefit from a time-varying set capturing the influence of the disturbance, for example along the lines of [26]. In a self-triggered scheme, the predicted size of this set would grow during the open-loop phase and shrink again when feedback is present, automatically adapting the tightening of the constraints to different values of $\hat{M}$.

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