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γ-Invasive Event-triggered and Self-triggered Control for Perturbed Linear Systems

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Abstract— **In this paper, aperiodic control of linear discrete-time systems subject to bounded additive disturbances is considered. We present a framework for event- and self-triggered controllers that achieve the same worst-case bounds on the system state (up to a predefined factor, which is a design parameter) as a given linear controller that is updated at every point in time. We demonstrate in an example that the proposed event- and self-triggered controllers achieve a considerable reduction in the communication rate even without increasing the guaranteed worst-case error bounds.**

I. INTRODUCTION

We consider aperiodic control schemes where the times at which the controller is updated with information from the sensors are dependent on the evolution of the system state. These types of control schemes have been found to provide a better trade-off between the closed-loop performance and the required average sampling rate than periodic control schemes with a fixed sampling rate [1], [2]. Especially for networked control systems where communication is costly in terms of energy, or takes place over a shared, bandwidth-limited network, limiting the average sampling rate (while maintaining a certain level of performance) is an important design objective. In particular, we focus on so-called event- and self-triggered controllers for a networked control system as depicted in Figure 1. In event-triggered control, the state of the system is measured periodically or continuously and an event (transmission) instant is generated if the current system state meets certain well-designed trigger conditions. In self-triggered control, the state of the system at the time of a sampling (transmission) instant is used to explicitly compute the next sampling (transmission) instant. This setup has the benefit of allowing the sensors of the system to be shut off between transmission instants, possibly saving additional energy. On the other hand, event-triggered control generally provides better disturbance attenuation, as the sensors are able to detect a disturbance at the same time it acts on the system. These and other aspects of event- and self triggered control are discussed in [3]–[5], which also give an overview of past and recent developments in the field.

In this paper we focus on linear plants subject to bounded additive disturbances. Under a nominally stabilizing periodically sampled linear controller, a compact subset of the state space can be shown to be stabilized, providing a worst-case asymptotic bound for the system state. Moreover, the system state evolves around a nominal trajectory with a precomputable error bound, see for example [6], [7], and the references therein. Such bounds are crucial in the controller design if hard constraints on the state and input are present. Here, we investigate conditions under which an aperiodic controller guarantees the same error bounds as given for a particular periodically updated controller, scaled by a factor $\gamma \geq 0$. We call such controllers γ-*invasive* with respect to a given periodically sampled controller. Aperiodic controllers with similarly guaranteed performance (with varying performance measures) were for example proposed in [5], [8], see also the references therein. In comparison with the present paper, [5] considers systems without disturbances while [8] considers stochastic (and possibly unbounded) disturbances. Surprisingly, it is possible to achieve a considerable reduction of the average sampling rate even for $\gamma = 0$, that is, without increasing the worst-case bounds, see [9], [10]. We provide a simple necessary and sufficient condition for controllers to be γ-invasive and propose event- and self-triggered controllers based on this condition. Our framework is of general nature and does not impose restrictions on the control input between transmission instants. For example, controllers that provide zero inputs between transmission instants, controllers with constant input, and controllers with model-based inputs along the lines of [11]–[13] can be employed, and in fact can be treated in the same manner.

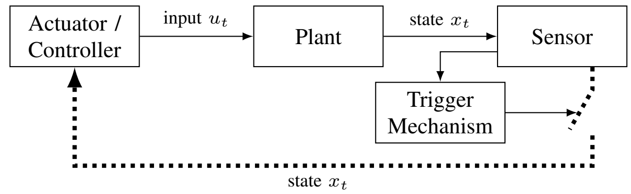

Fig. 1. Aperiodic control structure. Communication takes place between the sensor and the actuator/controller.

The remainder of the paper is structured in the following way. Section I closes with some remarks on notation. The problem setup is presented in Section II. In Section III we present general results on γ-invasive control. Event- and self-triggered controllers based on these results are proposed in Section IV. Comments on implementation are given in Section V. In Section VI we present a numerical example illustrating the efficacy of our approach. Several aspects of the scheme are discussed in Section VII and Section VIII concludes the paper.

Notation: The field of real numbers is denoted by $\mathbb{R}$. The set of nonnegative integers is denoted by $\mathbb{N}$. We define $\mathbb{N}_{\geq a} := \mathbb{N} \cap [a, \infty)$ for $a \in \mathbb{R}$. For $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{\infty\}$ we define $\mathbb{N}_{[a,b]} := \mathbb{N} \cap [a, b]$ if $b \in \mathbb{R}$ with $[a, b] := \emptyset$ for $a > b$ and $\mathbb{N}_{[a,b]} := \mathbb{N}_{\geq a}$ if $b = \infty$. For $x \in \mathbb{R}^n$, $|x|$ denotes the maximum norm. For $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$,

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$\alpha \in \mathbb{R}$, $M \in \mathbb{R}^{m\times n}$, and $x \in \mathbb{R}^n$ we define $\mathcal{A} \oplus \mathcal{B} := \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$, $\mathcal{A} \ominus \mathcal{B} := \{c \in \mathbb{R}^n \mid \{c\} \oplus \mathcal{B} \subseteq \mathcal{A}\}$, $\alpha\mathcal{A} := \{\alpha a \mid a \in \mathcal{A}\}$, $M\mathcal{A} := \{Ma \mid a \in \mathcal{A}\}$, and $|x|_{\mathcal{A}} := \inf_{a\in\mathcal{A}} |x-a|$. For sets $\mathcal{C}_i \subseteq \mathbb{R}^n$ with $i \in \mathbb{N}$ we define $\bigoplus_{i=a}^{b} \mathcal{C}_i := \left\{\sum_{i=a}^{b} c_i \mid c_i \in \mathcal{C}_i,\ i \in \mathbb{N}_{[a,b]}\right\}$, for any $a, b \in \mathbb{N} \cup \{\infty\}$, where the empty sum is zero by convention.

II. Problem Setup and Preliminaries

We consider linear discrete-time systems of the form

$$x_{t+1} = Ax_t + Bu_t + w_t, \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the input, and $w_t \in \mathcal{W} \subseteq \mathbb{R}^n$ is the disturbance, at time $t \in \mathbb{N}$. We assume that the disturbance w_t is unknown a priori, but that $\mathcal{W}$ is a known compact and convex set containing the origin. The state x_t is available as a measurement at any time $t \in \mathbb{N}$. Further, we assume that there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is Schur stable (eigenvalues in the interior of the unit disc).

Remark 1: If $\mathcal{W}$ does not contain the origin, it is always possible to transform the system variables into coordinates where this assumption holds.

Define

$$\mathcal{F}_t := \bigoplus_{j=0}^{t-1} (A + BK)^j \mathcal{W} \tag{2}$$

for all $t \in \mathbb{N} \cup \{\infty\}$. We call the controller defined by $u_t = Kx_t$, $t \in \mathbb{N}$, the "all-time triggered controller", as it requires transmissions from the sensor to the actuator at every point in time. If $u_t = Kx_t$ for all $t \in \mathbb{N}$, then for the system in (1) it holds that

$$x_t \in \{(A + BK)^t x_0\} \oplus \mathcal{F}_t \tag{3}$$

for any $x_0 \in \mathbb{R}^n$ and any realization of the disturbance sequence satisfying $w_t \in \mathcal{W}$ for all $t \in \mathbb{N}$. Furthermore, for this closed-loop system given by $x_{t+1} = (A + BK)x_t + w_t$, the set $\Omega := \mathcal{F}_\infty$ is an asymptotically stable compact set.

Lemma 1: It holds that $(A + BK)^t\Omega \oplus \mathcal{F}_t = \Omega$ for all $t \in \mathbb{N}$.

The statement follows directly from the definitions.

The results above are well known in the context of reachability of linear systems, see for example [6], [7], and the references therein.

Definition 1: Let $\gamma \in [0, \infty)$. A controller generating an input sequence $(u_t)_{t\in\mathbb{N}}$ for system (1) is called γ*-invasive* with respect to the all-time triggered controller defined by $u_t = Kx_t$, $t \in \mathbb{N}$, if

$$x_t \in \{(A + BK)^t x_0\} \oplus (\gamma + 1)\mathcal{F}_t \tag{4}$$

for all $x_0 \in \mathbb{R}^n$ and all $t \in \mathbb{N}$.

Our goal is to design γ-invasive controllers in a way such that the communication between the sensor and the actuator/controller is minimized. First, we state a result concerning the stability properties of a system under γ-invasive control.

Lemma 2: Let system (1) be controlled by a controller that is γ-invasive with respect to the controller defined by $u_t = Kx_t$ for $\gamma \in [0, \infty)$. Then, the set $(\gamma + 1)\Omega$ is asymptotically stable for the closed-loop system.

The proof is given in the appendix.

III. General Results

In this section, we investigate γ-invasive event- and self-triggered controllers. In particular, we present necessary and sufficient conditions for a certain class of event- and self-triggered controllers to be γ-invasive.

Considering event- or self-triggered control, let the time instants at which communication takes place, called *transmission instants* henceforth, be collected in the set

$$\mathcal{E} := \{t_i \mid i \in \mathbb{N}_{[0,i_{\max}]}\} \subseteq \mathbb{N}, \tag{5}$$

where $i_{\max} \in \mathbb{N} \cup \{\infty\}$ and for all $t_i, t_j \in \mathcal{E}$ it holds that $j > i \Leftrightarrow t_j > t_i$. We assume that $t_0 = 0$. For the case that $i_{\max}$ is finite, we define, with slight abuse of notation, $\mathbb{N}_{[t_{i_{\max}}, t_{i_{\max}+1}-1]} := \mathbb{N}_{\geq t_{i_{\max}}}$. We assume $u_t = \kappa(t - t_i, x_{t_i})$ for $t \in \mathbb{N}_{[t_i, t_{i+1}-1]}$ for a mapping $\kappa : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^m$. That is, between transmission instants the input may only depend on the system state at the last transmission instant. This makes transmission of information between the sensor and the actuator unnecessary at times between transmission instants, thereby reducing the overall amount of communication in the system. We place a restriction on the controller κ, namely that for all $x \in \mathbb{R}^n$ it holds that

$$\kappa(0, x) = Kx. \tag{6}$$

Particular cases falling into this framework are *sparse controllers*, where

$$\kappa(\tau, x) = \begin{cases} Kx, & \tau = 0,\ x \in \mathbb{R}^n \\ 0, & \text{else,} \end{cases} \tag{7a}$$

zero-order-hold controllers

$$\kappa(\tau, x) = Kx, \quad \tau \in \mathbb{N},\ x \in \mathbb{R}^n, \tag{7b}$$

and *model-based controllers* [11]–[13]

$$\kappa(\tau, x) = K(A + BK)^\tau x, \quad \tau \in \mathbb{N},\ x \in \mathbb{R}^n, \tag{7c}$$

see also [3], [5], [14]. The distinction between event- and self-triggered controllers is made only by how the transmission instants are generated. In particular, for event-triggered controllers and a given point in time $t \in \mathbb{N}$, the information about all states x_τ for $\tau \in \mathbb{N}_{[0,t]}$ may be used to determine whether t is a transmission instant or not. On the other hand, in self-triggered control, given a transmission instant t_i, the next transmission instant t_{i+1} is determined based only on the information about the states x_{t_j} for $j \in \mathbb{N}_{[0,i]}$. Hence, the transmission instant t_{i+1} can already be determined at time t_i.

Consider now system (1) in the equivalent form

$$x_{t+1} = (A + BK)x_t + B(u_t - Kx_t) + w_t \tag{8}$$

such that

$$x_t = (A + BK)^t x_0 + z_t, \tag{9}$$

where

$$z_t = \sum_{j=0}^{t-1} (A + BK)^{t-1-j}(B(u_j - Kx_j) + w_j) \tag{10}$$

with $z_{t+1} = (A + BK)z_t + B(u_t - Kx_t) + w_t$ for all $t \in \mathbb{N}$ and $z_0 = 0$. Note that the information necessary to compute $z_t = x_t - (A + BK)^t x_0$ is available to the trigger mechanism at time point t if required.

In the following, we state an immediate result on which the controllers proposed in this paper are based.

Lemma 3: A controller generating an input sequence $(u_t)_{t\in\mathbb{N}}$ for system (1) is γ-invasive with respect to the all-time triggered controller defined by $u_t = Kx_t$, $t \in \mathbb{N}$, for a $\gamma \in [0, \infty)$ if and only if $z_t \in (\gamma + 1)\mathcal{F}_t$ for all $t \in \mathbb{N}$, where z_t is defined by (9)-(10).

The statement follows directly from the definitions.

IV. γ-INVASIVE CONTROLLERS

In this section, we propose event- and self-triggered controllers based on Lemma 3. That is, the trigger conditions will be designed such that $z_t \in (\gamma+1)\mathcal{F}_t$ is guaranteed for all $t \in \mathbb{N}$.

A. γ-invasive event-triggered control

We propose an event-triggered controller that checks at every point in time $t \in \mathbb{N}$ whether the condition $z_{t+1} \in (\gamma+1)\mathcal{F}_{t+1}$ is guaranteed. Note that $z_{t+1} \in \{(A+BK)z_t + \{B(u_t - Kx_t)\} \oplus \mathcal{W}$. Let the control input be given by $u_t = \kappa(t-t_i, x_{t_i})$ for all $t \in \mathbb{N}_{[t_i, t_{i+1}-1]}$ and all $i \in \mathbb{N}$. Let the set $\mathcal{E}$ of transmission instants be determined in an event-triggered fashion by

$$t_0 = 0 \tag{11a}$$

$$\begin{aligned} t_{i+1} = \inf\{ &t \in \mathbb{N}_{\geq t_i+1} \mid \\ &(A+BK)z_t + B(\kappa(t-t_i, x_{t_i}) - Kx_t) \notin \mathcal{T}_t \ominus \mathcal{W}\}. \end{aligned} \tag{11b}$$

Here, the sets $\mathcal{T}_t \subseteq \mathbb{R}^n$, $t \in \mathbb{N}$ define the trigger conditions. The idea behind (11b) is to trigger an event at the first time instant t after the last transmission instant where $z_{t+1} \in (\gamma+1)\mathcal{F}_{t+1}$ is *not* guaranteed without updating the actuator. Note that the value of the term $(A+BK)z_t + B(\kappa(t-t_i, x_{t_i}) - Kx_t)$ can be computed from information available to the trigger mechanism at time step $t \in \mathbb{N}$.

Theorem 1: If $\mathcal{T}_t \subseteq (\gamma+1)\mathcal{F}_{t+1}$ for all $t \in \mathbb{N}$, for a $\gamma \in [0, \infty)$, then the resulting event-triggered controller is γ-invasive with respect to the all-time triggered controller defined by $u_t = Kx_t$, $t \in \mathbb{N}$.
The proof is given in the appendix.

Note that the condition on the threshold sets is independent of the chosen controller κ. The largest thresholds based on this result for a given γ are obtained by defining $\mathcal{T}_t = (\gamma+1)\mathcal{F}_{t+1}$ for all $t \in \mathbb{N}$. In Section V we provide alternative thresholds.

B. γ-invasive self-triggered control

We propose a self-triggered controller that determines, at the transmission instant t_i, the next transmission instant t_{i+1} such that $z_t \in (\gamma+1)\mathcal{F}_t$ for all $t \in \mathbb{N}_{[t_i+1, t_{i+1}]}$, for any realization of the disturbances w_t. With $x_{t+1} - z_{t+1} = (A+BK)(x_t - z_t)$ it holds that

$$\begin{aligned} z_t &= x_t + (A+BK)^{t-t_i}(z_{t_i} - x_{t_i}) \\ &\in \left\{ A^{t-t_i}x_{t_i} + \sum_{j=0}^{t-t_i-1} A^{t-t_i-1-j}Bu_{t_i+j} \right\} \\ &\quad \oplus \bigoplus_{j=0}^{t-t_i-1} A^j\mathcal{W} \oplus \{(A+BK)^{t-t_i}(z_{t_i} - x_{t_i})\} \end{aligned} \tag{12}$$

for $t \in \mathbb{N}_{[t_i+1, t_{i+1}]}$, $i \in \mathbb{N}_{[0, i_{\max}]}$. Let the control input be given by $u_t = \kappa(t-t_i, x_{t_i})$ for all $t \in \mathbb{N}_{[t_i, t_{i+1}-1]}$ and all $i \in \mathbb{N}_{[0, i_{\max}]}$. Let the set of transmission instants $\mathcal{E}$ be determined in a self-triggered fashion by

$$t_0 = 0 \tag{13a}$$

$$t_{i+1} = t_i + M(x_{t_i}, z_{t_i}, t_i), \tag{13b}$$

where

$$\begin{aligned} M(x_{t_i}, z_{t_i}, t_i) = \inf \Bigg\{ M \in \mathbb{N}_{\geq 1} \;\Bigg|\; \\ (A+BK)^{M+1}(z_{t_i} - x_{t_i}) + A^{M+1}x_{t_i} + \sum_{j=0}^{M} A^{M-j}B\kappa(j, x_{t_i}) \\ \notin \mathcal{T}_{M+1, t_i} \ominus \bigoplus_{j=0}^{M} A^j\mathcal{W} \Bigg\}. \end{aligned} \tag{13c}$$

Here $\mathcal{T}_{M,t_i} \subseteq \mathbb{R}^n$, $M, t_i \in \mathbb{N}$ are the sets defining the trigger conditions. The intuition behind (13c) is to schedule the next transmission instant at the first time t after the current transmission instant where $z_{t+1} \in (\gamma+1)\mathcal{F}_{t+1}$ is not guaranteed without updating the actuator.

Theorem 2: Let $\mathcal{T}_{M,t_i} \subseteq (\gamma+1)\mathcal{F}_{t_i+M}$ for all $M, t_i \in \mathbb{N}$ and a $\gamma \in [0, \infty)$. Then, the self-triggered controller is γ-invasive with respect to the all-time triggered controller defined by $u_t = Kx_t$, $t \in \mathbb{N}$.

The proof is given in the appendix.

As in the event-triggered case, the condition on the threshold sets is independent of the chosen controller κ. The largest thresholds based on this result for a given γ are obtained by defining $\mathcal{T}_{M,t_i} = (\gamma+1)\mathcal{F}_{t_i+M}$ for all $t_i, M \in \mathbb{N}$. In Section V we provide alternative thresholds.

V. IMPLEMENTATION

The trigger conditions with the largest thresholds proposed in Section IV are time-dependent and, in general, grow more complex as time increases. This limits their applicability. In this section, we propose several suboptimal trigger policies with bounded complexity.

A. Time-invariant trigger conditions

In this subsection, we propose thresholds that only depend on the time since the last transmission instant, that is $t - t_i$ for $t \in \mathbb{N}_{[t_i+1, t_{i+1}]}$, instead of the time t. If the time between transmission instants is limited by external means, for example by forcing a transmission instant at time t if $t - t_i \geq \Delta_{\max}$, then the maximal complexity of all employed threshold sets is guaranteed to be bounded.

Consider first event-triggered control. It holds that $z_t = (A+BK)^{t-t_i}z_{t_i} + x_t - (A+BK)^{t-t_i}x_{t_i}$ for $t \in \mathbb{N}_{[t_i, t_{i+1}]}$, $i \in \mathbb{N}_{[0, i_{\max}]}$. Hence,

$$\begin{aligned} z_{t+1} &= (A+BK)z_t + B(u_t - Kx_t) + w_t \\ &= (A+BK)\big((A+BK)^{t-t_i}z_{t_i} \\ &\qquad + x_t - (A+BK)^{t-t_i}x_{t_i}\big) + B(u_t - Kx_t) + w_t \\ &= (A+BK)^{t-t_i+1}z_{t_i} + A(x_t - (A+BK)^{t-t_i}x_{t_i}) \\ &\qquad + B(u_t - K(A+BK)^{t-t_i}x_{t_i}) + w_t \end{aligned} \tag{14}$$

for $t \in \mathbb{N}_{[t_i, t_{i+1}-1]}$, $i \in \mathbb{N}_{[0, i_{\max}]}$. If we replace our knowledge of the actual z_{t_i} with $z_{t_i} \in (\gamma+1)\mathcal{F}_{t_i}$, the condition in (11b) becomes

$$\begin{aligned} &A(x_t - (A+BK)^{t-t_i}x_{t_i}) \\ &\quad + B(\kappa(t-t_i, x_{t_i}) - K(A+BK)^{t-t_i}x_{t_i}) \\ &\qquad \notin \underbrace{\mathcal{T}_t \ominus (\gamma+1)(A+BK)^{t-t_i+1}\mathcal{F}_{t_i}}_{=:\bar{\mathcal{T}}_{t-t_i}} \ominus \mathcal{W}. \end{aligned}$$

Following the ideas in Section IV-B, if $\bar{\mathcal{T}}_{t-t_i} \subseteq (\gamma+1)\mathcal{F}_{t+1} \ominus (\gamma+1)(A+BK)^{t-t_i+1}\mathcal{F}_{t_i} = (\gamma+1)\mathcal{F}_{t-t_i+1}$, the resulting event-triggered controller is γ-invasive. Furthermore, the set $\bar{\mathcal{T}}_{t-t_i}$ indeed only depends on $t - t_i$.

Remark 2: Setting $\bar{\mathcal{T}}_{t-t_i} = \mathcal{F}_{t-t_i+1}$ returns the trigger conditions proposed in [9], [10] where a model based controller equivalent to $\kappa(t-t_i, x_{t_i}) = K(A+BK)^{t-t_i}x_{t_i}$ was employed. The setup in the present paper can be viewed as a generalization of the results in [9], [10] to (i) self-triggered control, (ii) more general controllers κ, and (iii) trigger conditions that depend on the evolution of the system state before the last transmission instant.

For the self-triggered case, replacing z_{t_i} with $(\gamma+1)\mathcal{F}_{t_i}$ in (13c) leads to the condition

$$A^{M+1}x_{t_i} - (A+BK)^{M+1}x_{t_i} + \sum_{j=0}^{M} A^{M-j}B\kappa(j, x_{t_i})$$
$$\notin \underbrace{\mathcal{T}_{M+1,t_i} \ominus (\gamma+1)(A+BK)^{M+1}\mathcal{F}_{t_i}}_{=:\bar{\mathcal{T}}_{M+1}} \ominus \bigoplus_{j=0}^{M} A^j\mathcal{W}. \quad (15)$$

Following the event-triggered case, if $\bar{\mathcal{T}}_{M+1} \subseteq (\gamma+1)\mathcal{F}_{t_i+M+1} \ominus (\gamma+1)(A+BK)^{M+1}\mathcal{F}_{t_i} = (\gamma+1)\mathcal{F}_{M+1}$, then the resulting self-triggered controller is γ-invasive. Furthermore, the condition on the set $\bar{\mathcal{T}}_{M+1}$ only depends on M.

B. Inner approximations

For all $k \in \mathbb{N}$ it holds that $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$. Furthermore, $\mathcal{F}_t$ converges to $\mathcal{F}_\infty = \Omega$ exponentially fast, see [6], [7]. Hence, in order to limit the complexity of the threshold sets, one might define

$$\mathcal{T}_t := \begin{cases} (\gamma+1)\mathcal{F}_{t+1} & t \in \mathbb{N}_{[0,T-1]} \\ (\gamma+1)\mathcal{F}_{T+1} & t \in \mathbb{N}_{\geq T} \end{cases} \quad (16)$$

and

$$\mathcal{T}_{M,t_i} := \begin{cases} (\gamma+1)\mathcal{F}_{M+t_i} & M+t_i \in \mathbb{N}_{[1,T]} \\ (\gamma+1)\mathcal{F}_{T+1} & M+t_i \in \mathbb{N}_{\geq T+1}, \end{cases} \quad (17)$$

for some reasonably large $T \in \mathbb{N}$.

Considering the set $\bigoplus_{j=0}^{M-1} A^j\mathcal{W}$, in the case of a Schur stable matrix A, one might replace $\bigoplus_{j=0}^{M-1} A^j\mathcal{W}$ with an outer approximation of $\bigoplus_{j=0}^{\infty} A^j\mathcal{W}$, see for example [15]. In the case of A not Schur stable and $\mathcal{W}$ containing the origin in its interior, the size[1] of the set $\bigoplus_{j=0}^{M-1} A^j\mathcal{W}$ grows unbounded with M. As $\mathcal{F}_t$ is bounded in t, this implies that for every $\gamma \in [0,\infty)$ there exists an $\bar{M} \in \mathbb{N}$, such that $(\gamma+1)\mathcal{F}_t \ominus \bigoplus_{j=0}^{M-1} A^j\mathcal{W} = \emptyset$ for all $t \in \mathbb{N}$ and all $M \in \mathbb{N}_{\geq \bar{M}}$. Hence, together with the condition $\mathcal{T}_{M,t_i} \subseteq (\gamma+1)\mathcal{F}_{t_i+M}$ we have $\mathcal{T}_{M,t_i} \ominus \bigoplus_{j=0}^{M-1} A^j\mathcal{W} = \emptyset$ for all $t_i \in \mathbb{N}$ and all $M \in \mathbb{N}_{\geq \bar{M}}$. Considering (13c), we can conclude that $M(x_t, z_t, t_i) < \bar{M}$ for all $x_{t_i}, z_{t_i} \in \mathbb{R}^n$ and all $t_i \in \mathbb{N}$, such that $M \in \mathbb{N}_{\geq \bar{M}}$ need not be considered when evaluating the condition in (13c). In combination with the inner approximation in (17), this implies an upper bound on the number of sets that have to be computed for the definition of the trigger condition in (13c).

C. Low-complexity approximations

In this subsection, we provide low-complexity inner approximations of the sets used in the trigger conditions, allowing an efficient implementation of the proposed controllers. Let $\mathcal{Q} \subseteq \mathcal{W}$ be a convex set satisfying $(A+BK)\mathcal{Q} \supseteq \lambda\mathcal{Q}$ for a $\lambda \in [0,1)$. Then it holds that

$$(\gamma+1)\frac{1-\lambda^i}{1-\lambda}\mathcal{Q} \subseteq (\gamma+1)\mathcal{F}_i \quad (18)$$

for $\gamma \in [0,\infty)$ and $i \in \mathbb{N}$. Let further a convex set $\mathcal{R} \subseteq \mathbb{R}^n$ be given[2] with $A\mathcal{R} \subseteq \mu\mathcal{R}$ for a $\mu \in [0,\infty)$ and $\mathcal{W} \subseteq \mathcal{R}$. Then it holds that

$$\bigoplus_{j=0}^{k-1} A^j\mathcal{W} \subseteq \left(\sum_{j=0}^{k-1} \mu^k\right)\mathcal{R}. \quad (19)$$

For efficient implementations, expressions of the form $(\gamma+1)\mathcal{F}_i \ominus \bigoplus_{j=0}^{k-1} A^j\mathcal{W}$ may be replaced by the inner approximation $(\gamma+1)\frac{1-\lambda^i}{1-\lambda}\mathcal{Q} \ominus \left(\sum_{j=0}^{k-1}\mu^j\right)\mathcal{R}$. The complexity of this set is, for polytopic sets $\mathcal{Q}$ and $\mathcal{R}$, uniformly bounded in i and k.

Remark 3: If $\mathcal{W}$ is a set of measure zero, then so must be $\mathcal{Q}$ and therefore $(\gamma+1)\frac{1-\lambda^i}{1-\lambda}\mathcal{Q}$ in these approximations. However, even if the measure of $\mathcal{W}$ is zero, the measure of $\mathcal{F}_i$ might be greater than zero for some $i \in \mathbb{N}$, due to the dynamics $A+BK$. Hence, from a measure-theoretic point of view, $(\gamma+1)\frac{1-\lambda^i}{1-\lambda}\mathcal{Q}$ might be a gross underapproximation of $(\gamma+1)\mathcal{F}_i$ in this case. Similarly, if w_t is a stochastic variable, the probability that a trigger condition based on a set Q with measure zero is *not* satisfied might be zero in this case, leading to a control scheme that triggers at every point in time with probability one. In case the measure of $\mathcal{W}$ is not zero but $A+BK$ is singular, it necessarily holds that Q is of measure zero but $\mathcal{R}$ is not, such that the approximative threshold set $(\gamma+1)\frac{1-\lambda^i}{1-\lambda}\mathcal{Q} \ominus \left(\sum_{j=0}^{k-1}\mu^j\right)\mathcal{R}$ is empty, necessarily leading to the resulting scheme triggering at every point in time.

VI. Numerical Example

Consider the sampled double integrator system defined by

$$x_{t+1} = \begin{bmatrix} 1 & 0.3 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0.045 \\ 0.3 \end{bmatrix} u_t + w_t, \quad (20)$$

where $|w_t| \leq 1$ for all $t \in \mathbb{N}$. We choose K as the LQ-optimal controller for the weighting matrices $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $R = 1$. We employ the largest possible threshold sets, that is $\mathcal{T}_t = (\gamma+1)\mathcal{F}_{t+1}$ in the event-triggered case and $\mathcal{T}_{M,t_i} = (\gamma+1)\mathcal{F}_{t_i+M}$ in the self-triggered case. For the self-triggered scheme, we limit the time between two transmission instants to $\Delta_{\max} = 10$. The system state is initialized with $x_0 = (40, 40)^\top$. We sample w_t independently for each $t \in \mathbb{N}$ from a uniform distribution on $\{w \in \mathbb{R}^2 \mid |w| \leq 1\}$.

It holds that $x_t - z_t \to 0$ as $t \to \infty$. Hence, for large t, the condition in (11b) approaches $Ax_t + B\kappa(t-t_i, x_{t_i}) \notin \mathcal{T}_t \ominus \mathcal{W}$ or, if $\mathcal{T}_t = (\gamma+1)\mathcal{F}_{t+1}$, $Ax_t + B\kappa(t-t_i, x_{t_i}) \notin (\gamma+1)\Omega \ominus \mathcal{W}$, which is a static threshold condition on x_t. We therefore distinguish the analysis of the transient behavior of the closed-loop system from the asymptotic (or quasi steady-state) behavior. In our simulations, the system state reaches the set Ω in approximately 13 to 14 time steps for $\gamma = 0$. Therefore, in order to evaluate the transient behavior, we only consider the first 14 time steps in the simulation. In Table I we list the observed average sampling frequency over 5000 simulation runs for different values of γ and different configurations of the control scheme (event- or self-triggered, different controller types κ, according to (7)). A sampling frequency of 1 corresponds to an update at every time instant t. For comparison, we list the observed average sampling frequency for the time-invariant threshold conditions from Section V-A in Table II. The reported sampling rates are higher[3], which may justify the use of the more involved trigger rules using the full information about z_t in some setups. In all cases, the trade-off between the achieved average sampling rate and the guaranteed performance in terms of γ is apparent. We want to emphasize that even for the case $\gamma = 0$ average sampling rates considerably lower than 1 can be achieved.

In Table III we report the asymptotic behavior, where the average is taken over 500 simulations by only considering the time span between step 100 and 200 in the simulations. Inner approximations of the thresholds as described in Section V-B for $T = 25$

[1] in the sense of a diameter

[2] One might choose $R = \eta\mathcal{Q}$ for an $\eta \in [0,\infty)$ for an even more efficient implementation.

[3] but still provide a substantial benefit over the all-time-triggered controller in many cases

TABLE I

AVERAGE SAMPLING FREQUENCIES FOR EVENT- AND SELF-TRIGGERED SCHEMES DEPENDING ON THE PARAMETER γ AND THE TYPE OF CONTROLLER: TRANSIENT BEHAVIOR.

Event-triggered control				Self-triggered control			
γ	sparse	hold	model	γ	sparse	hold	model
0.00	0.68	0.51	0.14	0.00	0.90	0.73	0.61
0.25	0.58	0.46	0.10	0.25	0.79	0.66	0.44
0.50	0.53	0.44	0.09	0.50	0.70	0.57	0.40
1.00	0.45	0.38	0.08	1.00	0.56	0.49	0.27
1.50	0.37	0.40	0.07	1.50	0.51	0.46	0.20
2.00	0.27	0.37	0.07	2.00	0.45	0.40	0.20
4.00	0.20	0.29	0.07	4.00	0.27	0.33	0.13

TABLE II

AVERAGE SAMPLING FREQUENCIES FOR EVENT- AND SELF-TRIGGERED SCHEMES: TRANSIENT BEAHVIOR FOR TIME-INVARIANT TRIGGER CONDITIONS.

Event-triggered control				Self-triggered control			
γ	sparse	hold	model	γ	sparse	hold	model
0.00	0.92	0.66	0.25	0.00	1.00	1.00	1.00
0.25	0.83	0.56	0.12	0.25	1.00	1.00	1.00
0.50	0.73	0.51	0.09	0.50	0.98	0.86	0.53
1.00	0.58	0.46	0.08	1.00	0.89	0.67	0.33
1.50	0.50	0.41	0.07	1.50	0.68	0.57	0.27
2.00	0.44	0.37	0.07	2.00	0.58	0.51	0.20
4.00	0.20	0.34	0.07	4.00	0.32	0.40	0.13

TABLE III

AVERAGE SAMPLING FREQUENCIES FOR EVENT- AND SELF-TRIGGERED SCHEMES: ASYMPTOTIC BEHAVIOR

Event-triggered control				Self-triggered control			
γ	sparse	hold	model	γ	sparse	hold	model
0.00	0.11	0.21	0.07	0.00	0.53	0.50	0.50
0.25	0.07	0.21	0.05	0.25	0.40	0.33	0.34
0.50	0.06	0.21	0.04	0.50	0.35	0.33	0.33
1.00	0.04	0.21	0.03	1.00	0.26	0.25	0.23
1.50	0.03	0.22	0.03	1.50	0.21	0.22	0.19
2.00	0.02	0.21	0.02	2.00	0.18	0.21	0.17
4.00	0.01	0.22	0.01	4.00	0.11	0.21	0.10

TABLE IV

AVERAGE SAMPLING FREQUENCIES FOR EVENT- AND SELF-TRIGGERED SCHEMES: ASYMPTOTIC BEHAVIOR FOR TIME-INVARIANT TRIGGER CONDITIONS.

Event-triggered control				Self-triggered control			
γ	sparse	hold	model	γ	sparse	hold	model
0.00	0.21	0.27	0.15	0.00	1.00	1.00	1.00
0.25	0.12	0.17	0.06	0.25	1.00	1.00	1.00
0.50	0.09	0.15	0.04	0.50	0.67	0.57	0.50
1.00	0.05	0.14	0.03	1.00	0.39	0.37	0.33
1.50	0.04	0.15	0.02	1.50	0.28	0.28	0.25
2.00	0.03	0.17	0.02	2.00	0.22	0.24	0.20
4.00	0.02	0.18	0.01	4.00	0.13	0.20	0.11

were employed. Surprisingly, in the case of the "to-hold" strategy, increasing γ does not lead to a decrease in the sampling rate beyond 0.21 for this example. Further, the closed-loop dynamics exhibit limit-cycle behavior (not shown) in this case. In Table IV, we report the results for the time-invariant threshold conditions from Section V-A.

VII. DISCUSSION

If we were to apply the static trigger condition $Ax_t + B\kappa(t - t_i, x_{t_i}) \notin (\gamma+1)\Omega \ominus \mathcal{W}$ mentioned in the previous section for all $t \in \mathbb{N}$, the trigger mechanism would create many consecutive events until the state x_t has approached the target set $(\gamma+1)\Omega$. Hence, the proposed event-triggered scheme can be interpreted as extending the local properties of a static threshold to the whole state space by taking into account the initial conditions in the trigger rule. The same applies for the self-triggered approach.

In order to further reduce the communication rate, it may be advantageous to let the controller κ also depend on z_{t_i}. If the condition in (6) is still satisfied, then all results in this paper hold without restrictions. However, it might be possible to even loosen the requirement in (6). This matter is subject to future research. In any case, Lemma 3 holds for any type of controller, without restrictions.

Finally, in order achieve a better trade-off between the complexity of the thresholds and the reduction of communication, one might want to define trigger conditions along similar lines as in Section V-A, where information from at most $\delta \in \mathbb{N}$ recent time steps is used (possibly also from times before t_i), where δ is an arbitrary parameter.

VIII. CONCLUSIONS AND OUTLOOK

We have presented a general framework for event- and self-trigged controllers which guarantee bounds on the system state which are the same as for a controller which is updated at every point in time, up to a factor $\gamma + 1$ that is a design parameter in the scheme. As demonstrated in the examples, the proposed control schemes require much less communication between the sensor and the actuator than controllers which are updated at every point in time. Future research will focus on extending the results to output feedback control and decentralized control. Further, we are going to investigate aperiodic model predictive control, where guaranteed bounds on the uncertainty are required in order to ensure robust constraint satisfaction.

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APPENDIX

We require the following technical result.

Lemma 4: Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$ be compact sets. Then for all $x, y \in \mathbb{R}^n$ it holds that $|x + y|_{\mathcal{A}\oplus\mathcal{B}} \leq |x|_{\mathcal{A}} + |y|_{\mathcal{B}}$.

The proof follows directly from the definitions.

A. *Proof of Lemma 2*

As $A + BK$ is Schur stable, there exists a matrix $P \in \mathbb{R}^{p\times n}$ with $p \in \mathbb{N}_{\geq 1}$ and scalars $a_1, a_2 \in (0, \infty)$, $\lambda \in [0, 1)$, such that $a_1|x| \leq |Px| \leq a_2|x|$ and $|P(A + BK)x| \leq \lambda|Px|$ for all $x \in \mathbb{R}^n$, see for example [19], [20]. Assume now that $x_t \in \{(A + BK)^t x_0\} \oplus (\gamma+1)\mathcal{F}_t$ for $t \in \mathbb{N}$ and some $x_0 \in \mathbb{R}^n$. It follows that there exists $z_t \in (\gamma+1)\mathcal{F}_t$ with $x_t = (A+BK)^t x_0 + z_t$ for all $t \in \mathbb{N}$. Hence,

$$
\begin{aligned}
|x_t|_{(\gamma+1)\Omega} &= |(A + BK)^t x_0 + z_t|_{(\gamma+1)\Omega} \\
&\stackrel{\text{Lemma 1}}{=} |(A + BK)^t x_0 + z_t|_{(\gamma+1)(A+BK)^t\Omega\oplus(\gamma+1)\mathcal{F}_t} \\
&\stackrel{\text{Lemma 4}}{\leq} |(A + BK)^t x_0|_{(\gamma+1)(A+BK)^t\Omega} + |z_t|_{(\gamma+1)\mathcal{F}_t} \\
&\stackrel{z_t\in(\gamma+1)\mathcal{F}_t}{=} |(A + BK)^t x_0|_{(\gamma+1)(A+BK)^t\Omega}
\end{aligned}
$$

$$
\begin{aligned}
&= \min_{y \in (\gamma+1)(A+BK)^t \Omega} |(A+BK)^t x_0 - y| \\
&= \min_{y \in (\gamma+1)\Omega} |(A+BK)^t (x_0 - y)| \\
&\leq \frac{1}{a_1} \min_{y \in (\gamma+1)\Omega} |P(A+BK)^t (x_0 - y)| \\
&\leq \lambda^t \frac{1}{a_1} \min_{y \in (\gamma+1)\Omega} |P(x_0 - y)| \\
&\leq \lambda^t \frac{a_2}{a_1} \min_{y \in (\gamma+1)\Omega} |x_0 - y| = \lambda^t \frac{a_2}{a_1} |x_0|_{(\gamma+1)\Omega} \qquad (21)
\end{aligned}
$$

for all $t \in \mathbb{N}$. As x_0 and $z_t \in (\gamma+1)\mathcal{F}_t$, $t \in \mathbb{N}$, were arbitrary, it follows that $(\gamma+1)\Omega$ is asymptotically stable for the closed-loop system, thereby completing the proof. ■

B. Proof of Theorem 1

By Lemma 3, we need to show that $z_t \in (\gamma+1)\mathcal{F}_t$ for all $t \in \mathbb{N}$. The proof is by induction where the induction base is provided by $z_0 = 0$. Assume now that $z_t \in (\gamma+1)\mathcal{F}_t$ for an arbitrary $t \in \mathbb{N}$. Let $t \in \mathbb{N}_{[t_i, t_{i+1}-1]}$ for some $i \in \mathbb{N}_{[0, i_{\max}]}$. If $t = t_i$, then, by (6), $u_t = Kx_t$, and, hence,

$$
\begin{aligned}
z_{t+1} &= (A+BK)z_t + B(Kx_t - Kx_t) + w_t \\
&\in (A+BK)(\gamma+1)\mathcal{F}_t \oplus \mathcal{W} \\
&\overset{\gamma \geq 0}{\subseteq} (\gamma+1)\left((A+BK)\mathcal{F}_t \oplus \mathcal{W}\right) = (\gamma+1)\mathcal{F}_{t+1}. \qquad (22)
\end{aligned}
$$

If $t \neq t_i$, then $u_t = \kappa(t - t_i, x_{t_i})$, such that, by the definition of the trigger conditions,

$$
\begin{aligned}
z_{t+1} &= (A+BK)z_t + B(\kappa(t-t_i, x_{t_i}) - Kx_t) + w_t \\
&\in (\mathcal{T}_t \ominus \mathcal{W}) \oplus \mathcal{W} \\
&\subseteq \mathcal{T}_t \subseteq (\gamma+1)\mathcal{F}_{t+1}. \qquad (23)
\end{aligned}
$$

Hence, $z_t \in (\gamma+1)\mathcal{F}_t$ for all $t \in \mathbb{N}$, thereby completing the proof. ■

C. Proof of Theorem 2

The proof proceeds along similar lines as the proof of Theorem 1. It holds that $z_0 = 0 \in (\gamma+1)\mathcal{F}_0$. Assume now that $z_{t_i} \in (\gamma+1)\mathcal{F}_{t_i}$ for an arbitrary $i \in \mathbb{N}_{[0, i_{\max}]}$ and $M^* = \inf\{M \in \mathbb{N}_{\geq 1} \mid (A+BK)^{M+1}(z_{t_i} - x_{t_i}) + A^{M+1}x_{t_i} + \sum_{j=0}^{M} A^{M-j}B\kappa(j, x_{t_i}) \notin \mathcal{T}_{M+1,t_i} \ominus \bigoplus_{j=0}^{M-1} A^j\mathcal{W}\} = M(x_{t_i}, z_{t_i}, t_i)$. Then, it holds that

$$
(A+BK)^k(z_{t_i} - x_{t_i}) + A^k x_{t_i} + \sum_{j=0}^{k-1} A^{k-1-j}B\kappa(j, x_{t_i}) \in \mathcal{T}_{k,t_i} \ominus \bigoplus_{j=0}^{k-1} A^j \mathcal{W} \qquad (24)
$$

for all $k \in \mathbb{N}_{[2,M^*]}$. By (6), it holds that $\kappa(0, x_{t_i}) = Kx_{t_i}$, and, hence,

$$
z_{t_i+1} = (A+BK)z_{t_i} + w_{t_i} \in (\gamma+1)\mathcal{F}_{t_i+1} \qquad (25)
$$

as in the proof of Theorem 1 (namely, (22)). Further, using (12), it holds that

$$
\begin{aligned}
z_t \in &\left\{A^{t-t_i}x_{t_i} + \sum_{j=0}^{t-t_i-1} A^{t-t_i-1-j}Bu_{t_i+j}\right\} \\
&\oplus \{(A+BK)^{t-t_i}(z_{t_i} - x_{t_i})\} \oplus \bigoplus_{j=0}^{t-t_i-1} A^j\mathcal{W} \\
&\overset{(24)}{\subseteq} \mathcal{T}_{t-t_i,t_i} \subseteq (\gamma+1)\mathcal{F}_t \qquad (26)
\end{aligned}
$$

for all $t \in \mathbb{N}_{[t_i+2, t_i+M^*]}$, where the last inclusion holds by assumption. Hence, considering that $t_{i+1} = t_i + M^*$, for all $t \in \mathbb{N}_{[t_i+1, t_{i+1}]}$ it holds that $z_t \in (\gamma+1)\mathcal{F}_t$, such that the statement is proved by induction. ■

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