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Event-triggered and self-triggered control for linear systems based on reachable sets☆

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ABSTRACT

We propose novel aperiodic control schemes for additively perturbed discrete-time linear systems based on the evaluation of set-membership conditions related to disturbance reachable sets. The goal is to reduce the rate of communication between the sensor and the actuator, while guaranteeing that a certain set in the state space is asymptotically stabilized. In particular, we prescribe this set to be the minimal robust positively invariant set under a given feedback law updated at every time, multiplied by a factor that acts as a tuning parameter. This way, we achieve a trade-off between the communication rate and the worst-case asymptotic bound on the system state in the closed-loop system. We employ a novel stability concept that captures how much the system dynamics are explicitly dependent on past system states. This allows us to quantitatively compare the stability properties guaranteed by an all-time updated (static) feedback controller with those guaranteed by a (dynamic) aperiodic controller. We use the proposed framework to design both event-triggered and self-triggered controllers under the assumption of state feedback or output feedback.

1. Introduction

In the study of networked control systems, the transmission of information between the different components such as sensors, controllers, and actuators is taken into account explicitly. If there is a (high) cost on communications, then it is reasonable to attempt a trade-off between the average communication rate and other control-related performance criteria for the closed-loop system. The cost on communication might result from the energy demand involved with transmitting information (especially if the network is wireless) or from a bandwidth limit, if the network has to be shared between different agents.

Two paradigms that have proven to be effective in this context are event-triggered control and self-triggered control. In the former, the state or output of the plant is measured periodically or continuously and information is transmitted over the network only if a certain event condition is met, such as the output deviating from the previously transmitted value beyond a given threshold; in self-triggered control, measurements are only taken at certain sampling instants, where at each sampling instant the next one is determined – online – based on given information at that time, compare Fig. 1. Event-triggered control has the advantage of being able to react immediately to unforeseen plant behavior, possibly due to disturbances, while in self-triggered control, the input to the plant is strictly open-loop between sampling instants. On the other hand, self-triggered control allows the sensors and the communication system to be completely shut off between sampling instants, allowing additional energy to be saved. Both paradigms generate aperiodic transmission behavior depending on the evolution of the system state. It has been shown quantitatively (Antunes & Heemels, 2014; Åström & Bernhardsson, 2002) that such aperiodic control schemes allow a better performance trade-off than schemes based on purely periodic sampling. See also Anta and Tabuada (2010), Cardoso de Castro (2012), Cassandras (2014), Gommans and Heemels (2015), Heemels, Johansson, and Tabuada (2012), and the references therein, for a discussion of event-triggered and self-triggered control.

In this paper, we consider the event-triggered and self-triggered control of linear discrete-time systems subject to bounded additive

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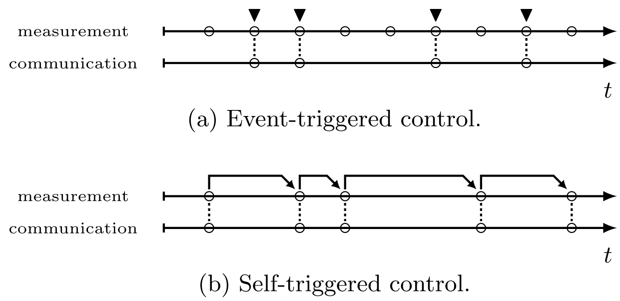

(a) Event-triggered control.

(b) Self-triggered control.

Fig. 1. Two aperiodic scheduling paradigms considered in the paper.

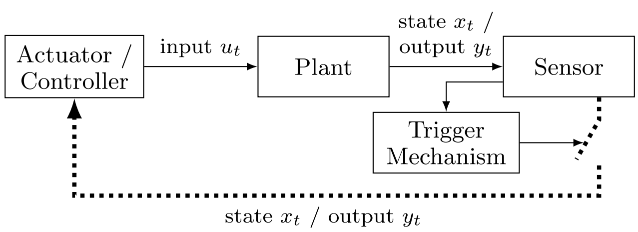

Fig. 2. Networked control system with a communication channel between the sensor and the controller.

disturbances, in a networked setting as depicted in Fig. 2. We assume that the actuators and the sensors of the plant are not collocated, thereby necessitating a networked control setup. The control objective is to stabilize a compact set in the state-space which is the minimal robust positively invariant set, multiplied by a constant factor, for the closed-loop dynamics under a given linear feedback that is updated at every time. This constant multiplicative factor acts as a tuning parameter, and, in some examples, can be chosen equal to 1 while still allowing a significant reduction in the communication rate, as we will see. In this way, we specify the performance of the closed-loop system – which we define here as the size of the guaranteed asymptotic bound on the state – in terms of the performance guaranteed by a controller that requires communication at every time step. We propose trigger conditions that are based on reachable-set considerations for linear systems with bounded disturbances, taking into account the past evolution of the system state up to a given horizon. In particular, the trigger conditions ensure that at each time the system state is contained in the disturbance forward reachable set (multiplied by the tuning factor) for the closed-loop system under feedback updated at every time, initialized at the system state at some time in the aforementioned horizon. Extending the horizon relaxes the trigger conditions, allowing a greater reduction in the communication rate. Hence, the proposed aperiodic control schemes constitute dynamic feedback controllers, as the input at a given time depends, explicitly through the trigger conditions, on system states at past times. In order to quantitatively compare the stability properties of such controllers with *static* feedback controllers, we employ a novel stability concept, which takes this dependence into account without requiring to extend the state-space under consideration. In this way, dynamic or static controllers with different dimensions defining the closed-loop system state can be compared quantitatively. A similar method for the analysis of the stability of aperiodically sampled systems is to model the closed-loop system as a time-delay system, see, for example, Hetel et al. (2017, Section 4.1) and the references therein.

Previous results on aperiodic control based on set-theoretic properties of dynamical systems can, for example, be found in Boisseau, Martinez, Raharijaona, Durand, and Marchand (2017), Grüne et al. (2010), Heemels, Sandee, and Van Den Bosch (2008) and Lunze and Lehmann (2010), where event-triggered controllers based on fixed trigger sets around the origin or the evolution of a nominal systems are employed; set-theoretic approaches are used to compute an asymptotic bound on the system state or to design the trigger sets such that a given bound is achieved. A similar approach is pursued in Kögel and Findeisen (2014), using forward reachable sets in order to guarantee state constraint satisfaction in a self-triggered context. Similar approaches to both event-triggered and self-triggered control are also featured in Nghiem (2012), additionally making use of backwards reachability sets and time-varying sets for event-triggered control. In Sijs, Lazar, and Heemels (2010), set-valued bounds on the estimation errors are obtained from an event-triggered observer and combined with a robust predictive control algorithm. Approaches using concepts such as input-to-state stability, (ultimate) boundedness, and $\mathcal{L}_p$ stability, among others, can, for example, be found in Mazo, Anta, and Tabuada (2010), Rabi and Johansson (2008), Tabuada (2007), Tiberi, Fischione, Johansson, and Di Benedetto (2013) and Wang and Lemmon (2009).

The major difference in our approach is to replace single trigger conditions (which in the existing approaches is most often defined by the state of the system being contained in a certain set around the origin or a nominal system state) by a *set* of (non-)trigger conditions, of which only one has to be fulfilled in order for the mechanism not to trigger a communication event. Moreover, the conditions we specify are explicitly based on the state of the system at past time instants within a specified horizon, instead of only on the state at the last trigger instant or even being defined by constant sets (as, for example, in Boisseau et al., 2017; Grüne et al., 2010; Heemels et al., 2008; Lunze & Lehmann, 2010). The combination of these two novelties enlarges, at each given time instant, the set of possible system states that do not trigger a communication event, thereby potentially allowing a greater reduction in the average communication rate. Finally, we tie the length of the backwards horizon used to generate the trigger conditions to a quantifiable stability property, enabling a trade-off between desired closed-loop system properties and communication rate in the controller design.

The remainder of the paper is structured in the following way. This introductory section concludes with some remarks on notation. The problem setup and the assumed structure of the aperiodic controllers are presented in Section 2. In Section 3, the novel aperiodic controllers based on reachable sets are presented. An extension to the output-feedback case is presented in Section 4. Section 5 contains some notes on implementation and complexity, links the framework to earlier aperiodic control schemes, and highlights certain peculiarities of self-triggered control. An academic example illustrating the results is presented in Section 6 and Section 8 concludes the paper. If not indicated otherwise, proofs for statements are given in the Appendix. The present paper is based on the preliminary works in Brunner and Allgöwer (2016) and Brunner, Heemels, and Allgöwer (2016). Here, we generalize the trigger conditions therein and extend the framework to the output-feedback case.

Notation: The set of non-negative integers is denoted by $\mathbb{N}$. For $a \in \mathbb{N}$, $\mathbb{N}_{\geq a}$ denotes the set of integers larger than or equal to a. The set of nonnegative real numbers is denoted by $\mathbb{R}_{\geq 0}$. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the maximum norm. For a vector $x \in \mathbb{R}^n$ and a compact set $\mathcal{Y} \subseteq \mathbb{R}^n$, $|x|_{\mathcal{Y}}$ denotes the distance $\min_{y \in \mathcal{Y}} |x - y|$ of x from $\mathcal{Y}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and it holds that $\alpha(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is of class $\mathcal{K}$ in its first argument for any fixed value of its second argument, is a decreasing function in its second argument for any value of its first argument, and $\beta(r, s) \to 0$ as $s \to \infty$ for any $r \in \mathbb{R}_{\geq 0}$.

We use 0 and I to denote the zero matrix and the identity matrix, respectively, where the dimensions are defined by context. For a set $\mathcal{X} \subseteq \mathbb{R}^n$, matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$, and a scalar $c \in \mathbb{R}$, we define $A\mathcal{X} = \{Ax \mid x \in \mathcal{X}\}$, $B^{-1}\mathcal{X} = \{x \in \mathbb{R}^m \mid Bx \in \mathcal{X}\}$, and $c\mathcal{X} = \{cx \mid x \in \mathcal{X}\}$. For sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$, the Minkowski sum is denoted by $\mathcal{X} \oplus \mathcal{Y} = \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$ and the Pontryagin difference by $\mathcal{X} \ominus \mathcal{Y} = \{x \in \mathbb{R}^n \mid \{x\} \oplus \mathcal{Y} \subseteq \mathcal{X}\}$. For a sequence $(\mathcal{X}_i)_{i\in\mathbb{N}}$ with $\mathcal{X}_i \subseteq \mathbb{R}^n$, we additionally define $\bigoplus_{k=i}^{j}\mathcal{X}_k = \{\sum_{k=i}^{j} x_k \mid x_k \in \mathcal{X}_k\, k \in \{i, i+1, \ldots, j\}\}$ for $i \in \mathbb{N}$ and $j \in \mathbb{N} \cup \{\infty\}$. The empty sum ($i > j$) is equal to $\{0\} \subseteq \mathbb{R}^n$, by convention.

2. Problem setup and preliminaries

2.1. System description and control objective

We consider disturbed linear discrete-time systems

$$x_{t+1} = Ax_t + Bu_t + w_t, \tag{1}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, $w_t \in \mathcal{W} \subseteq \mathbb{R}^n$ are the state, the input, and the disturbance, respectively, at time $t \in \mathbb{N}$. We make the assumption that x_t is available as a measurement at each time point $t \in \mathbb{N}$, which we will relax when treating the output-feedback case in Section 4. Further, we assume that the disturbance w_t is unknown at time t, but that the set $\mathcal{W}$ is a known compact and convex set containing the origin. Finally, we assume that a matrix $K \in \mathbb{R}^{m\times n}$ is given such that $A+BK$ is Schur (that is, the eigenvalues of $A + BK$ are contained in the interior of the complex unit disc).

Our goal is to design networked controllers – as depicted in Fig. 2 – for system (1) that robustly stabilize a compact subset of the state space while requiring as little communication between the sensor and the actuator as possible.

In the remainder of the section, we formalize the stability concept employed in this paper and make some structural assumptions on the controllers to be designed.

2.2. Stability concept

For a given dynamical system, the standard definition of uniform global asymptotic stability of the origin requires that for any given $\epsilon > 0$, there exists a $\delta > 0$ such that if the norm of the system state at any given point t^0 in time is bounded by δ, then the norm of the system state at any later time $t > t^0$ is bounded by ϵ. Further, for any initial condition the state must converge to the origin — the convergence rate being independent of the time of initialization. However, in the aperiodic control schemes considered in the subsequent sections, the input to the system at a given time t^0 does not only depend on the current system state, but also, explicitly, on the system state at earlier times. One would expect the system behavior to become more predictable if the time span in which such past states could influence the future system behavior was shorter. The following stability definition includes the allowed maximal length of this time span as the parameter θ.

Definition 1 (*θ-UGAS*)**.** Consider a dynamical system $\mathbf{p}$ mapping an initial condition and a sequence of disturbances to a sequence of states, that is, $\mathbf{p} : \mathbb{R}^n \times (\mathcal{W})^{\mathbb{N}} \to (\mathbb{R}^n)^{\mathbb{N}}$, $(x_0, (w_s)_{s\in\mathbb{N}}) \mapsto (x_s)_{s\in\mathbb{N}}$. Let further $\mathcal{Y}$ be a compact subset of $\mathbb{R}^n$. We call $\mathcal{Y}$ *θ-uniformly globally asymptotically stable* (θ-UGAS) for $\mathbf{p}$, where $\theta \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, if there exists a $\mathcal{KL}$-function β such that for every $(x_0, (w_s)_{s\in\mathbb{N}}) \in \mathbb{R}^n \times (\mathcal{W})^{\mathbb{N}}$, every $t^0 \in \mathbb{N}$, and every $t \in \mathbb{N}_{\geq t^0}$, $(x_s)_{s\in\mathbb{N}} = \mathbf{p}(x_0, (w_s)_{s\in\mathbb{N}})$ satisfies $|x_t|_{\mathcal{Y}} \leq \max_{\tau\in\{0,1,\ldots,\min\{t^0,\theta-1\}\}} \beta(|x_{t^0-\tau}|_{\mathcal{Y}}, t - t^0 + \tau)$.

A few comments are in order regarding this definition.

(i) For $\theta = 1$, Definition 1 coincides with the standard concept of uniform global asymptotic stability, that is, $|x_t|_{\mathcal{Y}} \leq \beta(|x_{t^0}|_{\mathcal{Y}}, t - t^0)$ for all $x_0 \in \mathbb{R}^n$, $t \in \mathbb{N}_{\geq t^0}$ (compare, for example, Hahn (1967, Chapter V), where β is the product of a $\mathcal{K}$-function and a strictly decreasing function).

(ii) If the dynamical system is described by a time-invariant difference equation of the form $x_{t+1} = f(x_t)$, then the stability property of Definition 1 is equivalent for *all* $\theta \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and, hence, also to the standard definition of (uniform) global asymptotic stability for such systems, due to point (i).

(iii) For any $\theta \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, θ-UGAS implies $\lim_{t\to\infty}|x_t|_{\mathcal{Y}} = 0$, by the choice of $t^0 = 0$, that is, the set $\mathcal{Y}$ is globally attractive and therefore provides an asymptotic bound for the system state.

(iv) Asymptotic stability of a subset of the state-space is closely related to the concept of ultimate boundedness, see for example Definition 4.6 in Khalil (2002). Note, however, that first, attractivity of a set does not imply its stability, and that second, ultimate boundedness implies convergence of the system state to a certain set in finite time whereas asymptotic stability only implies asymptotic convergence.

(v) For systems subject to additive disturbances, a commonly employed stability concept is that of input-to-state stability (ISS). An important difference between ISS and θ-UGAS is the gain provided in the ISS definition between the ℓ_∞ norm of the disturbance sequence $(w_t)_{t\in\mathbb{N}}$ and an asymptotic bound on the system state. Further, ISS implies that if the disturbances sequence converges to zero, so does the system state. Both properties are absent in the definition of θ-UGAS, which makes it an overall weaker system theoretic property. Note, however, that θ-UGAS being a weaker property implies that it is also easier to satisfy, which – potentially – allows communication to be suspended at a larger number of time instants.

(vi) For fixed θ, and the assumption that $x_\rho = x_0$ for $\rho \in \{-\theta + 1, \ldots, -1\}$, (or if $x_{-\theta+1}, \ldots, x_{-1}$ are not relevant for the system evolution[1]), Definition 1 is equivalent to the stability concept for time-delay systems with delay $\theta - 1$, or more generally, the stability concept for systems described by functional difference equations, where the condition for uniform global asymptotic stability, in terms of a $\mathcal{KL}$-function $\hat{\beta}$, reads $|x_t|_{\mathcal{Y}} \leq \hat{\beta}(\sup_{\tau\in\{0,\ldots,\theta-1\}}|x_{t^0-\tau}|_{\mathcal{Y}}, t - t^0)$, compare Liz and Ferreiro (2002), and also Hahn (1967, Section 44) for a continuous-time counterpart. Formally, we have the following result.

Lemma 2. *Let $\theta \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and consider a dynamical system $\mathbf{q}$ mapping an initial condition sequence and a sequence of disturbances to a sequence of states, that is, $\mathbf{q} : (\mathbb{R}^n)^\theta \times (\mathcal{W})^{\mathbb{N}} \to (\mathbb{R}^n)^{\mathbb{N}}$, $((x_{-\theta+1}, x_{-\theta+2}, \ldots, x_{-1}, x_0), (w_s)_{s\in\mathbb{N}}) \mapsto (x_s)_{s\in\mathbb{N}}$. Let further $\mathcal{Y}$ be a compact subset of $\mathbb{R}^n$. If there exists a $\mathcal{KL}$-function β such that for all $((x_{-\theta+1}, x_{-\theta+2}, \ldots, x_{-1}, x_0), (w_s)_{s\in\mathbb{N}}) \in (\mathbb{R}^n)^\theta \times (\mathcal{W})^{\mathbb{N}}$, every $t^0 \in \mathbb{N}$, and every $t \in \mathbb{N}_{\geq t^0}$, every $(x_s)_{s\in\mathbb{N}} = \mathbf{q}((x_{-\theta+1}, x_{-\theta+2}, \ldots, x_{-1}, x_0), (w_s)_{s\in\mathbb{N}})$ satisfies*

$$|x_t|_{\mathcal{Y}} \leq \max_{\tau\in\{0,1,\ldots,\min\{t^0,\theta-1\}\}} \beta(|x_{t^0-\tau}|_{\mathcal{Y}}, t - t^0 + \tau) \tag{2}$$

then there exists a $\mathcal{KL}$-function $\hat{\beta}$ such that for all $((x_{-\theta+1}, x_{-\theta+2}, \ldots, x_{-1}, x_0), (w_s)_{s\in\mathbb{N}}) \in (\mathbb{R}^n)^\theta \times (\mathcal{W})^{\mathbb{N}}$, every $t^0 \in \mathbb{N}$, and every $t \in \mathbb{N}_{\geq t^0}$, every $(x_s)_{s\in\mathbb{N}} = \mathbf{q}((x_{-\theta+1}, x_{-\theta+2}, \ldots, x_{-1}, x_0), (w_s)_{s\in\mathbb{N}})$ satisfies

$$|x_t|_{\mathcal{Y}} \leq \hat{\beta}\left(\sup_{\tau\in\{0,\ldots,\theta-1\}} |x_{t^0-\tau}|_{\mathcal{Y}}, t - t^0\right). \tag{3}$$

[1] For the dynamical systems considered in Definition 1, this is in fact the case.

If $x_\rho = x_0$ for $\rho \in \{-(\theta-1),\dots,-1\}$, that is, the truncation of τ at t^0 in (2) is irrelevant, then the converse also holds.

In the Appendix, we provide a counterexample for the case where $x_{-1} \neq x_0$, or, more precisely, where the influence of x_{-1} on $|x_t|$ never disappears, not even for arbitrarily large t. In particular, we show that there exists a system for which the origin is uniformly globally asymptotically stable in the time-delay sense, but is not θ-UGAS for any θ. This, together with Lemma 2, implies that θ-UGAS is a *stronger* stability notion than stability in the time-delay sense for dynamical systems $\mathbf{q}$ as considered in 2.

For dynamical systems $\mathbf{p}$ as in Definition 1, we can state the following result, which follows directly from Lemma 2.

Corollary 3. *For a dynamical system $\mathbf{p} : \mathbb{R}^n \times (\mathcal{W})^{\mathbb{N}} \to (\mathbb{R}^n)^{\mathbb{N}}$, $(x_0,(w_s)_{s\in\mathbb{N}}) \mapsto (x_s)_{s\in\mathbb{N}}$, a compact set $\mathcal{Y}$ is θ-UGAS if and only if it is uniformly globally asymptotically stable in the time-delay sense with delay $\theta-1$, that is, if and only if there exists a $\mathcal{KL}$-function β such that for every $(x_0,(w_s)_{s\in\mathbb{N}}) \in \mathbb{R}^n \times (\mathcal{W})^{\mathbb{N}}$, every $t^0 \in \mathbb{N}$, and every $t \in \mathbb{N}_{\geq t^0}$, $(x_s)_{s\in\mathbb{N}} = \mathbf{p}(x_0,(w_s)_{s\in\mathbb{N}})$ satisfies $|x_t|_{\mathcal{Y}} \leq \max_{\tau\in\{0,1,\dots,\min\{t^0,\theta-1\}\}} \beta(|x_{t^0-\tau}|_{\mathcal{Y}}, t-t^0+\tau)$, there exists a $\mathcal{KL}$-function $\hat{\beta}$ such that for every $(x_0,(w_s)_{s\in\mathbb{N}}) \in \mathbb{R}^n \times (\mathcal{W})^{\mathbb{N}}$, every $t^0 \in \mathbb{N}$, and every $t \in \mathbb{N}_{\geq t^0}$, $(x_s)_{s\in\mathbb{N}} = \mathbf{p}(x_0,(w_s)_{s\in\mathbb{N}})$ satisfies $|x_t|_{\mathcal{Y}} \leq \beta(\max_{\tau\in\{0,1,\dots,\min\{t^0,\theta-1\}\}} |x_{t^0-\tau}|_{\mathcal{Y}}, t-t^0)$.*

Finally, we emphasize that the case $\theta = \infty$ is of actual relevance; it arises if the input to the system at any time may depend explicitly on the state x_0 at initialization. As we will see, $\theta = \infty$ will come up in the controllers proposed in the paper as a case where communications are saved especially.

2.3. Event-triggered and self-triggered control

We are interested in controlling (1) over a communication network as depicted in Fig. 2. In order to reduce the number of transmissions over the network, we employ controllers that require communication from the sensor to the actuator/controller only at the *transmission instants* t_i, $i \in \mathbb{N}$. These times satisfy $t_i \in \mathbb{N}\cup\{\infty\}$ and $t_i \geq t_j + 1$ for all $i, j \in \mathbb{N}$ with $i > j$ (∞ is included in order to capture the case that no transmissions occur after a finite time $t_{\max}$, where we use the convention that $\infty + 1 = \infty$ and $\infty \geq \infty$). We require that in the time span $\{t_i,\dots,t_{i+1}-1\}$, the input u_t to the system is a function of the state x_{t_i} at time t_i and the time $t - t_i$ since the last transmission only. In this paper, event-triggered and self-triggered controllers are distinguished by the way the sequence $(t_i)_{i\in\mathbb{N}}$ is generated.

We consider event-triggered controllers of the form

$$u_t = \kappa(t-t_i, x_{t_i}),\ \text{if } t \in \{t_i,\dots,t_{i+1}-1\} \tag{4a}$$
$$t_0 = 0 \tag{4b}$$
$$t_{i+1} = \inf\{t \in \mathbb{N}_{\geq t_i+1} \mid \delta_t(t_i, x_0, x_1,\dots,x_t) = 1\} \tag{4c}$$

with the *event-generating functions* $\delta_t : \mathbb{N}\times\mathbb{R}^{(t+1)n} \to \{0,1\}$, $t \in \mathbb{N}$, and self-triggered controllers of the form

$$u_t = \kappa(t-t_i, x_{t_i}),\ \text{if } t \in \{t_i,\dots,t_{i+1}-1\} \tag{5a}$$
$$t_0 = 0 \tag{5b}$$
$$t_{i+1} = s_i(t_0, t_1\dots,t_i, x_{t_0}, x_{t_1}\dots,x_{t_i}) \tag{5c}$$

with the *scheduling functions* $s_i : \mathbb{N}^{i+1} \times \mathbb{R}^{(i+1)n} \to \mathbb{N}$, $i \in \mathbb{N}$. With these definitions, it is clear that:

- For $t \in \{t_i,\dots,t_{i+1}-1\}$, the input u_t is a function of x_{t_i} only, and, hence, no communication is required between the sensor and the actuator at these times.
- The event-triggered controller determines the transmission instants by constantly monitoring the system state x_t and generating an event at time t if x_t meets certain criteria, which are expressed by the event-generating functions δ_t.
- The self-triggered controller determines the transmission instants by computing t_{i+1} at time t_i as a function of (among others) x_{t_i}.
- The self-triggered controller does not require measurements of x_t for $t \in \{t_i+1,\dots,t_{i+1}-1\}$. Hence, the sensor system as well as the communication system may be shut down for these time points (as neither measurements or communications are required), potentially saving additional energy.

Note that for the event-triggered controller, the sensor system needs to be active at all times as the trigger mechanism needs to evaluate δ_t. Furthermore, at least the communication system at the actuator side needs to remain active as new information may arrive at any time.

We assume the function $\kappa : \mathbb{N}\times\mathbb{R}^n \to \mathbb{R}^m$ to be given and to satisfy $\kappa(0,x) = Kx$ for all $x \in \mathbb{R}^n$. Particular cases in this framework are *to-zero controllers*, where

$$\kappa(\tau,x) = \begin{cases} Kx, & \tau = 0,\ x \in \mathbb{R}^n \\ 0, & \text{otherwise}, \end{cases} \tag{6a}$$

to-hold controllers

$$\kappa(\tau,x) = Kx, \quad \tau \in \mathbb{N},\ x \in \mathbb{R}^n, \tag{6b}$$

and *model-based controllers* (Garcia & Antsaklis, 2013; Heemels & Donkers, 2013; Lunze & Lehmann, 2010)

$$\kappa(\tau,x) = K(A+BK)^\tau x, \quad \tau \in \mathbb{N},\ x \in \mathbb{R}^n, \tag{6c}$$

see also Gommans and Heemels (2015), Heemels et al. (2012) and Schenato (2009).

Remark 4. (i) The assumption that $\kappa(0,x) = Kx$ ensures that the input in the closed-loop system coincides with the all-time updated feedback $u_t = Kx_t$ at the transmission times. We will exploit this property in order to establish stability in the closed-loop systems. (ii) In the self-triggered setup, one may consider to change the control input additionally depending on the time until the next scheduled transmission instant. Further, the schemes presented here can easily be modified such that the assumption $\kappa(0,x) = Kx$ is only necessary if $t_{i+1} = t_i + 1$ in the self-triggered case. For simplicity of exposition and in order to treat event-triggered and self-triggered controllers in the same framework, we stick to the slightly more restrictive setup above. Note that the optimal choice of a schedule-dependent feedback is an open question.

In the following section, we consider the design of the event-generating functions δ_t, $t \in \mathbb{N}$, and the scheduling functions s_i, $i \in \mathbb{N}$, such that a certain compact set $\mathcal{Y} \subseteq \mathbb{R}^n$ is θ-UGAS for the dynamical systems which generate the closed-loop state sequence $(x_t)_{t\in\mathbb{N}}$, where $\mathcal{Y}$ and θ are design parameters.

3. Guaranteeing stability via set-membership constraints

In this section, we propose a means to guarantee that a certain set is θ-UGAS by exploiting reachability results. It is a well-known fact that the system

$$x_{t+1} = (A+BK)x_t + w_t \tag{7}$$

with $w_t \in \mathcal{W}$ satisfies

$$x_t \in \{(A+BK)^{t-t_0}x_{t^0}\} \oplus \mathcal{F}_{t-t^0} \tag{8}$$

for all $t \in \mathbb{N}$ and all $t^0 \in \{0,1,\dots,t\}$, where

$$\mathcal{F}_i := \bigoplus_{j=0}^{i-1} (A+BK)^j \mathcal{W} \tag{9}$$

for all $i \in \mathbb{N} \cup \{\infty\}$. Further, the compact and convex set $\mathcal{F}_\infty$ is 1-UGAS for the system (7). Finally, it holds that

$$(A+BK)^i \mathcal{F}_j \oplus \mathcal{F}_i = \mathcal{F}_{i+j} \tag{10}$$

for $i \in \mathbb{N}$ and $j \in \mathbb{N} \cup \{\infty\}$, with the convention $\infty + i = \infty$. These statements are discussed in Artstein and Raković (2008), Kolmanovsky and Gilbert (1995, 1998) and Raković (2007).

The following result allows to formulate sufficient conditions for the stability properties of the closed-loop system in terms of set-membership constraints.

Theorem 5. *Consider a dynamical system* $\mathbf{p} : \mathbb{R}^n \times (\mathcal{W})^{\mathbb{N}} \to (\mathbb{R}^n)^{\mathbb{N}}, (x_0, (w_s)_{s\in\mathbb{N}}) \mapsto (x_s)_{s\in\mathbb{N}}$. *Let there exist a* $\gamma \in [1,\infty)$ *and a* $\theta \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ *such that for every* $(x_0, (w_s)_{s\in\mathbb{N}}) \in \mathbb{R}^n \times (\mathcal{W})^{\mathbb{N}}$, $(x_s)_{s\in\mathbb{N}} = \mathbf{p}(x_0, (w_s)_{s\in\mathbb{N}})$ *and every* $t \in \mathbb{N}$,

$$\begin{aligned} &\exists \tau \in \{1,\ldots,\min\{\theta, t+1\}\}, \text{ such that} \\ &\quad x_{t+1} \in \{(A+BK)^\tau x_{t+1-\tau}\} \oplus \gamma \mathcal{F}_\tau . \end{aligned} \tag{11}$$

Then the set $\gamma \mathcal{F}_\infty$ *is* θ*-UGAS for* $\mathbf{p}$.

Note that we make no assumptions on the structure or description of the dynamical system $\mathbf{p}$ in question. Next, we present event-triggered and self-triggered controllers that guarantee that for $t \in \mathbb{N}$ the set-membership condition (11) indeed holds. The controllers contain implicit set-membership conditions that guarantee the stability properties of the closed-loop system. In Section 5 we show how these conditions reduce to checking whether a finite number of points is included in certain given polytopes, thereby showing their straight-forward implementation.

Note that the variables $\theta \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and $\gamma \in [1,\infty)$ are design parameters of the control schemes. Here, γ is used to parameterize the set $\mathcal{Y}$ which is stabilized by the controller according to Definition 1.

3.1. Set-based event-triggered control

Define for $t \in \mathbb{N}_{\geq 1}$ the function δ_t in (4a) by

$$\delta_t(t_i, x_0, x_1, \ldots, x_t) = \begin{cases} 0, & \text{if } \exists \tau \in \{1,\ldots,\min\{\theta, t+1\}\}, \text{ such that} \\ & \quad \forall w_{0|t} \in \mathcal{W}, \\ & \qquad x_{1|t} \in \{(A+BK)^\tau x_{t+1-\tau}\} \oplus \gamma \mathcal{F}_\tau \\ 1, & \text{otherwise,} \end{cases} \tag{12}$$

where $x_{1|t} := Ax_t + B\kappa(t-t_i, x_{t_i}) + w_{0|t}$ plays the role of a prediction of the state at time $t+1$ under the assumption that the input $\kappa(t-t_i, x_{t_i})$ is applied and the disturbance $w_{0|t}$ acts on the system.[2]

Theorem 6. *Let* $\mathbf{p}$ *be the dynamical system generating* $(x_s)_{s\in\mathbb{N}}$ *for the closed loop consisting of* (1) and (4a), *where* δ_t, $t \in \mathbb{N}_{\geq 1}$, *is defined as in* (12). *Then, the set* $\gamma \mathcal{F}_\infty$ *is* θ*-UGAS for* $\mathbf{p}$.

3.2. Set-based self-triggered control

Define for $i \in \mathbb{N}$ the function s_i in (4a) by

$$\begin{aligned} &s_i(t_0,\ldots,t_i, x_{t_0},\ldots,x_{t_i}) \\ &\quad = \sup \Big\{ t \in \mathbb{N}_{\geq t_i+1} \,\Big|\, \forall l \in \{1,\ldots,t-t_i\} \\ &\qquad \exists \tau_l \in \{1,\ldots,\min\{\theta, t_i+l\}\}, \text{ such that} \\ &\qquad \forall (w_{t_j-t_i|t_i},\ldots,w_{0|t_i}, w_{1|t_i},\ldots,w_{l-1|t_i}) \\ &\qquad\quad \in \mathcal{W} \times \cdots \times \mathcal{W}, \\ &\qquad x_{l|t_i} \in \{(A+BK)^{\tau_l} x_{l-\tau_l|t_i}\} \oplus \gamma \mathcal{F}_{\tau_l} \Big\}, \end{aligned} \tag{13}$$

where $x_{k+1|t_i} := Ax_{k|t_i} + B\kappa(k-t_i, x_{t_i}) + w_{k|t_i}$ for all $k \in \{0,\ldots,l-1\}$, $x_{k+1|t_i} := Ax_{k|t_i} + B\kappa(k-t_j, x_{t_j}) + w_{k|t_i}$ for all $k \in \{t_j - t_i, \ldots, t_i + l - \tau_l - t_j\}$ with j such that $t_i + l - \tau_l \in \{t_j, \ldots, t_{j+1}-1\}$, $x_{0|t_i} := x_{t_i}$ and $x_{t_j - t_i|t_i} := x_{t_j}$. (For simplicity, we omit the dependence of j on τ_l in the notation). Similar to the event-triggered case, $w_{k|t_i}$, $k \in \{t_j - t_i, \ldots, l-1\}$, are assumed disturbances acting on the system and $x_{k|t_i}$ plays the role of a predicted future state (for $k \in \mathbb{N}_{\geq 1}$); however, as we assumed that measurements of the state are only available at transmission times, also states in the past need to be "predicted", starting from an earlier transmission time t_j. This becomes necessary if the time point $t_i + l - \tau_l$ used as a reference lies in the past, that is, $t_j < t_i$. Note that we only use the bounds $w_{k|t_i} \in \mathcal{W}$ for the disturbances in the scheduling functions s_i, both in the future and in the past. For the past disturbances, in the time span $\{t_j,\ldots,t_i-1\}$, one could obtain tighter bounds by evaluating the (known) states x_{t_j} and x_{t_i}. For example, if $t_i = t_j + 1$, then it holds that $w_{t_j} = x_{t_i} - Ax_{t_j} - Bu_{t_j}$, that is, w_{t_j} is known exactly. For simplicity, we do not make use of this additional information here.

Theorem 7. *Let* $\mathbf{p}$ *be the dynamical system generating* $(x_s)_{s\in\mathbb{N}}$ *for the closed-loop system consisting of* (1) and (5), *where* s_i, $i \in \mathbb{N}$, *is defined as in* (13). *Then, the set* $\gamma \mathcal{F}_\infty$ *is* θ*-UGAS for* $\mathbf{p}$.

Proof. The statement follows similarly as the proof of Theorem 6, noting that $x_{t_i+1} = (A+BK)x_{t_i} + w_{t_i} \in \{(A+BK)x_{t_i}\} \oplus \gamma \mathcal{F}_1$ in any case. □

Remark 8. (i) The rate of convergence, linked to the function β in the definition of θ-UGAS, depends only on the matrix $A+BK$, compare the proof of Theorem 5, and is independent of both θ and γ. (ii) The results in this section apply independently of the chosen κ generating the input to the system, as long as the assumption $\kappa(0,x) = Kx$ holds. In particular, the results hold for all the examples of κ in (6). (iii) Both the event-generating function in (12) and the scheduling function in (13) require knowledge of the way the input applied to the system is computed. One method to implement this is to include a copy of the controller (in the form of κ) in the trigger mechanism in Fig. 2. Alternatively, the inputs u_t for $t \in \{t_i,\ldots,t_{i+1}-1\}$ may be computed at time t_i at the sensor side and transmitted (in one packet) at time t_i to the actuator, such that the controller does not have to be implemented at the actuator side in any form, compare Lješnjanin, Quevedo, and Nešić (2014) and the references therein for similar architectures. Note that this is possible as, by assumption, the inputs in the time span $\{t_i,\ldots,t_{i+1}-1\}$ are a function of x_{t_i} (for the event-triggered case, one would need to additionally enforce an *a priori* bound on $t_{i+1}-t_i$, as t_{i+1} is not known at time t_i). As a third alternative, this input sequence might also be computed, at time t_i, at the controller side (after receiving x_{t_i} from the sensor) and then transmitted back to the trigger mechanism.

4. Output feedback

In this section, we show how the results obtained so far can be extended to the case that x_t is not directly available as a measurement. We restrict ourselves here to the event-triggered case. However, similar results can be obtained for self-triggered control. Consider the system

$$x_{t+1} = Ax_t + Bu_t + w_t \tag{14a}$$

$$y_t = C_t x_t + v_t, \tag{14b}$$

[2] With $w_{k|t}$ we denote a hypothetical disturbance, employed in the evaluation of a trigger condition at time t and which is assumed to act on the system at time $t+k$.

where, in addition to the definitions in Section 2, it holds that $y_t \in \mathbb{N}^q$ and $v_t \in \mathcal{V} \subseteq \mathbb{R}^q$ for $t \in \mathbb{N}$, $\mathcal{V}$ being a convex and compact set that contains the origin. Additionally, we assume that there exists a matrix $L \in \mathbb{R}^{n\times q}$ such that $A + LCA$ is Schur. Hence, the dynamic output-feedback controller

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t - L(y_{t+1} - C(A\hat{x}_t + Bu_t)) \tag{15a}$$
$$u_t = K\hat{x}_t \tag{15b}$$

stabilizes system (14). In particular, it holds that

$$\begin{bmatrix} x_t \\ \hat{x}_t \end{bmatrix} \in \left\{ \tilde{A}^{t-t^0} \begin{bmatrix} x_{t^0} \\ \hat{x}_{t^0} \end{bmatrix} \right\} \oplus \mathcal{H}_{t-t^0} \tag{16}$$

for the closed-loop system consisting of (14) and (15), and all $t \in \mathbb{N}$, $t^0 \in \{0, \dots, t\}$. Here,

$$\tilde{A} := \begin{bmatrix} A & BK \\ -LCA & A + LCA + BK \end{bmatrix} \tag{17}$$

and, for $i \in \mathbb{N} \cup \{\infty\}$ (compare Chisci & Zappa, 2002),

$$\mathcal{H}_i := \bigoplus_{j=0}^{i-1} \tilde{A}^j \begin{bmatrix} I & 0 \\ -LC & -L \end{bmatrix} (\mathcal{W} \times \mathcal{V}). \tag{18}$$

The results of Section 3 could be applied directly to the stabilization of the joint dynamics consisting of the system (14) and the observer (15a) if the joint state $(x_t, \hat{x}_t)$ was available as a measurement. As this is obviously not the case, we make the assumption that *set-valued* estimates in the form of sets $\mathcal{X}_t \subseteq \mathbb{R}^{(t+1)n}$ are available, where $(x_0, \dots, x_t) \in \mathcal{X}_t$ for all $t \in \mathbb{N}$. In Section 5.3, we provide an example of such an estimator.

4.1. Output-feedback event-triggered control

The event-triggered output-feedback controller takes the form

$$u_t = \kappa(t - t_i, \hat{x}_{t_i}), \text{ if } t \in \{t_i, \dots, t_{i+1} - 1\} \tag{19a}$$
$$t_0 = 0 \tag{19b}$$
$$t_{i+1} = \inf\{t \in \{t_i + 1, \dots\} \mid \delta_t(t_i, \hat{x}_0, \dots, \hat{x}_t, \mathcal{X}_t) = 1\} \tag{19c}$$

with

$$\delta_t(t_i, \hat{x}_0, \dots, \hat{x}_t, \mathcal{X}_t) = \begin{cases} 0 & \text{if } \exists \tau \in \{1, \dots, \min\{\theta, t+1\}\}, \text{ such that} \\ & \quad \forall w_{0|t} \in \mathcal{W},\ \forall v_{1|t} \in \mathcal{V},\ \forall (x'_0, \dots, x'_t) \in \mathcal{X}_t, \\ & \quad (x_{1|t}, \hat{x}_{1|t}) \in \{\tilde{A}^\tau (x'_{t+1-\tau}, \hat{x}_{t+1-\tau})\} \oplus \gamma\mathcal{H}_\tau \\ 1 & \text{otherwise,} \end{cases} \tag{20}$$

where[3]

$$\begin{bmatrix} x_{1|t} \\ \hat{x}_{1|t} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -LCA & A + LCA \end{bmatrix} \begin{bmatrix} x'_t \\ \hat{x}_t \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} \kappa(t - t_i, \hat{x}_{t_i}) + \begin{bmatrix} I & 0 \\ -LC & -L \end{bmatrix} \begin{bmatrix} w_{0|t} \\ v_{1|t} \end{bmatrix}. \tag{21}$$

Theorem 9. *Let* $\mathbf{p}$ *be the dynamical system generating* $((x_s, \hat{x}_s))_{s\in\mathbb{N}}$ *for the closed loop consisting of (14), (15a), (19), and (20). Then, the set* $\gamma\mathcal{H}_\infty$ *is* θ*-UGAS for* $\mathbf{p}$.

The proof follows analogously to the state-feedback case by noting that the scheme guarantees that for all $t \in \mathbb{N}$ there exists a $\tau \in \{1, \dots, \min\{\theta, t+1\}\}$ such that $(x_{t+1}, \hat{x}_{t+1}) \in \{\tilde{A}^{t+1-\tau}(x_{t+1-\tau}, \hat{x}_{t+1-\tau})\} \oplus \gamma\mathcal{H}_\tau$.

[3] Note that we use $(x_{1|t}, \hat{x}_{1|t})$ and $\begin{bmatrix} x_{1|t} \\ \hat{x}_{1|t} \end{bmatrix}$ interchangeably.

Remark 10. We emphasize here that the *quality* of the state estimates, that is, the size of the sets $\mathcal{X}_t$, is not important in order to guarantee stability, as long as it is guaranteed that the true system states are contained in $\mathcal{X}_t$. However, larger (that is, more conservative) estimates $\mathcal{X}_t$ make it more likely that the set-membership tests in the trigger conditions fail, resulting in shorter inter-transmission times $t_{i+1} - t_i$, and, presumably, in a higher average communication rate.

5. Implementation and simplifications

In this section, we describe how the proposed aperiodic controllers may be implemented and what type of computations have to be performed in order to evaluate the trigger conditions. In Section 5.2.1, we highlight some properties of model-based control and link our framework to earlier aperiodic schemes.

5.1. Bounded computation time

In order to implement the proposed algorithms, it is necessary to ensure that the maximum number of cases to be checked in the various trigger conditions is bounded. For the self-triggered setup, this requires a bound on the inter-transmission time $t_{i+1} - t_i$, which may have to be artificially enforced by replacing, in the case of state-feedback control, the expression $\sup\left\{t \in \mathbb{N}_{\geq t_i+1} \,\middle|\, \dots\right\}$ in (13) with $\sup\left\{t \in \{t_i+1, t_i+2, \dots, t_i+M\} \,\middle|\, \dots\right\}$ for some fixed $M \in \mathbb{N}$. Note that this does not change the stability guarantees of the closed-loop system; compare similar discussions in Barradas Berglind, Gommans, and Heemels (2012, Section 4).

The additional question arises how the case $\theta = \infty$ is to be handled. First note that checking the trigger conditions only for a *subset* of $\{1, \dots, \min\{t+1, \theta\}\}$ (and triggering if the set membership can never be ensured), still guarantees stability. Further, from the proof of Theorem 5, we see that the proposed trigger conditions ensure $x_{t+1} \in \{(A+BK)^{t+1}x_0\} \oplus \gamma\mathcal{F}_{t+1}$ for all $t \in \mathbb{N}$. Hence, if $\theta = \infty$, we can always reduce the trigger conditions to checking only the case $\tau = t+1$. Note, however, that this might lead to conservatism if information about the disturbances in the time-interval $\{0, \dots, t_i - 1\}$ is neglected, compare Section 3.2.

5.2. Set membership conditions in state-feedback case

We first consider the state-feedback case, where the trigger conditions reduce to checking whether a finite number of vectors are contained in convex sets.

For the event-triggered setup, the condition $\forall w_{0|t} \in \mathcal{W}, x_{1|t} \in \{(A+BK)^\tau x_{t+1-\tau}\} \oplus \gamma\mathcal{F}_\tau$ in (12) is equivalent to

$$Ax_t + B\kappa(t - t_i, x_{t_i}) - (A+BK)^\tau x_{t+1-\tau} \in \gamma\mathcal{F}_\tau \ominus \mathcal{W}, \tag{22}$$

where the left hand side can be computed from quantities known at time t. The set on the right hand side can be computed[4] offline when θ is finite. Hence, the event-generating function in (12) can equivalently (and more explicitly) written as

$$\delta_t(t_i, x_0, x_1, \dots, x_t) = \begin{cases} 0, & \text{if } \exists \tau \in \{1, \dots, \min\{\theta, t+1\}\}, \text{ such that} \\ & \quad Ax_t + B\kappa(t - t_i, x_{t_i}) - (A+BK)^\tau x_{t+1-\tau} \\ & \qquad \in \gamma\mathcal{F}_\tau \ominus \mathcal{W}, \\ 1, & \text{otherwise.} \end{cases} \tag{23}$$

[4] At least for the case of polytopic $\mathcal{W}$ reasonably efficient numerical methods exist.

For the self-triggered setup, consider first the case $t_i + l - \tau_l \geq t_i$, which implies

$$x_{l|t_i} = A^{\tau_l} x_{l-\tau_l|t_i} + \sum_{k=t_i+l-\tau_l}^{t_i+l-1} A^{t_i+l-1-k}(B\kappa(k-t_i, x_{t_i}) + w_{k-t_i|t_i}) \tag{24}$$

and

$$x_{l-\tau_l|t_i} = A^{l-\tau_l} x_{t_i} + \sum_{k=t_i}^{t_i+l-\tau_l-1} A^{t_i+l-\tau_l-1-k}(B\kappa(k-t_i, x_{t_i}) + w_{k-t_i|t_i}). \tag{25}$$

The set-membership condition in (13), in this case, is therefore equivalent to

$$\begin{aligned}&(A^{\tau_l} - (A+BK)^{\tau_l})\Bigg(\Bigg\{A^{l-\tau_l} x_{t_i} \\ &\quad + \sum_{k=t_i}^{t_i+l-\tau_l-1} A^{t_i+l-\tau_l-1-k} B\kappa(k-t_i, x_{t_i})\Bigg\} \oplus \mathcal{G}_{l-\tau_l}\Bigg) \\ &\quad \oplus \Bigg\{\sum_{k=t_i+l-\tau_l}^{t_i+l-1} A^{t_i+l-1-k} B\kappa(k-t_i, x_{t_i})\Bigg\} \oplus \mathcal{G}_{\tau_l} \subseteq \gamma \mathcal{F}_{\tau_l}\end{aligned} \tag{26}$$

where $\mathcal{G}_i = \bigoplus_{j=0}^{i-1} A^j \mathcal{W}$ for $i \in \mathbb{N}$. This set-inclusion condition can be equivalently expressed as a set-membership condition regarding the set $\gamma\mathcal{F}_{\tau_l} \ominus \left((A^{\tau_l} - (A+BK)^{\tau_l})\mathcal{G}_{l-\tau_l} \oplus \mathcal{G}_{\tau_l}\right)$, analogous to (22) in the event-triggered setup.

Consider now the case $t_i + l - \tau_l < t_i$. Here, we simplify the derivations by disregarding information that can be inferred for the bound on past disturbances from the knowledge of both x_{t_j} and $x_{t_{j+1}}$, where $j < i$ and $t_i + l - \tau_l \in \{t_j, \ldots, t_{j+1} - 1\}$. Hence, we use

$$x_{l|t_i} = A^l x_{t_i} + \sum_{k=t_i}^{t_i+l-1} A^{t_i+l-1-k}(B\kappa(k-t_i, x_{t_i}) + w_{k-t_i|t_i}) \tag{27}$$

and

$$x_{l-\tau_l|t_i} = A^{t_i+l-\tau_l-t_j} x_{t_j} + \sum_{k=t_j}^{t_i+l-1-\tau_l} A^{t_i+l-\tau_l-1-k}(B\kappa(k-t_j, x_{t_j}) + w_{k-t_i|t_i}), \tag{28}$$

while assuming $w_{k-t_i|t_i} \in \mathcal{W}$ for $k \in \{t_j, \ldots, t-1\}$. This leads to the condition

$$\begin{aligned}&\Bigg\{A^l x_{t_i} + \sum_{k=t_i}^{t_i+l-1} A^{t_i+l-1-k} B\kappa(k-t_i, x_{t_i}) \\ &\quad - (A+BK)^{\tau_l}\Bigg(A^{t_i+l-\tau_l-t_j} x_{t_j} + \sum_{k=t_j}^{t_i+l-\tau_l-1} A^{t_i+l-\tau_l-1-k} B\kappa(k-t_j, x_{t_j})\Bigg)\Bigg\} \\ &\quad \oplus (-(A+BK)^{\tau_l})\mathcal{G}_{t_i+l-\tau_l-t_j} \oplus \mathcal{G}_l \subseteq \gamma\mathcal{F}_{\tau_l}.\end{aligned} \tag{29}$$

In summary, the scheduling function in (13) becomes

$$\begin{aligned}&s_i(t_0, \ldots, t_i, x_{t_0}, \ldots, x_{t_i}) \\ &= \sup \Bigg\{ t \in \mathbb{N}_{\geq t_i+1} \;\Bigg|\; \forall l \in \{1, \ldots, t-t_i\}, \\ &\quad \exists \tau_l \in \{1, \ldots, \min\{\theta, l\}\}, \text{ such that} \\ &\quad (A^{\tau_l} - (A+BK)^{\tau_l})\Bigg(A^{l-\tau_l} x_{t_i} \\ &\quad + \sum_{k=t_i}^{t_i+l-\tau_l-1} A^{t_i+l-\tau_l-1-k} B\kappa(k-t_i, x_{t_i})\Bigg) \\ &\quad + \sum_{k=t_i+l-\tau_l}^{t_i+l-1} A^{t_i+l-1-k} B\kappa(k-t_i, x_{t_i}) \\ &\quad \in \gamma\mathcal{F}_{\tau_l} \ominus \left((A^{\tau_l} - (A+BK)^{\tau_l})\mathcal{G}_{l-\tau_l} \oplus G_{\tau_l}\right), \\ &\text{or} \quad \exists \tau_l \in \{\min\{\theta, l+1\}, \ldots, \min\{\theta, t_i+l\}\}, \\ &\quad \text{such that} \\ &\quad A^l x_{t_i} + \sum_{k=t_i}^{t_i+l-1} A^{t_i+l-1-k} B\kappa(k-t_i, x_{t_i}) \\ &\quad - (A+BK)^{\tau_l}\Bigg(A^{t_i+l-\tau_l-t_j} x_{t_j} \\ &\quad + \sum_{k=t_j}^{t_i+l-\tau_l-1} A^{t_i+l-\tau_l-1-k} B\kappa(k-t_j, x_{t_j})\Bigg) \\ &\quad \in \gamma\mathcal{F}_{\tau_l} \ominus \left((-(A+BK)^{\tau_l})\mathcal{G}_{t_i+l-\tau_l-t_j} \oplus \mathcal{G}_l\right), \\ &\quad \text{where } j \text{ is such that } t_i + l - \tau_l \in \{t_j, \ldots, t_{j+1} - 1\}\Bigg\}.\end{aligned} \tag{30}$$

Hence, similar to the event-triggered setup, the scheduling functions s_i can be evaluated by checking a finite number of set-memberships, where the points in question can be easily computed online and the sets in question can be computed offline.

5.2.1. The special case of model-based control

In this subsection, we take a closer look at some special cases of the framework presented in this paper; in particular, we assume in the following that κ is chosen model-based as in (6c).

Consider first the event-triggered case and the additional restriction that $\tau = t + 1 - t_i$ (requiring enforced triggering at t if otherwise $t_{i+1} - t_i > \theta$). Then, the condition in (22) becomes

$$x_t - (A+BK)^{t-t_i} x_{t_i} \in A^{-1}(\gamma\mathcal{F}_{t+1-t_i} \ominus \mathcal{W}), \tag{31}$$

which is a threshold-based trigger condition on the error between the state at time t and the state of a simulated undisturbed closed-loop system, initialized at the last transmission instant t_i, compare Brunner, Heemels, and Allgöwer (2015) and Lunze and Lehmann (2010).

This demonstrates that our trigger conditions provide a generalization of certain existing event-triggered schemes, as the above mentioned restriction on τ is, in general, not required in our approach. In other words, our trigger conditions provide a greater opportunity of *not* communicating at a given time.

Consider now the self-triggered case. For κ being chosen model-based, the inclusion in (26) reduces to $(A^{\tau_l} - (A+BK)^{\tau_l})\mathcal{G}_{l-\tau_l} \oplus \mathcal{G}_{\tau_l} \subseteq \gamma\mathcal{F}_{\tau_l}$, that is, it becomes independent of x_{t_i}. This implies that the self-triggered scheme results in *periodic* triggering, independent of the initial state or disturbances, in the case of κ chosen model-based and τ_l restricted by $t_i + l - \tau_l \geq t_i$.

On the other hand, for model-based κ, (29) becomes

$$\begin{aligned}&(A+BK)^l(x_{t_i} - (A+BK)^{t_i-t_j} x_{t_j}) \\ &\quad \in \gamma\mathcal{F}_{\tau_l} \ominus \left((-(A+BK)^{\tau_l})\mathcal{G}_{t_i+l-\tau_l-t_j} \oplus \mathcal{G}_l\right).\end{aligned}$$

Hence, a smaller deviation between x_{t_i} and $(A+BK)^{t_i-t_j} x_{t_j}$, related to smaller disturbances in the interval $\{t_j, \ldots, t_i - 1\}$, makes it more likely that the set membership holds and a larger value of $t_{i+1} - t_i$ is realized. Further, the trigger behavior does not necessarily become periodic (which will, in fact, be shown in the example section). This discussion shows that in order to exploit "benign" past disturbance realizations in (model-based) self-triggered control, the past evolution of the system has to be taken into account. Otherwise, the conditions in (26) only evaluate worst-case *future* disturbance realizations, which are independent of the current system state.

5.2.2. Complexity of involved sets

While all sets appearing in this section so far can be computed[5] offline if θ and, in the self-triggered case, additionally $t_{i+1} - t_i$, are bounded, their complexity might grow to undesirable levels. It is, however, always possible to replace the right hand side in (22) with a subset thereof, without destroying the stability guarantees of the closed-loop systems. The same holds analogously for the set-membership conditions in the self-triggered setup. If these inner approximations are of simple shapes, such as boxes or ellipsoids, the effort involved with evaluating the trigger conditions remains low. Further, we want to point out that due to $0 \in \mathcal{W}$, it holds that $\mathcal{F}_i \subseteq \mathcal{F}_j$ for $i \leq j$. Hence, one may replace $\mathcal{F}_\tau$ with $\mathcal{F}_{\bar{\tau}}$ for $\tau \geq \bar{\tau}$ and a fixed $\bar{\tau}$ in the implementation. Considering that $\mathcal{F}_i \overset{i\to\infty}{\to} \mathcal{F}_\infty$ anyway, a reasonably small $\bar{\tau}$ might be chosen without introducing too much conservatism.

5.3. Set membership conditions in output-feedback case

In the output-feedback case, we have to take into account the unknown system state x_t. In the event-triggered setup, measurements are available at every time, such that the estimation error $x_t - \hat{x}_t$ remains bounded. In fact, if bounds on the estimation error at time 0 are known *a priori*, it is possible to compute a compact set $\mathcal{E} \subseteq \mathbb{R}^n$ satisfying $x_t \in \{\hat{x}_t\} \oplus \mathcal{E}$ for all $t \in \mathbb{N}$, compare Chisci and Zappa (2002). Then, the set $\mathcal{X}_t$ takes the form $(\{\hat{x}_0\} \oplus \mathcal{E}) \times \cdots \times (\{\hat{x}_t\} \oplus \mathcal{E})$. Therefore, the set-membership constraints in the event-generating function (20) can be obtained in a similar fashion as those in Section 5.2; in particular, for $\tau \in \mathbb{N}_{\geq 2}$, we arrive at the condition

$$\forall x'_{t+1-\tau} \in \{\hat{x}_{t+1-\tau}\} \oplus \mathcal{E},\ \forall x'_t \in \{\hat{x}_t\} \oplus \mathcal{E},$$
$$\left\{\begin{bmatrix} A & 0 \\ -LCA & A+LCA \end{bmatrix}\begin{bmatrix} x'_t \\ \hat{x}_t \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix}\kappa(t-t_i, \hat{x}_{t_i})\right\}$$
$$\oplus \begin{bmatrix} I & 0 \\ -LC & -L \end{bmatrix}(\mathcal{W}\times\mathcal{V}) \subseteq \left\{\tilde{A}^\tau \begin{bmatrix} x'_{t+1-\tau} \\ \hat{x}_{t+1-\tau} \end{bmatrix}\right\} \oplus \gamma\mathcal{H}_\tau, \tag{32}$$

which is equivalent to the set-membership condition

$$\begin{bmatrix} I \\ I \end{bmatrix}\left(A\hat{x}_t + B\kappa(t-t_i, \hat{x}_{t_i}) - (A+BK)^\tau \hat{x}_{t+1-\tau}\right)$$
$$\in \gamma\mathcal{H}_\tau \ominus \left(\begin{bmatrix} I & 0 \\ -LC & -L \end{bmatrix}(\mathcal{W}\times\mathcal{V})\right.$$
$$\left.\oplus \begin{bmatrix} A & 0 \\ -LCA & A+LCA \end{bmatrix}(\mathcal{E}\times\{0\}) \oplus (-\tilde{A}^\tau)(\mathcal{E}\times\{0\})\right). \tag{33}$$

The left-hand side of this set-membership is, as in the state-feedback case, easily computable from known quantities at time t; the right-hand side is a set that only depends on τ and can be computed offline.

For $\tau = 1$ we can derive the less conservative condition

$$\begin{bmatrix} I \\ I \end{bmatrix}\left(A\hat{x}_t + B\kappa(t-t_i, \hat{x}_{t_i}) - (A+BK)\hat{x}_{t+1-\tau}\right)$$
$$\in \gamma\mathcal{H}_1 \ominus \left(\begin{bmatrix} I & 0 \\ -LC & -L \end{bmatrix}(\mathcal{W}\times\mathcal{V})\right.$$
$$\left.\oplus \left(\begin{bmatrix} A & 0 \\ -LCA & A+LCA \end{bmatrix} - \tilde{A}\right)(\mathcal{E}\times\{0\})\right), \tag{34}$$

equivalent to (using $x_{t+1-\tau} = x_t$),

$$\begin{bmatrix} I \\ I \end{bmatrix}\left(A\hat{x}_t + B\kappa(t-t_i, \hat{x}_{t_i}) - (A+BK)\hat{x}_{t+1-\tau}\right) \in (\gamma-1)\mathcal{H}_1. \tag{35}$$

[5] At least in the case of polytopic $\mathcal{W}$.

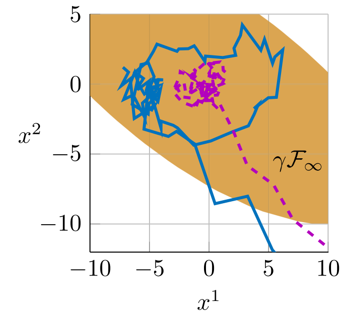

Fig. 3. Closed-loop trajectories under event-triggered control ("to-zero"-type) for $\gamma = 1.5$, $\theta = 1$ (dashed, magenta) and $\theta = \infty$ (solid, blue). An approximation of $\gamma\mathcal{F}_\infty$ is depicted in orange.

6. Numerical example

Consider the system

$$x_{t+1} = \begin{bmatrix} 1.0 & 0.3 \\ 0.0 & 1.0 \end{bmatrix} x_t + \begin{bmatrix} 0.045 \\ 0.300 \end{bmatrix} u_t + w_t, \tag{36}$$

obtained by discretizing a continuous double-integrator with a step size of 0.3, and where $w_t \in \mathcal{W} = [-1, 1] \times [-1, 1]$. The feedback gain K was computed to be LQ-optimal for the weighting matrices $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $R = 1$, that is, $K = [-0.77 \quad -1.46]$. We applied the controllers proposed in Section 3 for different choices of the parameters γ and θ, for all three controller types in (6). We chose $x_0 = (100, 100)^\top$ and simulated each parameter pairing for 10 random realizations of the disturbance sequence, where w_t was sampled independently and uniformly on $\mathcal{W}$ for all $t \in \{0, \ldots, T_{\text{sim}} - 1\}$ and the simulation horizon T_{sim} was chosen to 1000. To limit the computational complexity, we restricted $t_{i+1} - t_i$ in the self-triggered case to 20 and also replaced $\mathcal{F}_\tau$ with $\mathcal{F}_{20}$ for $\tau \in \mathbb{N}_{\geq 20}$ in the threshold definitions. In the evaluation of the scheduling function (13) we allowed both $t_i + l - \tau_l \geq t_i$ and $t_i + l - \tau_l < t_i$.

The resulting average communication rates (where a rate of 1 implies communication at every time) are reported in Tables 1 and 2. For the choice $\gamma = 1$ and $\theta = 1$, no communications can be saved. This is not surprising, as, considering only event-triggered control, the right hand side of (22) is $\{0\}$ for $\gamma = 1$ and $\tau = 1$, such that almost surely an event is generated at every given time (with the disturbances being uniformly distributed). However, the increase of *either* of the two parameters γ and θ allows a significant reduction of the average communication rate, for the price of worsening the stability guarantees (increased size of $\gamma\mathcal{F}_\infty$ for larger γ) and increasing the computational effort (more cases have to be checked in the trigger conditions for larger θ).

In order to illustrate the effect of the parameter θ on the stability properties of the closed-loop system, consider the state trajectories depicted in Fig. 3, resulting from the same disturbance realization (uniformly distributed on $\mathcal{W}$), but with different values for θ (we chose $\gamma = 1.5$ here). Consistent with Definition 1, for the $\theta = 1$ the state trajectory does not leave $\gamma\mathcal{F}_\infty$ once it has entered the set. For $\theta = \infty$, however, $\gamma\mathcal{F}_\infty$ is not positively invariant (which still is consistent with Definition 1).

Finally, we illustrate the points made on model-based self-triggered control in Section 5.2.1. For this, we simulated the closed-loop systems under self-triggered control both for an arbitrary selection of $\tau_l \in \{1, \ldots, \min\{\theta, t_i + l\}\}$ and for $\tau_l = l$ fixed. Here, we restricted the maximum time between transmission instants to $t_{i+1} - t_i \leq \theta - 1$, such that for both closed-loop systems the set $\gamma\mathcal{F}_\infty$ is guaranteed to be θ-UGAS. In particular, we chose

Table 1
Average communication rates in the closed loop with event-triggered control and uniformly distributed white-noise disturbances.

θ	to-zero control						θ	to-hold control						θ	model-based control					
	γ							γ							γ					
	1	1.25	1.5	2	2.5	3		1	1.25	1.5	2	2.5	3		1	1.25	1.5	2	2.5	3
1	1.00	0.56	0.33	0.15	0.10	0.08	1	1.00	0.51	0.28	0.19	0.18	0.19	1	1.00	0.47	0.21	0.09	0.06	0.05
2	0.48	0.26	0.17	0.11	0.07	0.06	2	0.47	0.25	0.20	0.19	0.20	0.20	2	0.44	0.19	0.11	0.07	0.04	0.04
5	0.20	0.14	0.10	0.07	0.05	0.05	5	0.22	0.20	0.20	0.21	0.21	0.21	5	0.14	0.08	0.06	0.04	0.03	0.03
10	0.15	0.10	0.08	0.06	0.04	0.04	10	0.21	0.21	0.21	0.22	0.22	0.22	10	0.09	0.06	0.05	0.03	0.03	0.02
∞	0.12	0.09	0.07	0.05	0.04	0.03	∞	0.21	0.21	0.22	0.22	0.22	0.22	∞	0.07	0.05	0.04	0.03	0.02	0.02

Table 2
Average communication rates in the closed loop with self-triggered control and uniformly distributed white-noise disturbances.

θ	to-zero control						θ	to-hold control						θ	model-based control					
	γ							γ							γ					
	1	1.25	1.5	2	2.5	3		1	1.25	1.5	2	2.5	3		1	1.25	1.5	2	2.5	3
1	1.00	1.00	1.00	0.62	0.48	0.37	1	1.00	1.00	1.00	0.52	0.46	0.36	1	1.00	1.00	1.00	0.50	0.33	0.33
2	1.00	1.00	0.68	0.51	0.38	0.31	2	1.00	1.00	0.58	0.50	0.36	0.30	2	1.00	1.00	0.50	0.50	0.33	0.25
5	0.81	0.57	0.48	0.36	0.29	0.23	5	0.69	0.50	0.45	0.34	0.28	0.24	5	0.72	0.50	0.42	0.33	0.25	0.20
10	0.59	0.49	0.39	0.29	0.24	0.20	10	0.50	0.44	0.34	0.28	0.24	0.22	10	0.51	0.44	0.33	0.25	0.20	0.17
∞	0.54	0.42	0.36	0.27	0.22	0.18	∞	0.50	0.34	0.34	0.26	0.23	0.21	∞	0.50	0.34	0.33	0.24	0.20	0.17

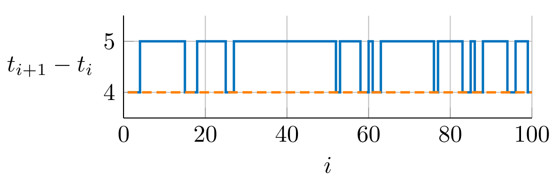

Fig. 4. Inter-transmission times $t_{i+1} - t_i$ for an arbitrary selection of τ_l (blue, solid) and for τ fixed (orange, dashed).

$\theta = 11$, $\gamma = 2.25$ and sampled the disturbance uniformly on $\mathcal{W}$. The resulting average transmission rate for an arbitrary selection of τ_l was 0.2095 and, for τ_l fixed, 0.25. Further, as shown in Fig. 4, the trigger behavior in the latter case is strictly periodic, whereas it is aperiodic in the former case.

YALMIP (Löfberg, 2004), the Multi-Parametric Toolbox 3.0 (Herceg, Kvasnica, Jones, & Morari, 2013), and IBM ILOG CPLEX Optimization Studio (IBM, 2014) were used in the simulations.

7. Discussion

The proposed controllers demonstrably achieve a reduction in communication with a trade-off between the stability properties of the closed-loop system (defined by the parameters θ and γ) and the average communication rate. Compared with earlier approaches (as, for example those in Boisseau et al., 2017; Grüne et al., 2010; Heemels et al., 2008; Lunze & Lehmann, 2010) the trigger conditions proposed in the present paper require, in general, a higher amount of storage and communication. For simplicity, we compare our approaches to one where (i) only a single trigger condition based on the state at the last communication instant is used and (ii) the trigger condition amounts to the evaluation of the norm of a point in the state space. Further, we restrict the discussion to the state-feedback event-triggered case. As shown in Section 5, the evaluation of the trigger conditions at each time step amounts to checking the set-membership (22) for every $\tau \in \{1, \ldots, \theta\}$. Here, the computation of the left-hand side is approximately of the same complexity as earlier "model-based" event-triggered schemes for example proposed in Lunze and Lehmann (2010). The right-hand side can be computed offline and represents a compact subset of the state-space. The complexity of sets of this form is known to grow exponentially with θ, leading to a far greater storage requirement when compared to approaches, where the trigger conditions are based on simple, constant sets based on vector norms. However, as pointed out in Section 5.2.2, one may employ *inner approximation* of right-hand side of (22), possibly based on vector norms, without deteriorating the stability guarantees of the closed-loop system. With such approximations in place, the storage and (online) computation effort involved with our schemes is approximately θ times that of earlier approaches such as those in Lunze and Lehmann (2010), while at the same time increasing the number of (non-)trigger conditions (of which only one needs to be fulfilled in order to prevent communication) also by a factor of θ.

8. Conclusions

We have presented a general framework for aperiodic control of perturbed discrete-time linear systems based on checking set-membership conditions. The framework encompasses many cases of interest and led to the design of both event-triggered and self-triggered controllers and both state-feedback and output-feedback schemes, and can be based on several control input generators including to-hold, to-zero and model-based predictions of input signals between event/transmission times. In particular, the output-feedback case was handled by considering the extended state space describing the original system and the state estimate obtained by well-designed observer structures. In fact, to describe the stability of the resulting closed-loop systems a new stability concept (θ-UGAS) for sets was introduced and connections to more classical notions for stability were revealed. The tuning parameters in our scheme are related to the size of the asymptotic state set (γ) and the complexity of the scheme (θ) and directly influence the transmission rates. Design trade-offs between these parameters, the asymptotic state set and the resulting transmission rates and comparison to existing schemes (such as periodic triggering and the model-based ETC schemes in Heemels & Donkers, 2013; Lunze & Lehmann, 2010) naturally emerge in our framework. The results were illustrated by numerical examples showing a significant reduction in the average transmission rate while ensuring an *a priori* chosen worst-case bound on the system state. Future research directions include the treatment of multiplicative disturbances, that is, time-varying uncertainties in the system matrices, the inclusion of disturbance estimators, and the extension to distributed control problems. Another research direction is the quantification of the computational effort involved with control schemes of the form presented here and the ensuing trade-off between control performance, communication, and computation. Finally, the impact of effects such as quantization, packet loss, and delay in various points in the network on the qualitative and quantitative properties of the control schemes should be investigated.

Appendix A. Equivalence of stability notions

A.1. Proof of Lemma 2

Let arbitrary $\theta \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and $(x_t)_{t\in\mathbb{N}}$ be given. Assume first that β is a $\mathcal{KL}$-function such that for all $t^0 \in \mathbb{N}$ and all $t \in \mathbb{N}_{\geq t^0}$ it holds that

$$|x_t|_{\mathcal{Y}} \leq \max_{\tau\in\{0,1,\dots,\min\{t^0,\theta-1\}\}} \beta(|x_{t^0-\tau}|_{\mathcal{Y}}, t - t^0 + \tau). \tag{A.1}$$

Let $t^0 \in \mathbb{N}$ and $t \in \mathbb{N}_{\geq t^0}$ be arbitrary but fixed. As β is decreasing in its second argument, it follows that $|x_t|_{\mathcal{Y}} \leq \max_{\tau\in\{0,1,\dots,\min\{t^0,\theta-1\}\}} \beta(|x_{t^0-\tau}|_{\mathcal{Y}}, t - t^0)$. Further, as β is increasing in its first argument, it also holds that $|x_t|_{\mathcal{Y}} \leq \beta(\max_{\tau\in\{0,1,\dots,\min\{t^0,\theta-1\}\}} |x_{t^0-\tau}|_{\mathcal{Y}}, t - t^0)$ and, hence, also that $|x_t|_{\mathcal{Y}} \leq \beta(\max_{\tau\in\{0,1,\dots,\theta-1\}} |x_{t^0-\tau}|_{\mathcal{Y}}, t - t^0)$. This shows that the implication holds as claimed, with $\hat{\beta} = \beta$.

Assume second that $\hat{\beta}$ is a $\mathcal{KL}$-function such that for all $t^0 \in \mathbb{N}$ and all $t \in \mathbb{N}_{\geq t^0}$ it holds that

$$|x_t|_{\mathcal{Y}} \leq \hat{\beta}\left(\max_{\tau\in\{0,1,\dots,\theta-1\}} |x_{t^0-\tau}|_{\mathcal{Y}}, t - t^0\right). \tag{A.2}$$

We consider the cases of finite and infinite θ separately; for now, we assume $\theta \in \mathbb{N}_{\geq 1}$. Let $t^0 \in \mathbb{N}$ and $t \in \mathbb{N}_{\geq t^0}$ be arbitrary but fixed. By the assumption on x_ρ for negative ρ, it follows that $|x_t|_{\mathcal{Y}} \leq \hat{\beta}(\max_{\tau\in\{0,1,\dots,\min\{t^0,\theta-1\}\}} |x_{t^0-\tau}|_{\mathcal{Y}}, t - t^0)$ and, using the fact that $\hat{\beta}$ is increasing in its first argument, that $|x_t|_{\mathcal{Y}} \leq \max_{\tau\in\{0,1,\dots,\min\{t^0,\theta-1\}\}} \hat{\beta}(|x_{t^0-\tau}|_{\mathcal{Y}}, t - t^0)$. Consider now the $\mathcal{KL}$-function β defined by

$$\beta(r,s) := \begin{cases} \dfrac{\theta-1}{s+1}\hat{\beta}(r,0) & s \in \{0,\dots,\theta-2\} \\ \hat{\beta}(r, s-\theta+1) & s \in \mathbb{N}_{\geq \theta-1} \end{cases} \tag{A.3}$$

for all $r \in \mathbb{R}_{\geq 0}$ and all $s \in \mathbb{N}$. With this definition, for all $r \in \mathbb{R}_{\geq 0}$ and all $s \in \mathbb{N}$ it holds that $\beta(r, s+\tau) \geq \hat{\beta}(r,s)$ for all $\tau \in \{0,\dots,\theta-1\}$. Hence, it holds that $|x_t|_{\mathcal{Y}} \leq \max_{\tau\in\{0,1,\dots,\min\{t^0,\theta-1\}\}} \beta(|x_{t^0-\tau}|_{\mathcal{Y}}, t - t^0 + \tau)$, showing the claimed implication also in the reverse direction for finite θ. For $\theta = \infty$, it is sufficient to show that (A.1) holds for $\tau = t_0$, that is $|x_t|_{\mathcal{Y}} \leq \beta(|x_0|, t)$. With the assumption that $x_\rho = x_0$ for $\rho \in \{-(\theta-1),\dots,-1\}$, however, (A.2) implies $|x_t|_{\mathcal{Y}} \leq \hat{\beta}\left(\max_{\tau\in\{0,1,\dots,\theta-1\}} |x_{0-\tau}|_{\mathcal{Y}}, t-0\right) = \hat{\beta}\left(|x_0|_{\mathcal{Y}}, t\right)$, such that the claimed implication holds with $\beta = \hat{\beta}$, thereby completing the proof. □

A.2. Counterexample for $x_{-1} \neq x_0$

In the following, we drop the assumption that $x_\rho = x_0$ for $\rho \in \{-(\theta-1),\dots,-1\}$ and provide an example where the origin is uniformly globally asymptotically stable when the system is viewed as a time-delay system with delay 1 but the origin is *not* θ-UGAS for the corresponding $\theta = 2$. Consider a dynamical system $\mathbf{q} : \mathbb{R}^2 \times (\mathcal{W})^{\mathbb{N}} \to (\mathbb{R})^{\mathbb{N}}$, with $x_{t+1} = \frac{1}{2}x_{t-1}$, $t \in \mathbb{N}$, that is

$$x_t = \begin{cases} (\dfrac{1}{2})^{\frac{t}{2}} & x_0 & t \text{ even} \\ (\dfrac{1}{2})^{\frac{t+1}{2}} & x_{-1} & t \text{ odd} \end{cases} \tag{A.4}$$

for all $t \in \mathbb{N}$. It follows that

$$\begin{aligned} |x_t| &\leq \max\left\{ \left(\frac{1}{2}\right)^{\frac{t-t^0}{2}} |x_{t^0}|, \left(\frac{1}{2}\right)^{\frac{t-t^0+1}{2}} |x_{t^0-1}| \right\} \\ &\leq \left(\frac{1}{2}\right)^{\frac{t-t^0}{2}} \max\{|x_{t^0}|, |x_{t^0-1}|\} \end{aligned} \tag{A.5}$$

for all $t^0 \in \mathbb{N}$ and all $t \in \mathbb{N}_{\geq t^0}$. Hence, the origin is uniformly globally asymptotically stable in the time-delay sense for the system. The definition of θ-UGAS (for any $\theta \in \mathbb{N}_{\geq 1} \cup \{\infty\}$) requires that there exists a $\mathcal{KL}$-function β with $|x_t| \leq \beta(|x_0|, t)$ for all $t \in \mathbb{N}$ (indeed, take $t^0 = 0$), implying that for all $x_0 \in \mathbb{R}$ there exists a $T \in \mathbb{N}$ with $|x_t| \leq \frac{1}{2}|x_0|$ for all $t \geq T$. For the system in (A.4), however, it holds that $\lim_{|x_{-1}|\to\infty} |x_t| = \infty$ for all odd $t \in \mathbb{N}$ and, hence, a $T \in \mathbb{N}$ with the properties above which depends only on x_0, but which is independent of x_{-1}, does not exist. Hence, the origin cannot be θ-UGAS for any θ for this system and, hence is, in particular, not 2-UGAS.

Appendix B. Proof of Theorem 5

We prove the statement by first establishing that the stated conditions imply that for all $t^0 \in \mathbb{N}$ and all $t \in \mathbb{N}_{\geq t^0}$, there exists a $t^{-1} \in \mathbb{N} \cap \{t^0 - \theta + 1, \dots, t^0\}$ such that $x_t \in \{(A+BK)^{t-t^{-1}} x_{t^{-1}}\} \oplus \gamma\mathcal{F}_{t-t^{-1}}$. Note that we may equivalently establish that the statement holds for all $t \in \mathbb{N}$ and all $t^0 \in \{0,\dots,t\}$, simplifying the following reasoning based on strong induction on $t \in \mathbb{N}$. In particular, we will prove that for all $t \in \mathbb{N}$ the following hypothesis holds:

$$\begin{aligned} &\forall t_0 \in \{0,\dots,t\},\ \exists t^{-1} \in \mathbb{N} \cap \{t^0 - \theta + 1, \dots, t_0\}, \\ &\quad x_t \in \{(A+BK)^{t-t^{-1}} x_{t^{-1}}\} \oplus \gamma\mathcal{F}_{t-t^{-1}}. \end{aligned} \tag{B.1}$$

Let $(x_s)_{s\in\mathbb{N}} = \mathbf{p}(x_0, (w_s)_{s\in\mathbb{N}})$ be arbitrary. The claimed set-membership condition $x_t \in \{(A+BK)^{t-t^{-1}} x_{t^{-1}}\} \oplus \gamma\mathcal{F}_{t-t^{-1}}$ holds for $t = 0$ and $t^0 = 0$ with $t^{-1} = 0$, providing the base case. Assume now that the hypothesis in (B.1) holds for all $\hat{t} \in \{0,\dots,t\}$ and an arbitrary $t \in \mathbb{N}$, that is, for all $\hat{t} \in \{0,\dots,t\}$ and all $\hat{t}^0 \in \{0,\dots,\hat{t}\}$ we have $x_{\hat{t}} \in \{(A+BK)^{\hat{t}-\hat{t}^{-1}} x_{\hat{t}^{-1}}\} \oplus \gamma\mathcal{F}_{\hat{t}-\hat{t}^{-1}}$ for some $\hat{t}^{-1} \in \mathbb{N} \cap \{\hat{t}^0 - \theta + 1, \dots, \hat{t}^0\}$. Consider the time point $t+1$, for which, by (11), it holds that $x_{t+1} \in \{(A+BK)^\tau x_{t+1-\tau}\} \oplus \gamma\mathcal{F}_\tau$ for some $\tau \in \{1,\dots,\min\{\theta, t+1\}\}$. Hence, for $t^0 \in \{t+1-\tau,\dots,t+1\}$ the choice $t^{-1} = t+1-\tau$ is sufficient for providing the inductive step for this case. Assume now that $t^0 \in \{0,\dots,t-\tau\}$. By the induction hypothesis (B.1), noting that $t+1-\tau \in \{0,\dots,t\}$, there exists a $\hat{t}^{-1} \in \mathbb{N} \cap \{t^0 - \theta + 1, \dots, t^0\}$ such that $x_{t+1-\tau} \in \{(A+BK)^{t+1-\tau-\hat{t}^{-1}} x_{t^{-1}}\} \oplus \gamma\mathcal{F}_{t+1-\tau-\hat{t}^{-1}}$. Using again the assumption that (11) holds for x_{t+1}, we obtain $x_{t+1} \in (A+BK)^\tau \Big(\{(A+BK)^{t+1-\tau-\hat{t}^{-1}} x_{t^{-1}}\} \oplus \gamma\mathcal{F}_{t+1-\tau-\hat{t}^{-1}}\Big) \oplus \gamma\mathcal{F}_\tau$. With (10), this implies $x_{t+1} \in \{(A+BK)^{t+1-\hat{t}^{-1}} x_{t^{-1}}\} \oplus \gamma\mathcal{F}_{t+1-\hat{t}^{-1}}$, such that $\hat{t}^{-1}$ has the desired properties, completing the inductive step.

Before we continue, we state the following fact which follows immediately from the definitions.

Lemma 11. *Let $x, y \in \mathbb{R}^n$ and let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$ be compact sets. Then it holds that $|x+y|_{\mathcal{X}\oplus\mathcal{Y}} \leq |x|_{\mathcal{X}} + |y|_{\mathcal{Y}}$.*

We proceed with the proof of Theorem 5. The following analysis is similar to the proof of Lemma 3.2 in Ghaemi, Sun, and Kolmanovsky (2008). As the matrix $A+BK$ is Schur, there exist scalars $c_1, c_2 \in (0,\infty)$, a matrix $P \in \mathbb{R}^{p\times n}$, and a scalar $\lambda \in [0,1)$ such that for all $x \in \mathbb{R}^n$ it holds that $c_1|x| \le |Px| \le c_2|x|$ and $|P(A+BK)x| \le \lambda|Px|$, see for example Lazar (2010) and Molchanov and Pyatnitskiy (1989). The condition $x_t \in \{(A+BK)^{t-t^{-1}}x_{t^{-1}}\} \oplus \gamma\mathcal{F}_{t-t^{-1}}$ implies that $x_t = (A+BK)^{t-t^{-1}}x_{t^{-1}} + f$ for some $f \in \gamma\mathcal{F}_{t-t^{-1}}$. From the stated properties above, we obtain

$$
\begin{aligned}
&|x_t|_{\gamma\mathcal{F}_\infty}\\
&\overset{(10)}{=} |(A+BK)^{t-t^{-1}}x_{t^{-1}} + f|_{\gamma(A+BK)^{t-t^{-1}}\mathcal{F}_\infty \oplus \gamma\mathcal{F}_{t-t^{-1}}}\\
&\overset{\text{Lemma 11}}{\le} |(A+BK)^{t-t^{-1}}x_{t^{-1}}|_{\gamma(A+BK)^{t-t^{-1}}\mathcal{F}_\infty} + |f|_{\gamma\mathcal{F}_{t-t^{-1}}}\\
&\overset{f\in\gamma\mathcal{F}_{t-t^{-1}}}{=} |(A+BK)^{t-t^{-1}}x_{t^{-1}}|_{\gamma(A+BK)^{t-t^{-1}}\mathcal{F}_\infty}\\
&= \min_{y\in\gamma(A+BK)^{t-t^{-1}}\mathcal{F}_\infty} |(A+BK)^{t-t^{-1}}x_{t^{-1}} - y|\\
&= \min_{y\in\gamma\mathcal{F}_\infty} |(A+BK)^{t-t^{-1}}(x_{t^{-1}} - y)|\\
&\le \frac{1}{c_1}\min_{y\in\gamma\mathcal{F}_\infty} |P(A+BK)^{t-t^{-1}}(x_{t^{-1}} - y)|\\
&\le \lambda^{t-t^{-1}}\frac{1}{c_1}\min_{y\in\gamma\mathcal{F}_\infty} |P(x_{t^{-1}} - y)|\\
&\le \lambda^{t-t^{-1}}\frac{c_2}{c_1}\min_{y\in\gamma\mathcal{F}_\infty} |x_{t^{-1}} - y|\\
&= \lambda^{t-t^{-1}}\frac{c_2}{c_1}|x_{t^{-1}}|_{\gamma\mathcal{F}_\infty}\\
&\le \max_{\tau\in\{0,\ldots,\min\{t^0,\theta-1\}\}} \lambda^{t-t^0+\tau}\frac{c_2}{c_1}|x_{t^0-\tau}|_{\gamma\mathcal{F}_\infty},
\end{aligned}
\tag{B.2}
$$

where the last line follows from the fact that $t^{-1} \in \mathbb{N} \cap \{t^0-\theta+1,\ldots,t^0\}$. Hence, as $(x_s)_{s\in\mathbb{N}}$, $t^0 \in \mathbb{N}$, and $t \in \mathbb{N}_{\ge t^0}$ were arbitrary, the requirements of Definition 1 hold with $\beta : (s,r) \mapsto \lambda^r\frac{c_2}{c_1}s$. $\square$

Appendix C. Proof of Theorem 6

In order to prove the statement, we establish that the conditions in Theorem 5 are satisfied. If $\delta_t(t_i, x_0, \ldots, x_t) = 0$, the condition $x_{t+1} \in \{(A+BK)^\tau x_{t+1-\tau}\} \oplus \gamma\mathcal{F}_\tau$ for some $\tau \in \{1,\ldots,\min\{\theta, t+1\}\}$ holds by definition. If $\delta_t(t_i, x_0, \ldots, x_t) = 1$, we have $t = t_i$, and, by assumption, $u_t = \kappa(0, x_t) = Kx_t$. Hence, it holds that $x_{t+1} = (A+BK)x_t + w_t \in \{(A+BK)x_t\} \oplus \mathcal{W}$. By convexity of $\mathcal{W}$ and the assumption that $0 \in \mathcal{W}$, it holds that $\mathcal{W} \subseteq \gamma\mathcal{F}_1$, such that $x_{t+1} \in \{(A+BK)^\tau x_{t+1-\tau}\} \oplus \gamma\mathcal{F}_\tau$ holds with $\tau = 1$, thereby completing the proof. $\square$

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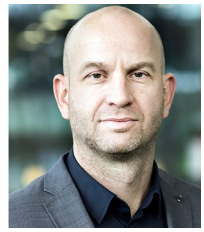

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