Automatica 72 (2016) 73–83

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Robust self-triggered MPC for constrained linear systems: A tube-based approach[☆]

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ARTICLE INFO

Article history:
Received 7 May 2015
Received in revised form
30 October 2015
Accepted 20 April 2016

Keywords:
Optimization-based controller synthesis
Networked control systems

ABSTRACT

We propose a robust self-triggered control algorithm for constrained linear discrete-time systems subject to additive disturbances based on MPC. At every sampling instant, the controller provides both the next sampling instant, as well as the inputs that are applied to the system until the next sampling instant. By maximizing the inter-sampling time subject to bounds on the MPC value function, the average sampling frequency in the closed-loop system is decreased while guaranteeing an upper bound on the performance loss when compared with an MPC scheme sampling at every point in time. Robust constraint satisfaction is achieved by tightening input and state constraints based on a Tube MPC approach. Moreover, a compact set in the state space, which is a parameter in the MPC scheme, is shown to be robustly asymptotically stabilized.

1. Introduction

For control systems where the communication between system and controller constitutes a considerable effort in terms of energy or infrastructure, the performance of the control system must be weighed against the amount of communication necessary to achieve this performance. In this context, it has been found that controllers with aperiodic scheduling of input and measurement updates may achieve a better trade-off between performance and overall communication load than controllers with periodic scheduling, see for example Heemels, Johansson, and Tabuada (2012) and You and Xie (2013) and the references therein. In particular, *event-triggered* and *self-triggered* control schemes have been proposed, where in the first class of controllers a new input is computed and communicated to the system only if certain conditions on the state of the system are met (defining an "event"), and in the second class the next update time is calculated explicitly at the current update time based on the current state of the system. The main difference between the two classes of controllers is that event-triggered control requires periodic or continuous measurement of the system state (or output) while in self-triggered control the sensors may be shut down completely between updates. For a recent overview of event-triggered and self-triggered control we refer the interested reader to Heemels et al. (2012). While self-triggered control schemes have the advantage of requiring overall less information from the system in general, this advantage at the same time makes these schemes more susceptible to disturbances and uncertainties when compared to event-triggered control schemes.

In this paper, we present a robust self-triggered MPC method based on ideas from Tube MPC (Chisci, Rossiter, & Zappa, 2001; Langson, Chryssochoos, Raković, & Mayne, 2004). MPC is a control method where the control input at each sampling instant is defined as the first part of the solution of a finite-horizon optimal control problem. MPC is especially suited for setups with hard constraints on the input and states, as these constraints can be explicitly taken into account in the definition of the optimization problem. For an overview of MPC, please refer to Mayne (2014), Mayne, Rawlings, Rao, and Scokaert (2000) and Rawlings and Mayne (2009). For linear time-invariant systems subject to bounded additive disturbances, Tube MPC has proven to be an effective way of robustifying MPC. Tube MPC is based on set-valued predictions of the state and input of the system taking the effect of the disturbances into account. A key ingredient in Tube MPC is the assumption that *feedback* is present at every point in time, reducing the effect of the disturbances and thereby restricting the growth of the predicted sets (Chisci et al., 2001). In the present work, the

[☆] The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Giancarlo Ferrari-Trecate under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2016.05.004

assumption of feedback at every point in time is not satisfied as we are explicitly designing controllers with extended periods of open-loop control. This limitation leads to a stronger growth of the uncertainty in the prediction. However, it holds that feedback will be present at *some* time in the future which still restricts the growth of the uncertainty in the predictions, as we will see. This knowledge is used in the construction of the tightened constraint sets, which extend those employed in Chisci et al. (2001). An additional challenge when adapting Tube MPC methods to a self-triggered setup is the fact that the asymptotic bound on the system state depends on the times between control updates, which are determined online. In standard Tube MPC, the times between control updates are uniform. We address this problem by carefully designing the MPC cost function and determining the times between control updates according to the evolution of this function. This enables us to provide an offline a priori asymptotic bound on the system state which is also a tuning parameter of the MPC scheme. Note that in Eqtami (2013), and the references therein, robust event- and self-triggered MPC schemes are proposed based on tubes where *no* feedback is assumed in the predictions. For open-loop unstable systems this has the drawback of leading to an exponential growth of the predicted uncertainty, thereby imposing an upper bound on the maximal prediction horizon if state constraints are present and reducing the feasible region of the MPC scheme. Inspired by Barradas Berglind, Gommans, and Heemels (2012) and Gommans, Antunes, Donkers, Tabuada, and Heemels (2014), the self-triggered controller in the present paper maximizes, at each sampling instant, the time until the next sampling instant subject to constraints on the associated MPC cost function and addresses the mentioned issue of exponentially growing uncertainty under open-loop predictions. These constraints on the MPC cost will enable us to prove that the cost of our new self-triggered MPC scheme is bounded by the cost associated with the solution of a standard periodically triggered MPC scheme multiplied by a positive factor which is a tuning knob of our scheme. Another tuning knob is the size of the set that is robustly stabilized. As a consequence, the proposed self-triggered MPC scheme allows trade-offs between closed-loop performance, the asymptotic bound on the system state, and the average communication rate.

Alternative MPC-based self-triggered control schemes are available. In Henriksson (2014) and Henriksson, Quevedo, Sandberg, and Johansson (2012), an MPC scheme for undisturbed systems is considered, where the sampling rate is part of the MPC cost function. In Antunes and Heemels (2014), an optimization-based scheme is proposed where at each sampling instant the input is decided by selecting an optimal scheduling *sequence* with respect to a quadratic cost function. Note that both Barradas Berglind et al. (2012), Henriksson (2014) and Henriksson et al. (2012) do not consider disturbances, while Antunes and Heemels (2014) and Gommans et al. (2014) consider disturbances but no constraints on the state or input. In Kögel and Findeisen (2014), a self-triggered scheme for disturbed systems under constraints was presented based on robust control-invariant sets. However, neither stability, nor performance is addressed. In earlier results on self-triggered MPC for disturbed systems (Aydiner, 2014; Brunner, Heemels, & Allgöwer, 2014), the asymptotic bound depended on the optimal MPC cost function, which is usually not easily obtainable. In Aydiner, Brunner, Heemels, and Allgöwer (2015), a robust self-triggered MPC scheme based on Raković, Kouvaritakis, Findeisen, and Cannon (2012) was presented, which allows a similar a priori determination of the asymptotic bound, while employing a conceptionally different way of describing the uncertainties in the prediction. The MPC schemes proposed in Eqtami (2013) allow an a priori determination of the guaranteed asymptotic bound in the form of an ellipsoidal set, which is a conservative restriction for the linear systems considered in the present paper.

The remainder of the paper is structured in the following way. Some notes on notation and some preliminary results and definitions are given in Section 2. The problem setup is stated in Section 3. In Section 4, a Tube MPC optimization problem is defined, where the first steps in the prediction horizon are assumed to be applied in an open-loop fashion. The main results of the paper are given in Section 5, where the robust self-triggered scheme and its properties are presented. Section 6 contains some notes on the implementation and the complexity of the algorithm. Section 7 concludes the paper.

For the sake of readability, most of the proofs are located in the Appendix.

2. Notation and preliminaries

Let $\mathbb{N}$ denote the set of non-negative integers. For $q, s \in \mathbb{N} \cup \{\infty\}$, let $\mathbb{N}_{\geq q}$ and $\mathbb{N}_{[q,s]}$ denote the sets $\{r \in \mathbb{N} \mid r \geq q\}$ and $\{r \in \mathbb{N} \mid q \leq r \leq s\}$, respectively. The set of non-negative real numbers is denoted by $\mathbb{R}_+$. For $n \in \mathbb{N}$, I_n denotes the $n \times n$ identity matrix. A matrix with zero entries is denoted by 0, where the dimension is defined by context. Given sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$, a scalar α, and matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$ we define $\alpha\mathcal{X} := \{\alpha x \mid x \in \mathcal{X}\}$, $A\mathcal{X} := \{Ax \mid x \in \mathcal{X}\}$, and $B^{-1}\mathcal{X} := \{x \in \mathbb{R}^m \mid Bx \in \mathcal{X}\}$. The Minkowski set addition is defined by $\mathcal{X} \oplus \mathcal{Y} := \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$. Given a vector $x \in \mathbb{R}^n$ we define $\mathcal{X} \oplus x := x \oplus \mathcal{X} := \{x\} \oplus \mathcal{X}$. The Pontryagin set difference (Kolmanovsky & Gilbert, 1995, 1998) is defined by $\mathcal{X} \ominus \mathcal{Y} := \{z \in \mathbb{R}^n \mid z \oplus \mathcal{Y} \subseteq \mathcal{X}\}$. Given a (finite or infinite) sequence of sets $\mathcal{X}_i$ for $i \in \mathbb{N}_{[a,b]}$ with $a \in \mathbb{N}$ and $b \in \mathbb{N} \cup \{\infty\}$, we define $\bigoplus_{i=a}^{b} \mathcal{X}_i := \left\{\sum_{i=a}^{b} x_i \mid x_i \in \mathcal{X}_i\right\}$. By convention, the empty sum is equal to $\{0\}$. Similarly, for any vectors $v_i \in \mathbb{R}^n$, $i \in \mathbb{N}$, we define $\sum_{i=a}^{b} v_i = 0$ for any $a, b \in \mathbb{N}$ if $a > b$. We call a compact, convex set containing the origin a *C*-set. A *C*-set containing the origin in its (non-empty) interior is called a *PC*-set. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. If additionally $\alpha(s) \to \infty$ as $s \to \infty$, α is said to belong to class $\mathcal{K}_\infty$. The Euclidean norm of a vector $v \in \mathbb{R}^n$ is denoted by $|v|$. Given any compact set $\mathbb{S} \subseteq \mathbb{R}^n$, the distance between v and $\mathbb{S}$ is defined by $|v|_{\mathbb{S}} := \min_{s\in\mathbb{S}} |v - s|$. The convex hull of a set $\mathcal{X} \subseteq \mathbb{R}^n$ is denoted by $\mathrm{convh}(\mathcal{X})$. Define finally the Euclidean unit ball by $\mathcal{B} := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

Lemma 1. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^n$ be compact convex sets. Let further $A \in \mathbb{R}^{m\times n}$. Then it holds that $\mathcal{X} \oplus \mathcal{Y} = \mathcal{Y} \oplus \mathcal{X}$, $\mathcal{X} \ominus (\mathcal{Y} \oplus \mathcal{Z}) = (\mathcal{X} \ominus \mathcal{Y}) \ominus \mathcal{Z}$, $(\mathcal{X} \oplus \mathcal{Y}) \ominus \mathcal{Y} = \mathcal{X}$, $(\mathcal{X} \ominus \mathcal{Y}) \oplus \mathcal{Y} \subseteq \mathcal{X}$, $A(\mathcal{X} \oplus \mathcal{Y}) = A\mathcal{X} \oplus A\mathcal{Y}$, and $A(\mathcal{X} \ominus \mathcal{Y}) \subseteq (A\mathcal{X} \ominus A\mathcal{Y})$.*

Next, we define stability properties of dynamical systems subject to disturbances of the form

$$(x_{k+1}^\top, z_{k+1}^\top)^\top = f(x_k, z_k, w_k), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathcal{W} \to \mathbb{R}^n$, $k \in \mathbb{N}$, are given, and $x_k \in \mathbb{R}^n$ and $w_k \in \mathcal{W} \subseteq \mathbb{R}^n$, are the state and disturbance at time $k \in \mathbb{N}$, and $z_k \in \mathbb{R}^p$ is an internal state of the controller with $z_0 = 0$.

Definition 2 (*Robust Lyapunov Stability of Sets*)**.** A set $\mathcal{Y} \subseteq \mathbb{R}^n$ is *robustly Lyapunov stable* for System (1) if there exist a $\mathcal{K}$-function γ and a $\delta > 0$ such that for any initial condition $x_0 \in \{x \in \mathbb{R}^n \mid |x|_{\mathcal{Y}} \leq \delta\}$ and any disturbances with $w_k \in \mathcal{W}$, $k \in \mathbb{N}$, it holds that $|x_k|_{\mathcal{Y}} \leq \gamma(|x_0|_{\mathcal{Y}})$ for all $k \in \mathbb{N}$.

Definition 3 (*Robust Asymptotic Stability of Sets*)**.** A set $\mathcal{Y} \subseteq \mathbb{R}^n$ is *robustly asymptotically stable* for System (1) with $\hat{\mathcal{X}} \subseteq \mathbb{R}^n$ *belonging to its region of attraction* if it is robustly Lyapunov stable for this system and $\lim_{k\to\infty} |x_k|_{\mathcal{Y}} = 0$ for all $x_0 \in \hat{\mathcal{X}}$, and any disturbances with $w_k \in \mathcal{W}$, $k \in \mathbb{N}$.

Definition 4 (*Robustly Positive Invariant Sets*). A set $\mathcal{Y} \subseteq \mathbb{R}^n$ is *robustly positive invariant* (RPI) for System (1) with $p = 0$, if for every $x \in \mathcal{Y}$, every $z \in \mathbb{R}^p$, and every $w \in \mathcal{W}$ it holds that $f(x, z, w) \in \mathcal{Y}$.

3. Problem formulation

We consider discrete-time linear time-invariant systems subject to bounded additive disturbances given by

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, and $w_k \in \mathcal{W} \subseteq \mathbb{R}^n$ denote the state, control input, and unknown disturbance, respectively, at discrete time $k \in \mathbb{N}$. Moreover, $\mathcal{W}$ is a known C-set. The state x_k is available as a measurement.

Our goal is to robustly asymptotically stabilize a set containing the origin of (2) while satisfying the constraints $x_k \in \mathcal{X}$ and $u_k \in \mathcal{U}$ for all $k \in \mathbb{N}$, where $\mathcal{X}$ and $\mathcal{U}$ are PC-sets. We want to achieve this goal by implementing a control law for which only sporadic measurements of the system state are necessary, thereby reducing the overall communication load in the control system. Furthermore, we want to guarantee an upper bound for the closed-loop performance that is not worse than the upper bound guaranteed by a periodically updated MPC scheme, multiplied by a positive factor of our choice. In the closed-loop system, the input is given by

$$u_k = \kappa(x_{k_j}, k - k_j), \quad k \in \mathbb{N}_{[k_j,k_{j+1}-1]}, \ j \in \mathbb{N}, \tag{3}$$

for a function $\kappa : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m$. The state measurements are only updated at the time points $k_j \in \mathbb{N}$, called sampling instants henceforth. These sampling instants shall be determined in a self-triggered fashion, that is,

$$k_0 = 0, \quad k_{j+1} = k_j + \mu(x_{k_j}), \quad j \in \mathbb{N}, \tag{4}$$

for a function $\mu : \mathbb{R}^n \to \mathbb{N}_{\geq 1}$. Hence, the problem addressed in this paper is finding functions κ and μ such that the closed-loop system (2)–(4) exhibits the properties mentioned above. Note that the closed-loop system composed of (2), (3), and (4) can be written in the form of (1) with $z_k := (x_{k_j}^\top - x_k^\top, k - k_j)^\top$ for all $k \in \mathbb{N}_{[k_j,k_{j+1}-1]}$ and $j \in \mathbb{N}$.

Remark 5. We allow the input to be time-varying between sampling instants. However, the input is open-loop in the sense that it is only allowed to depend on the state at the last sampling instant. If the definition of the feedback law in (3) is changed to $u_k = \kappa(x_{k_j}, k_j)$, $k \in \mathbb{N}_{[k_j,k_{j+1}-1]}$, $j \in \mathbb{N}$, then the input only changes at the sampling instants, which further reduces the amount of communication needed in the system (in this case in the controller-to-actuator channel). Changing the requirement to $u_k = \kappa(x_{k_j}, k_j)$, $k \in \{k_j \mid j \in \mathbb{N}\}$, $u_k = 0$, else, promotes sparsity in the input signal in addition to reducing the amount of communication. Please refer to Gommans and Heemels (2015, Sections 3.1 and 3.2), for an extended discussion of this matter.

4. *M*-step open-loop tube MPC

In this paper, we present a solution to the problem described in Section 3 based on MPC. In particular, at a given sampling instant k_j, and a system state x_{k_j}, the controller will provide a positive integer $M = \mu(x_{k_j})$ and a *sequence* of control inputs $(u_{k_j}, \ldots, u_{k_j+M-1}) = (\kappa(x_{k_j}, 0), \ldots, \kappa(x_{k_j}, M-1))$, which are then applied to the system (2) at the time points $k_j, \ldots, k_j + M - 1$. Hence, at the time points $k_j+1, \ldots, k_j+M-1$ the system can be considered to be controlled in an open-loop fashion. At the next sampling instant, defined by $k_{j+1} = k_j + M$, the next sequence of inputs is calculated. The input sequences are obtained by solving MPC problems, that is, they each are defined as the first M inputs of the solution to a finite-horizon optimal control problem, parameterized by the state x_{k_j}. The inter-sampling time M is chosen to be maximal, subject to constraints on the worst-case performance of the closed-loop system.

In order to guarantee robust constraint satisfaction, Tube MPC methods as proposed in Chisci et al. (2001) are employed. In Tube MPC, set-valued predictions of the (uncertain) system state are made under the assumption that the input at future time instances will include feedback reacting to the disturbances. The assumption of feedback is incorporated by parameterizing the input as $u_k = \bar{u}_k + Kx_k$, $k \in \mathbb{N}$, with the stabilizing feedback matrix $K \in \mathbb{R}^m \times \mathbb{R}^n$, and the new input $\bar{u}_k$. The following assumption holds in the remainder of the paper.

Assumption 6. The eigenvalues of the matrix $A+BK$ are contained in the interior of the unit disc.

The main problem when applying Tube MPC methods in a self-triggered setup is that the inputs are necessarily applied in an open-loop fashion until the next scheduled sampling instant, and hence cannot react to disturbances during this time span. This requires modifications of the standard Tube MPC approaches.

In Section 4.1, constraints on the predicted input sequence are defined that guarantee robust constraint satisfaction. The first M inputs in the sequence are assumed to be applied to the system in an open-loop fashion. For the remaining $N - M$ steps in the prediction horizon N it is assumed that feedback, based on the affinely parameterized control law mentioned above, is present. In order to guarantee robust stability properties and performance bounds on the closed-loop system, a cost function is introduced in Section 4.2. The finite-horizon optimal control problems used in the MPC scheme are discussed in Section 4.3. In Section 4.4, properties of the optimal cost functions are provided, useful for guaranteeing stability and performance properties of the closed-loop system later.

4.1. Feasibility problem

We first only consider the feasibility problem, that is, we define constraints on the predicted input sequence that guarantee recursive feasibility and robust constraint satisfaction for the closed-loop system.

Let the decision variable of the finite-horizon feasibility problem at time point k be given by

$$\mathbf{d}_k^{\mathrm{fp}} = ((x_{0|k}, \ldots, x_{N|k}), (u_{0|k}, \ldots, u_{N-1|k})) \in \mathbb{D}_N^{\mathrm{fp}}, \tag{5}$$

where $\mathbb{D}_N^{\mathrm{fp}} = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$ and $N \in \mathbb{N}_{\geq 1}$ is the prediction horizon.

Depending on the number $M \in \mathbb{N}_{[1,N]}$ of open-loop steps, different constraints are imposed on $\mathbf{d}_k^{\mathrm{fp}}$. In particular, for a given system state x_k at time point k we impose the constraints

$$x_{0|k} = x_k, \tag{6a}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}, \ x_{i+1|k} = Ax_{i|k} + Bu_{i|k}, \tag{6b}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}, \quad x_{i|k} \in \mathcal{X}_i^M, \tag{6c}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}, \quad u_{i|k} \in \mathcal{U}_i^M, \tag{6d}$$

$$x_{N|k} \in \mathcal{X}_{\mathrm{f}}^M \tag{6e}$$

on the decision variable $\mathbf{d}_k^{\mathrm{fp}}$, where the variables $x_{i|k}$ represent a predicted trajectory for the undisturbed system generated by the inputs $u_{i|k}$. The sets $\mathcal{X}_i^M$ and $\mathcal{U}_i^M$, $i \in \mathbb{N}_{[0,N-1]}$, are tightened constraint sets, each depending on the step i in the prediction and the number M of open-loop steps. The sets $\mathcal{X}_{\mathrm{f}}^M$ is a terminal set.

Define the set of all feasible decision variables for a given point $x_k \in \mathbb{R}^n$ and a fixed M by

$$\mathcal{D}_N^{M,\mathrm{fp}}(x_k) = \{\mathbf{d}_k^{\mathrm{fp}} \in \mathbb{D}_N^{\mathrm{fp}} \mid \text{(6a)–(6e)}\}. \tag{7}$$

In the following, the tightened constraint sets $\mathcal{X}_i^M$ and $\mathcal{U}_i^M$ will be defined, such that the application of the predicted control inputs ensures satisfaction of the constraints $(x_{k+i}, u_{k+i}) \in \mathcal{X} \times \mathcal{U}$ for all predicted time steps $k+i$, $i \in \mathbb{N}_{[0,N-1]}$. In particular, it is assumed that for the first M time steps in the prediction the inputs $u_{i|k}$ are applied in an open-loop fashion, while for all time steps after that feedback, defined by the matrix K (see Assumption 6), is present. Define for all $M \in \mathbb{N}_{[1,N]}$

$$\mathcal{X}_i^M := \mathcal{X} \ominus \mathcal{F}_i^M, \quad i \in \mathbb{N}_{[0,N-1]} \tag{8a}$$

$$\mathcal{U}_i^M := \begin{cases} \mathcal{U}, & i \in \mathbb{N}_{[0,M-1]} \\ \mathcal{U} \ominus K\mathcal{F}_i^M, & i \in \mathbb{N}_{[M,N-1]}. \end{cases} \tag{8b}$$

The sets $\mathcal{F}_i^M \subseteq \mathbb{R}^n$, $i \in \mathbb{N}_{[0,N-1]}$, are used to describe a tube containing all possible future trajectories around the nominal state trajectory given by $x_{i|k}$, produced by the nominal inputs $u_{i|k}$.

For all $M \in \mathbb{N}_{[1,N]}$ define

$$\mathcal{F}_i^M := \begin{cases} \bigoplus_{j=0}^{i-1} A^j \mathcal{W}, & i \in \mathbb{N}_{[0,M]} \\ (A+BK)^{i-M}\left(\bigoplus_{j=0}^{M-1} A^j \mathcal{W}\right) & \\ \quad \oplus \bigoplus_{j=0}^{i-M-1} (A+BK)^j \mathcal{W}, & i \in \mathbb{N}_{\geq M+1}. \end{cases} \tag{9}$$

For $M = 1$, the definition of $\mathcal{F}_i^M$, i.e. $\mathcal{F}_i^1 = \bigoplus_{j=0}^{i-1}(A+BK)\mathcal{W}$, matches the definition of the tube in Chisci et al. (2001). The satisfaction of the tightened constraints in (6) guarantees robust satisfaction of the state and input constraints.

Lemma 7. *Let any $M \in \mathbb{N}_{[1,N]}$ and any decision variable $\mathbf{d}_k^{fp} = ((x_{0|k}, \ldots, x_{N|k}), (u_{0|k}, \ldots, u_{N-1|k})) \in \mathbb{D}_N^{fp}$ satisfying (6a)–(6d) be given. Let further $x_{k+i+1} = Ax_{k+i} + Bu_{k+i} + w_{k+i}$, where $w_{k+i} \in \mathcal{W}$ and*

$$u_{k+i} = \begin{cases} u_{i|k}, & i \in \mathbb{N}_{[0,M-1]} \\ u_{i|k} + K(x_{k+i} - x_{i|k}), & i \in \mathbb{N}_{[M,N-1]} \end{cases} \tag{10}$$

for $i \in \mathbb{N}_{[0,N-1]}$. Then it holds that $x_{k+i} \in x_{i|k} \oplus \mathcal{F}_i^M$ for $i \in \mathbb{N}_{[0,N]}$ and $u_{k+i} \in u_{i|k} \oplus K\mathcal{F}_i^M$ for $i \in \mathbb{N}_{[M,N-1]}$. Furthermore, it holds that $x_{k+i} \in \mathcal{X}$ and $u_{k+i} \in \mathcal{U}$ for $i \in \mathbb{N}_{[0,N-1]}$.

Remark 8. From (9) it follows that for all $M \in \mathbb{N}_{[1,N]}$ and all $i \in \mathbb{N}_{\geq M}$ it holds that

$$\mathcal{F}_i^M = (A+BK)^{i-M}\mathcal{F}_M^M \oplus \mathcal{F}_{i-M}^1. \tag{11}$$

The terminal constraint (6e) is included in order to make the constraints in (6) *recursively feasible* in the following sense. If, starting at a given state x_k at time point k, and an input sequence satisfying the constraints in (6) for a given M is applied to the system in (2) for M steps, then, at the resulting state x_{k+M}, the existence of *some* input sequence satisfying the constraints in (6) for $M = 1$ is guaranteed. In order to ensure this property, the terminal sets $\mathcal{X}_\mathrm{f}^M$ are required to satisfy the following assumption.

Assumption 9. For all $M \in \mathbb{N}_{[1,N]}$ it holds that $\mathcal{X}_\mathrm{f}^M$ is a compact and convex set that satisfies

$$\forall i \in \mathbb{N}_{[0,M-1]}, \quad (A+BK)^i\mathcal{X}_\mathrm{f}^M \oplus \mathcal{F}_{N+i}^M \subseteq \mathcal{X} \tag{12a}$$

$$\forall i \in \mathbb{N}_{[0,M-1]}, \quad K(A+BK)^i\mathcal{X}_\mathrm{f}^M \oplus K\mathcal{F}_{N+i}^M \subseteq \mathcal{U} \tag{12b}$$

$$(A+BK)^M\mathcal{X}_\mathrm{f}^M \oplus (A+BK)^N\mathcal{F}_M^M \subseteq \mathcal{X}_\mathrm{f}^1 \tag{12c}$$

Remark 10. For the set $\mathcal{X}_\mathrm{f}^1$, which is equivalent to the terminal set used in Chisci et al. (2001), Assumption 9, requires $\mathcal{X}_\mathrm{f}^1$ to be an RPI set for the dynamics defined by $(A+BK)$ and the disturbance set $(A+BK)^N\mathcal{W}$. If the set $\mathcal{X}_\mathrm{f}^1$ has been defined, the other terminal sets $\mathcal{X}_\mathrm{f}^M$, $M \in \mathbb{N}_{\geq 2}$, can be calculated by simply intersecting the constraints defined in (12).

The property of recursive feasibility is formalized in the following lemma.

Lemma 11. *Let any $M \in \mathbb{N}_{[1,N]}$ and any decision variable $\mathbf{d}_k^{fp} = ((x_{0|k}, \ldots, x_{N|k}), (u_{0|k}, \ldots, u_{N-1|k})) \in \mathcal{D}_N^{M,fp}(x_k)$ be given. Let further $x_{k+i+1} = Ax_{k+i} + Bu_{k+i} + w_{k+i}$, where $w_{k+i} \in \mathcal{W}$ and $u_{k+i} = u_{i|k}$ for $i \in \mathbb{N}_{[0,M-1]}$. Then $\mathcal{D}_N^{1,fp}(x_{k+M}) \neq \emptyset$.*

4.2. Cost functions

For a disturbed system, the best result that can be achieved in terms of stability is the robust stabilization of an RPI set $\mathcal{Y} \subseteq \mathbb{R}^n$. We expect the size of this set to be traded off with the average inter-sampling time in the closed-loop system. In order to make this trade-off accessible in the design phase, the set $\mathcal{Y}$ is chosen to be a parameter in the MPC scheme. For simplicity, we choose $\mathcal{Y}$ to be an RPI set for system (2) in closed-loop with the feedback law $u_k = Kx_k$ (see Assumption 6). Both the performance specification and the cost function are defined in terms of this set. In particular, we consider the infinite-horizon performance index

$$V_\infty(x_0) := \sum_{k=0}^{\infty} \min_{\substack{y_k \in \mathcal{Y} \\ v_k \in K\mathcal{Y}}} \ell(x_k - y_k, u_k - v_k) \tag{13}$$

for system (2) in closed-loop with the self-triggered controller and initial condition x_0, with the stage cost function $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. We expect the performance of the closed-loop system to depend on the length M of the open-loop phase. In order to make the finite-horizon cost functions for different M comparable in the MPC scheme, we use different stage and terminal cost functions for different values of M. Essentially, we will define the stage and terminal cost functions such that the worst case deviation (defined by the function ℓ) of the cross section of the tube from the sets $\mathcal{Y}$ and $K\mathcal{Y}$, respectively, is penalized. Define the stage cost functions $\bar{\ell}_i^M : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i \in \mathbb{N}_{[0,N-1]}$, and the terminal cost functions $\bar{V}_\mathrm{f}^M : \mathbb{R}^n \to \mathbb{R}$, each for $M \in \mathbb{N}_{[1,N]}$, where

$$\bar{\ell}_i^M(x,u) := \begin{cases} \min\limits_{\substack{y \in \mathcal{Y}_i^M \\ v \in \mathcal{V}_i^M}} \max\limits_{e \in \mathcal{E}_i^M} \ell_i^M(x - y + e, u - v) & i \in \mathbb{N}_{[0,M-1]} \\ \min\limits_{y \in \mathcal{Y}_i^M} \max\limits_{e \in \mathcal{E}_i^M} \ell_i^M(x - y + e, u - Ky + Ke) & \\ \qquad i \in \mathbb{N}_{[M,N-1]} & \end{cases} \tag{14a}$$

$$\bar{V}_\mathrm{f}^M(x) := \min_{y \in \mathcal{Y}_N^M} V_\mathrm{f}^M(x - y), \tag{14b}$$

for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, where the sets $\mathcal{Y}_i^M \subseteq \mathbb{R}^n$, $i \in \mathbb{N}_{[0,N]}$, the sets $\mathcal{V}_i^M$, $i \in \mathbb{N}_{[0,M-1]}$, the sets $\mathcal{E}_i^M$, $i \in \mathbb{N}_{[0,N]}$, and the functions $\ell_i^M : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $V_\mathrm{f}^M : \mathbb{R}^n \to \mathbb{R}$ will be defined next. That is, for all $M \in \mathbb{N}_{[1,N]}$ define

$$\mathcal{Y}_i^M := \begin{cases} (\mathcal{Y} \oplus \mathcal{E}_i^M) \ominus \mathcal{F}_i^M, & i \in \mathbb{N}_{[0,M-1]} \\ ((\mathcal{Y} \ominus \mathcal{F}_{i-M}^1) \oplus \mathcal{E}_i^M) & i \in \mathbb{N}_{[M,N]} \\ \quad \ominus (A+BK)^{i-M}\mathcal{F}_M^M, & \end{cases} \tag{15a}$$

$$\mathcal{V}_i^M := K\mathcal{Y}, \quad i \in \mathbb{N}_{[0,M-1]} \tag{15b}$$

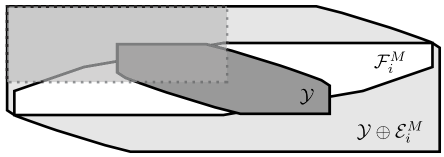

Fig. 1. Relationship between the sets $\mathcal{Y}$, $\mathcal{F}_i^M$, and $\mathcal{E}_i^M$ as described in (17b). The set with dotted border is obtained by translating the set $\mathcal{E}_i^M$.

for all $M \in \mathbb{N}_{[1,N]}$. We will use the following assumption on the sets $\mathcal{Y}$ and $\mathcal{E}_i^M$, $i \in \mathbb{N}_{[0,N]}$.

Assumption 12. The set $\mathcal{Y}$ is a C-set such that

$$(A + BK)\mathcal{Y} \oplus \mathcal{W} \subseteq \mathcal{Y}. \quad (16)$$

Further, for all $M \in \mathbb{N}_{[1,N]}$ and all $i \in \mathbb{N}_{[0,N]}$ it holds that $\mathcal{E}_i^M$ is a C-set and

$$\forall i \in \mathbb{N}_{[0,N]}, \quad \mathcal{E}_i^1 = \{0\}, \quad (17a)$$

$$\forall i \in \mathbb{N}_{[0,M-1]}, \quad \mathcal{F}_i^M \subseteq \mathcal{Y} \oplus \mathcal{E}_i^M \quad (17b)$$

$$\forall i \in \mathbb{N}_{[M,N]}, \quad (A + BK)^{i-M}\mathcal{F}_M^M \subseteq (\mathcal{Y} \ominus \mathcal{F}_{i-M}^1) \oplus \mathcal{E}_i^M. \quad (17c)$$

This assumption ensures that the sets $\mathcal{Y}_i^M$ are non-empty for all $M \in \mathbb{N}_{[1,N]}$ and all $i \in \mathbb{N}_{[0,N]}$. Essentially, the sets $\mathcal{E}_i^M$ are used to (over-) estimate the size of the uncertainty in the prediction when compared to the set $\mathcal{Y}$. The relationship between the sets $\mathcal{Y}$, $\mathcal{F}_i^M$ and $\mathcal{E}_i^M$ for the inclusion (17b) is sketched in Fig. 1.

Remark 13. From (16) it follows directly that $(A+BK)^i\mathcal{Y}\oplus\mathcal{F}_i^1 \subseteq \mathcal{Y}$ and, hence, $\mathcal{F}_i^1 \subseteq \mathcal{Y}$ for all $i \in \mathbb{N}_{[0,N]}$. Methods for constructing the sets $\mathcal{Y}$ and $\mathcal{E}_i^M$, $M \in \mathbb{N}_{[1,M]}$, $i \in \mathbb{N}_{[0,N]}$, are discussed in Section 6.3.

For the sake of exposition, for all $M \in \mathbb{N}_{[1,N]}$, all $i \in \mathbb{N}_{[1,N-1]}$, all $\bar{x} \in \mathbb{R}^n$, and all $\bar{u} \in \mathbb{R}^m$, the functions ℓ_i^M and V_i^M are defined by

$$\ell_i^M(\bar{x}, \bar{u}) = \begin{cases} \max\limits_{e \in \mathcal{E}_i^M} \ell(\bar{x} + e, \bar{u}), & i \in \mathbb{N}_{[0,M-1]} \\ \max\limits_{e \in \mathcal{E}_i^M} \ell(\bar{x} + e, \bar{u} + Ke), & i \in \mathbb{N}_{[M,N-1]} \end{cases} \quad (18a)$$

$$V_{\mathrm{f}}^M(\bar{x}) = \max_{e \in \mathcal{E}_N^M} V_{\mathrm{f}}(\bar{x} + e). \quad (18b)$$

Remark 14. With the cost functions defined in this way, the overall optimization problem defined below becomes similar to the optimization problems arising in min–max MPC approaches, see for example Campo and Morari (1987).

Lemma 15. *For all $i \in \mathbb{N}$ it holds that*

$$(A + BK)(\mathcal{Y} \ominus \mathcal{F}_i^1) \subseteq (\mathcal{Y} \ominus \mathcal{F}_{i+1}^1). \quad (19)$$

Lemma 16. *The sets $\mathcal{Y}_i^M$, $i \in \mathbb{N}_{[0,N]}$, are non-empty. Further, it holds that*

$$\forall i \in \mathbb{N}_{[0,N]}, \quad (A + BK)^i\mathcal{Y} \subseteq \mathcal{Y}_i^1. \quad (20)$$

Finally, the following assumptions are required to hold.

Assumption 17. The functions ℓ and V_{f} are continuous and positive definite. Furthermore, for all $x \in \mathbb{R}^n$

$$\ell(x, Kx) + V_{\mathrm{f}}((A + BK)x) \leq V_{\mathrm{f}}(x). \quad (21)$$

Finally, there exist $\mathcal{K}_\infty$-functions α_1, α_2, such that for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ it holds that

$$\ell(x, u) \geq \alpha_1(|x|), \quad (22a)$$

$$V_{\mathrm{f}}(x) \leq \alpha_2(|x|). \quad (22b)$$

Lemma 18. *For any $i \in \mathbb{N}_{\geq 1}$ and any $x \in \mathbb{R}^n$ it holds that $\sum_{j=0}^{i-1} \ell((A + BK)^jx, K(A + BK)^jx) + V_f((A + BK)^ix) \leq V_f(x)$.*

Proof. It follows immediately from Assumption 17 by induction. ∎

Lemma 19. *For all $M \in \mathbb{N}_{[1,N]}$ and all $i \in \mathbb{N}_{[0,N-1]}$ the functions ℓ_i^M and V_f^M are continuous and satisfy $\ell_i^M(x, u) \geq \alpha_1(|x|)$ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.*

Proof. It follows directly from the definition of the functions, Assumptions 12, and 17. ∎

4.3. Finite-horizon optimal control problem

With the stage and terminal cost functions defined above, we are now ready to define the finite-horizon optimal control problems used in the MPC scheme. For all $M \in \mathbb{N}_{[1,N]}$ and a fixed constant $\beta \geq 1$, define for all $\mathbf{d}_k^{\mathrm{fp}} \in \mathbb{D}_N^{\mathrm{fp}}$, $k \in \mathbb{N}$, the finite-horizon cost function

$$\bar{J}_N^M(\mathbf{d}_k^{\mathrm{fp}}) = \sum_{i=0}^{M-1} \frac{1}{\beta}\bar{\ell}_i^M(x_{i|k}, u_{i|k}) + \sum_{i=M}^{N-1} \bar{\ell}_i^M(x_{i|k}, u_{i|k}) + \bar{V}_{\mathrm{f}}^M(x_{N|k}), \quad (23)$$

which is inspired by the cost function proposed in Barradas Berglind et al. (2012). The parameter β allows a trade-off between the performance (in terms of the infinite horizon cost function) and the average sampling rate, see Barradas Berglind et al. (2012) and Gommans et al. (2014). For all $M \in \mathbb{N}_{[1,N]}$ and any $x_k \in \mathbb{R}^n$, define the finite-horizon optimization problem

$$\bar{V}_N^M(x_k) := \min_{\mathbf{d}_k^{\mathrm{fp}} \in \mathcal{D}_N^{M,\mathrm{fp}}(x_k)} \bar{J}_N^M(\mathbf{d}_k^{\mathrm{fp}}). \quad (24)$$

The optimization problem in (24) contains inner optimization problems due to the definition of the stage and terminal cost functions in (14). For the discussions pertaining to the optimal cost function in the next subsection, it is convenient to remove these inner optimization problems. Hence, define the augmented decision variable at time point k by

$$\mathbf{d}_k^M = ((x_{0|k}, \ldots, x_{N|k}), (u_{0|k}, \ldots, u_{N-1|k}), (y_{0|k}, \ldots, y_{N|k}), (v_{0|k}, \ldots, v_{M-1|k})) \in \mathbb{D}_N^M, \quad (25)$$

where $\mathbb{D}_N^M = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$. The additional variables are subject to the constraints

$$\forall i \in \mathbb{N}_{[0,N]}, \quad y_{i|k} \in \mathcal{Y}_i^M, \quad (26a)$$

$$\forall i \in \mathbb{N}_{[0,M-1]}, \quad v_{i|k} \in \mathcal{V}_i^M, \quad (26b)$$

such that the set of all feasible decision variables for a given point $x_k \in \mathbb{R}^n$ is given by

$$\mathcal{D}_N^M(x_k) = \{\mathbf{d}_k^M \in \mathbb{D}_N^M \mid \text{(6a)–(6e), (26a), (26b)}\}, \quad (27)$$

for a given $M \in \mathbb{N}_{[1,N]}$. The cost function for the augmented decision variable reads

$$J_N^M(\mathbf{d}_k^M) = \sum_{i=0}^{M-1} \frac{1}{\beta}\ell_i^M(x_{i|k} - y_{i|k}, u_{i|k} - v_{i|k}) + \sum_{i=M}^{N-1} \ell_i^M(x_{i|k} - y_{i|k}, u_{i|k} - Ky_{i|k}) + V_{\mathrm{f}}^M(x_{N|k} - y_{N|k}). \quad (28)$$

For a given $M \in \mathbb{N}_{[1,N]}$, the finite-horizon optimal control problem with the augmented decision variable is given by

$$V_N^M(x_k) := \min_{\mathbf{d}_k^M \in \mathcal{D}_N^M(x_k)} J_N^M(\mathbf{d}_k^M). \tag{29}$$

The optimal control problems in (24) and (29) are equivalent, which is expressed in the following lemma.

Lemma 20. *For a given $x_k \in \mathbb{R}^n$, a solution to the optimization problem in (24) exists, if and only if a solution to the optimization problem in (29) exists. Furthermore, if solutions to the problems exist, it holds that $V_N^M(x_k) = \bar{V}_N^M(x_k)$.*

Proof. It follows directly from the definitions. ■

4.4. Properties of the optimal cost function

In the following, several properties of the optimal cost function will be established.

Lemma 21. *Let any $k \in \mathbb{N}$, any $x_k \in \mathbb{R}^n$, any $M \in \mathbb{N}_{[1,N]}$, and any $\mathbf{d}_k^M = ((x_{0|k}, \dots, x_{N|k}), (u_{0|k}, \dots, u_{N-1|k}), (y_{0|k}, \dots, y_{N|k}), (v_{0|k}, \dots, v_{M-1|k})) \in \mathcal{D}_N^M(x_k)$ be given. Let further for all $i \in \mathbb{N}_{[0,M-1]}$ $x_{k+i+1} = Ax_{k+i} + Bu_{i|k} + w_{k+i}$, where $w_{k+i} \in \mathcal{W}$. Then, there exists a $\mathbf{d}_{k+M}^1 = ((x_{0|k+M}, \dots, x_{N|k+M}), (u_{0|k+M}, \dots, u_{N-1|k+M}), (y_{0|k+M}, \dots, y_{N|k+M}), v_{0|k+M}) \in \mathcal{D}_N^1(x_{k+M})$ such that $J_N^1(\mathbf{d}_{k+M}^1) \le J_N^M(\mathbf{d}_k^M) - \sum_{i=0}^{M-1} \frac{1}{\beta}\ell_i^M(x_{i|k} - y_{i|k}, u_{i|k} - v_{i|k})$.*

In order to derive properties of the optimal cost functions V_N^M, we need some further technical results.

Lemma 22. *Define the set where the optimization problem for $M = 1$ is feasible by $\hat{\mathcal{X}}_N := \{x_k \in \mathbb{R}^n \mid \mathcal{D}_N^1(x_k) \neq \emptyset\}$. It holds that $\hat{\mathcal{X}}_N$ is a compact set.*

Define a set $\tilde{\mathcal{X}}_N$ where the optimization problem for $M = 1$ is feasible with an input defined by the linear control law $u = Kx$. In particular, define

$$\begin{aligned}\tilde{\mathcal{X}}_N := \{x \in \mathbb{R}^n \,|\, &(A+BK)^N x \in \mathcal{X}_f^1\\ &\forall i \in \mathbb{N}_{[0,N-1]},\ (A+BK)^i x \oplus \mathcal{F}_i^1 \subseteq \mathcal{X},\\ &\forall i \in \mathbb{N}_{[0,N-1]},\ K(A+BK)^i x \oplus K\mathcal{F}_i^1 \subseteq \mathcal{U}, \}.\end{aligned} \tag{30}$$

Lemma 23. *It holds that $(A+BK)\tilde{\mathcal{X}}_N \oplus \mathcal{W} \subseteq \tilde{\mathcal{X}}_N$.*

Lemma 24. *If there exists an $\eta > 0$ such that $\eta\mathcal{B} \oplus \mathcal{Y} \subseteq \tilde{\mathcal{X}}_N$, then there exists a class $\mathcal{K}$-function α_3, such that for all $x_k \in \hat{\mathcal{X}}_N$ it holds that*

$$V_N^1(x_k) \le \alpha_3(|x_k|_{\mathcal{Y}}). \tag{31}$$

Proof. The upper bound on V_N^1 can be constructed by extending the proof of Theorem III.2 in Lazar, Heemels, Weiland, and Bemporad (2006). ■

5. Robust self-triggered control

In this section, we propose a solution to the problem statement in Section 3 in the form of an MPC controller which maximizes the number of steps until the next control update, subject to certain conditions that will guarantee robust constraint satisfaction, stability, and performance properties. In particular, for any $k \in \mathbb{N}$ and any $x_k \in \mathbb{R}^n$ define the optimization problem

$$M^\star(x_k) := \max\Big\{M \in \mathbb{N}_{[1,M_{\max}]} \,\Big|\, \mathcal{D}_N^1(x_k) \neq \emptyset,\ \mathcal{D}_N^M(x_k) \neq \emptyset,\ V_N^M(x_k) \le V_N^1(x_k)\Big\} \tag{32a}$$

$$\mathbf{d}_k^\star(x_k) := \operatorname*{argmin}_{\mathbf{d}_k \in \mathcal{D}_N^{M^\star(x_k)}(x_k)} J_N^{M^\star(x_k)}(\mathbf{d}_k) \tag{32b}$$

with

$$\begin{aligned}\mathbf{d}_k^\star(x_k) = (&(x_{0|k}^\star(x_k), \dots, x_{N|k}^\star(x_k)),\\ &(u_{0|k}^\star(x_k), \dots, u_{N-1|k}^\star(x_k)),\\ &(y_{0|k}^\star(x_k), \dots, y_{N|k}^\star(x_k))\\ &(v_{0|k}^\star(x_k), \dots, v_{M^\star(x_k)-1|k}^\star(x_k))),\end{aligned} \tag{33}$$

where $M_{\max} \in \mathbb{N}_{[1,N]}$ is a chosen maximal length of the open-loop phase.

Remark 25. We do not concern ourselves here with the possible non-uniqueness of the minimizer. In the following it is assumed that $\mathbf{d}_k^\star(x_k)$ is any solution to the optimization problem.

The optimization problem in (32) forms the basis of the following control algorithm.

Algorithm 1 Self-Triggered Tube MPC

1: Set $k = 0$.
2: At time k, obtain the current state x_k of system (2).
3: Solve the optimization problems in (32), obtain $M^\star(x_k)$ and $\mathbf{d}_k^\star(x_k)$.
4: Apply $u_{k+i} = u_{i|k}^\star(x_k)$ to the system for $i \in \mathbb{N}_{[0,M^\star(x_k)-1]}$.
5: At time $k + M^\star(x_k)$, set $k = k + M^\star(x_k)$.
6: Go to 2.

The set of states where Algorithm 1 is feasible is $\hat{\mathcal{X}}_N = \{x \in \mathbb{R}^n \mid \mathcal{D}_N^1(x) \neq \emptyset\}$. The closed-loop system resulting from the application of Algorithm 1 is

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k + w_k\\ &\quad \text{where } u_k = \kappa(x_{k_j}, k - k_j) \quad \text{if } k \in \mathbb{N}_{[k_j, k_{j+1}-1]}\end{aligned} \tag{34a}$$

$$k_{j+1} = k_j + \mu(x_{k_j}), \tag{34b}$$

for $j \in \mathbb{N}$, $k_0 = 0$, $x_0 \in \hat{\mathcal{X}}_N$, and $w_k \in \mathcal{W}$ for all $k \in \mathbb{N}$, where the functions κ and μ are given by

$$\kappa(x_{k_j}, k - k_j) := u_{k-k_j|k_j}^\star(x_{k_j}) \quad \text{if } k \in \mathbb{N}_{[k_j, k_{j+1}-1]} \tag{35a}$$

$$\mu(x_{k_j}) := M^\star(x_{k_j}). \tag{35b}$$

Theorem 26 (*Recursive Feasibility*)**.** *For all $x_0 \in \hat{\mathcal{X}}_N$, the closed-loop system (34) is well defined, that is, if $x_0 \in \hat{\mathcal{X}}_N$, then for all $j \in \mathbb{N}$ and all k_j the optimization problem in (32) admits a solution for x_{k_j}. Furthermore, if $x_0 \in \hat{\mathcal{X}}_N$, then for all $k \in \mathbb{N}$ and all $j \in \mathbb{N}$ it holds that $x_k \in \mathcal{X}$ and $u_k \in \mathcal{U}$ for any disturbances $w_k \in \mathcal{W}$, $k \in \mathbb{N}$.*

Proof. The statement follows directly from Lemmas 7 and 11. ■

Theorem 27 (*Performance Bound*)**.** *For any $x_0 \in \hat{\mathcal{X}}_N$ and any disturbances $w_k \in \mathcal{W}$, $k \in \mathbb{N}$, the closed-loop dynamics (34) satisfy the performance bound*

$$\sum_{k=0}^{\infty} \min_{\substack{y_k \in \mathcal{Y}\\ v_k \in K\mathcal{Y}}} \ell(x_k - y_k, u_k - v_k) \le \beta V_N^1(x_0) \tag{36}$$

defined in terms of the stage cost function ℓ, the set $\mathcal{Y}$, the parameter β, and the optimal cost function V_N^1 for the MPC scheme were sampling is assumed to occur at every point in time.

Theorem 28 (*Asymptotic Bound*). *For the closed-loop dynamics (34) and any $x_0 \in \hat{\mathcal{X}}_N$ it holds that x_k converges to the set $\mathcal{Y}$ as k approaches infinity in the sense that $\lim_{k\to\infty} |x_k|_{\mathcal{Y}} = 0$ for any disturbances with $w_k \in \mathcal{W}$, $k \in \mathbb{N}$.*

Theorem 29 (*Robust Asymptotic Stability*). *If there exists an $\eta > 0$ such that $\eta\mathcal{B} \oplus \mathcal{Y} \subseteq \tilde{\mathcal{X}}_N$, then the set $\mathcal{Y}$ is robustly asymptotically stable for the closed-loop dynamics (34) and the set $\hat{\mathcal{X}}_N$ belongs to its region of attraction.*

If the assumption that $\eta\mathcal{B} \oplus \mathcal{Y} \subseteq \tilde{\mathcal{X}}_N$ for an $\eta > 0$ is not satisfied, it is not guaranteed that the upper bound in Lemma 24 can be established. Hence, without this assumption, robust Lyapunov stability of $\mathcal{Y}$ is not ensured. However, Theorems 26, 27, and 28 still hold.

6. Implementation

In this section we discuss some issues regarding implementation and computation.

6.1. Implementation via quadratic or linear programming

If the stage and terminal cost functions are convex and can be written as the sum of quadratic and piecewise linear functions and if additionally all involved sets are polytopes, then the optimization problems in this paper can be solved via quadratic (or, if the quadratic term is zero, by linear) programming, compare Campo and Morari (1987) and Ramírez and Camacho (2001). Hence the algorithm in (32) can be evaluated by solving at most $M_{\max}$ quadratic programs or linear programs at each sampling instants, see also Barradas Berglind et al. (2012) for additional discussion.

6.2. Sparsity promoting constraints

In addition to the reduction of communication, the MPC scheme described in Section 5 can be used to promote sparsity in the inputs applied to the system. This can be achieved by adding additional constraints on the predicted input trajectory in the open-loop phase. Such constraints might be for example Aydiner et al. (2015), Barradas Berglind et al. (2012) and Brunner et al. (2014),

$$\forall i \in \mathbb{N}_{[1,M-1]},\ u_{i|k} = 0, \tag{37a}$$

$$\text{or} \quad \forall i \in \mathbb{N}_{[1,M-1]},\ u_{i|k} = u_{0|k}. \tag{37b}$$

6.3. Parameterization of $\mathcal{Y}$ and $\mathcal{E}_i^M$

In general, one would want to choose the set $\mathcal{Y}$ as small as possible, as the asymptotic bound on the system state guaranteed by Theorem 28 is defined in terms of $\mathcal{Y}$. However, if the assumption that $\eta\mathcal{B} \oplus \mathcal{Y} \subseteq \tilde{\mathcal{X}}_N$ for an $\eta > 0$ is satisfied, for all states contained in $\mathcal{Y}$ it holds that V_N^1 is identical to zero as a feasible solution with zero cost is given by just applying the control law $u = Kx$ at each predicted time step. As the constraints in the optimization problem are different for larger M, especially when considering the constraints discussed in Section 6.2, it does not necessarily hold that V_N^M is also zero on $\mathcal{Y}$, implying that for large times in the closed-loop system the inputs have to be updated at every time-step due to the way the self-triggering algorithm is defined in (32). *Enlarging* $\mathcal{Y}$ leads to a general decrease of the cost functions V_N^M and, hence, allows both V_N^1 and V_N^M to be zero at the same time for an enlarged set of states, in turn leading to an enlarged set of states in the state space where the algorithm in (32) decides on an $M^\star$ larger than one. This interdependency leads to a trade-off between the asymptotic bound $\mathcal{Y}$ on the system state and the average sampling frequency for large times.

In the following, an approach of parameterizing $\mathcal{Y}$ in terms of a single scalar parameter will be presented. The approach relies on a set $\bar{\mathcal{Y}}$, which is an RPI outer approximation of the minimal RPI set (Raković, Kerrigan, Kouramas, & Mayne, 2005). In particular, assume that

$$(A + BK)\bar{\mathcal{Y}} \oplus \mathcal{W} \subseteq \bar{\mathcal{Y}} \tag{38}$$

and let $\mathcal{Y} := c_1\bar{\mathcal{Y}}$, where $c_1 \geq 1$. If there exists an $\bar{\eta} > 0$ such that $\bar{\mathcal{Y}} \oplus \bar{\eta}\mathcal{B} \subseteq \tilde{\mathcal{X}}_N$, there also exist $c_1 > 1$ and $\eta > 0$ such that $c_1\bar{\mathcal{Y}} \oplus \eta\mathcal{B} \subseteq \tilde{\mathcal{X}}_N$. Assumption 12 is satisfied by construction with this approach.

After $\mathcal{Y}$ has been designed, the sets $\mathcal{E}_i^M$ can for example be constructed by parameterizing a *PC*-set and choosing the parameters such that Assumption 12 is satisfied. A simple choice is to fix any *PC*-set $\mathcal{E}$ and define

$$\mathcal{E}_i^M := \rho_i^M \mathcal{E}, \tag{39}$$

where $\rho_i^M \in \mathbb{R}_+$ is chosen as small as possible under the constraint that Assumption 12 is satisfied, for all $M \in \mathbb{N}_{[1,N]}$.

6.4. Computational complexity

Considering that, as far as polytopes are concerned, the complexity of a set $\mathcal{A} \ominus \mathcal{B}$ is not higher than that of $\mathcal{A}$ (see, for example, Section 2 of Kolmanovsky and Gilbert (1998)), constraints (6a)–(6e) are not more complex than for a standard MPC scheme. Additional complexity does arise from the non-standard cost function employed in this paper, i.e. via the minimization over the sets $\mathcal{Y}_i^M$ and $\mathcal{V}_i^M$ in (15) and the maximization over the sets $\mathcal{E}_i^M$ in (18). Note, however, that the complexity of these sets does not depend on the prediction horizon, and, in the case of the sets $\mathcal{E}_i^M$, can be determined by the user. Overall, the computational complexity of the scheme in terms of the number of decision variables and scalar valued inequalities grows linearly in the prediction horizon and is mainly determined by the complexity of the sets $\mathcal{Y}$ and $\mathcal{E}_i^M$ in (15).

Remark 30. Due to space limitations we could not include a numerical example. However, we will present numerical results that show the effectiveness of our novel control algorithm elsewhere.

7. Conclusions

In this paper, we have proposed a self-triggered controller that robustly stabilizes a compact set and guarantees constraint satisfaction while reducing the average communication between the controller and the system which is subject to additive disturbances. Performance guarantees are given in the form of a guaranteed upper bound on the performance loss when compared to an MPC scheme that is updated at every point in time. An asymptotic bound on the system state for the closed-loop system under disturbances can be determined a priori and is in fact a parameter in the MPC scheme. The exact interdependence between the closed-loop performance, the asymptotic bound and the average communication rate is subject to future research.

Acknowledgments

The authors would like to thank the German Research Foundation (DFG) for financial support of the project within the Cluster of Excellence in Simulation Technology (EXC 310/2) at the University of Stuttgart. The authors would also like to thank the DFG for their financial support within the research grant AL

316/9-1. This work is also supported by the Innovational Research Incentives Scheme under the VICI grant "Wireless control systems: A new frontier in automation" (no. 11382) awarded by NWO (Netherlands Organization for Scientific Research) and STW (Dutch Science Foundation).

Appendix A. Proof of Lemma 1

Only the last property, that is $A(\mathcal{X} \ominus \mathcal{Y}) \subseteq (A\mathcal{X} \ominus A\mathcal{Y})$, is proven here, as the other properties are proven in Kolmanovsky and Gilbert (1998) and Najfeld, Vitale, and Davis (1980). Let $z \in \mathbb{R}^m$ with $z \in A(\mathcal{X} \ominus \mathcal{Y})$ be arbitrary. Then it holds that $z = As$ for an $s \in \mathbb{R}^n$ where $s \oplus \mathcal{Y} \subseteq \mathcal{X}$. It follows that $As \oplus A\mathcal{Y} \subseteq A\mathcal{X}$, such that $z \oplus A\mathcal{Y} \subseteq A\mathcal{X}$, and therefore $z \in (A\mathcal{X} \ominus A\mathcal{Y})$. ■

Appendix B. Proof of Lemma 7

For $i \in \mathbb{N}_{[0,M]}$ it holds that

$$\begin{aligned} x_{k+i} &= A^i x_k + \sum_{j=0}^{i-1} A^j B u_{i-j-1|k} + \sum_{j=0}^{i-1} A^j w_{k+i-j-1} \\ &= x_{i|k} + \sum_{j=0}^{i-1} A^j w_{k+i-j-1} \\ &\in x_{i|k} \oplus \bigoplus_{j=0}^{i-1} A^j \mathcal{W} = x_{i|k} \oplus \mathcal{F}_i^M. \end{aligned} \tag{B.1}$$

Further, for $i \in \mathbb{N}_{[M,N-1]}$ it holds that $x_{k+i+1} - x_{i+1|k} = A(x_{k+i} - x_{i|k}) + BK(x_{k+i} - x_{i|k}) + w_{k+i}$ and, hence,

$$\begin{aligned} x_{k+i} - x_{i|k} &= (A+BK)^{i-M}(x_{k+M} - x_{M|k}) \\ &\quad + \sum_{j=0}^{i-M-1} (A+BK)^j w_{k+i-j-1} \\ &\in (A+BK)^{i-M} \bigoplus_{j=0}^{M-1} A^j \mathcal{W} \oplus \bigoplus_{j=0}^{i-M-1} (A+BK)^j \mathcal{W} \\ &= \mathcal{F}_i^M \end{aligned} \tag{B.2}$$

for $i \in \mathbb{N}_{[M+1,N]}$. Similarly, it holds that $u_{k+i} - u_{i|k} = K(x_{k+i} - x_{i|k}) \in K\mathcal{F}_i^M$. for $i \in \mathbb{N}_{[M,N-1]}$, thereby proving that $x_{k+i} \in x_{i|k} \oplus \mathcal{F}_i^M$ for $i \in \mathbb{N}_{[0,N]}$ and $u_{k+i} \in u_{i|k} \oplus K\mathcal{F}_i^M$ for $i \in \mathbb{N}_{[M,N-1]}$. The second part of the statement, that is, $x_{k+i} \in \mathcal{X}$ and $u_{k+i} \in \mathcal{U}$ for $i \in \mathbb{N}_{[0,N-1]}$, follows directly from the above and the definition of the tightened constraints in (6) and (8). ■

Appendix C. Proof of Lemma 11

Define

$$x_{j|k} := (A+BK)^{j-N} x_{N|k} \tag{C.1}$$

for $j \in \mathbb{N}_{[N+1,N+M]}$ and

$$u_{j|k} := K(A+BK)^{j-N} x_{N|k} \tag{C.2}$$

for $j \in \mathbb{N}_{[N,N+M-1]}$. Define further

$$x_{i|k+M} := x_{i+M|k} + (A+BK)^i (x_{k+M} - x_{M|k}) \tag{C.3}$$

for $i \in \mathbb{N}_{[0,N]}$ and

$$u_{i|k+M} := u_{i+M|k} + K(A+BK)^i (x_{k+M} - x_{M|k}) \tag{C.4}$$

for $i \in \mathbb{N}_{[0,N-1]}$. With these choices, it follows that $x_{i|k+M}$ and $u_{i|k+M}$ satisfy constraints (6a) and (6b) in $\mathcal{D}_N^{1,\mathrm{fp}}(x_{k+M})$. Further, by Lemma 7 it holds that

$$x_{k+M} - x_{M|k} \in \mathcal{F}_M^M, \tag{C.5}$$

such that for all $i \in \mathbb{N}_{[0,N]}$

$$\begin{aligned} x_{i|k+M} \oplus \mathcal{F}_i^1 &\overset{(C.3)}{\subseteq} x_{i+M|k} \oplus (A+BK)^i \mathcal{F}_M^M \oplus \mathcal{F}_i^1 \\ &\overset{(11)}{=} x_{i+M|k} \oplus \mathcal{F}_{i+M}^M. \end{aligned} \tag{C.6}$$

Similarly, for all $i \in \mathbb{N}_{[0,N-1]}$ it holds that

$$\begin{aligned} u_{i|k+M} \oplus K\mathcal{F}_i^1 &\overset{(C.4)}{\subseteq} u_{i+M|k} \oplus K(A+BK)^i \mathcal{F}_M^M \oplus K\mathcal{F}_i^1 \\ &= u_{i+M|k} \oplus K\mathcal{F}_{i+M}^M. \end{aligned} \tag{C.7}$$

Hence, for $i \in \mathbb{N}_{[0,N-M]}$, the satisfaction of constraint (6c) and for $i \in \mathbb{N}_{[0,N-M-1]}$, the satisfaction of constraint (6d) in $\mathcal{D}_N^{1,\mathrm{fp}}(x_{k+M})$ follows directly from the definitions of $\mathcal{D}_N^{M,\mathrm{fp}}(x_k)$ and the tightened constraint sets in (8).

By the definition of $\mathcal{D}_N^{M,\mathrm{fp}}(x_k)$, it holds that $x_{N|k} \in \mathcal{X}_\mathrm{f}^M$, such that by (C.1) and (12a) it also holds that $x_{i+M|k} \oplus \mathcal{F}_{i+M}^M \subseteq \mathcal{X}$ for $i \in \mathbb{N}_{[N-M+1,N]}$ and by (C.2) and (12b) that $u_{i+M|k} \oplus K\mathcal{F}_{i+M}^M \subseteq \mathcal{U}$ for $i \in \mathbb{N}_{[N-M+1,N-1]}$, such that by (C.6) and (C.7) it follows that constraint (6c) is also satisfied for $i \in \mathbb{N}_{[N-M+1,N]}$ and constraint (6d) is also satisfied for $i \in \mathbb{N}_{[N-M,N-1]}$ in $\mathcal{D}_N^{1,\mathrm{fp}}(x_{k+M})$. Further, again exploiting $x_{N|k} \in \mathcal{X}_\mathrm{f}^M$ we obtain

$$\begin{aligned} x_{N|k+M} &\overset{(C.5),(C.3)}{\in} x_{N+M|k} \oplus (A+BK)^N \mathcal{F}_M^M \\ &\overset{(C.1)}{=} (A+BK)^M x_{N|k} \oplus (A+BK)^N \mathcal{F}_M^M. \end{aligned} \tag{C.8}$$

Using now (12c) gives that constraint (6e) in $\mathcal{D}_N^{1,\mathrm{fp}}(x_{k+M})$ is satisfied, such that $\mathbf{d}_{k+M}^{\mathrm{fp}} := ((x_{0|k+M}, \ldots, x_{N|k+M}), (u_{0|k+M}, \ldots, u_{N-1|k+M})) \in \mathcal{D}_N^{1,\mathrm{fp}}(x_{k+M})$, as needed. ■

Appendix D. Proof of Lemma 15

By Assumption 12 it holds that $(A+BK)\mathcal{Y} \subseteq \mathcal{Y} \ominus \mathcal{W}$, such that for all $i \in \mathbb{N}$ it follows that

$$\begin{aligned} (A+BK)(\mathcal{Y} \ominus \mathcal{F}_i^1) &\subseteq (A+BK)\mathcal{Y} \ominus (A+BK)\mathcal{F}_i^1 \\ &\subseteq \mathcal{Y} \ominus \mathcal{W} \ominus (A+BK)\mathcal{F}_i^1 \\ &= \mathcal{Y} \ominus (\mathcal{W} \oplus (A+BK)\mathcal{F}_i^1) \\ &= \mathcal{Y} \ominus \mathcal{F}_{i+1}^1, \end{aligned} \tag{D.1}$$

thereby completing the proof. ■

Appendix E. Proof of Lemma 16

It follows directly from Assumption 12 that the sets $\mathcal{Y}_i^M$ are non-empty for $i \in \mathbb{N}_{[0,N]}$. The inclusions in (20) follow from Remark 13, Remark 8, and the fact that $\mathcal{E}_i^1 = \{0\}$. ■

Appendix F. Proof of Lemma 21

Consider again the variables $x_{0|k+M}, \ldots, x_{N|k+M}$, and $u_{0|k+M}, \ldots, u_{N-1|k+M}$ chosen in the proof of Lemma 11. By (C.3), there exists a $g_M \in \mathcal{F}_M^M$ such that

$$x_{i|k+M} = x_{i+M|k} + (A+BK)^i g_M, \quad i \in \mathbb{N}_{[0,N]}. \tag{F.1}$$

Hence, considering (15a) and (26a), for all $i \in \mathbb{N}_{[0,N-M]}$ there exists an $e_{i+M} \in \mathcal{E}_{i+M}^M$ such that

$$y_{i+M|k} + (A+BK)^i g_M \in (\mathcal{Y} \ominus \mathcal{F}_i^1) \oplus e_{i+M}. \tag{F.2}$$

Defining

$$y_{i|k+M} := y_{i+M|k} + (A+BK)^i g_M - e_{i+M} \tag{F.3}$$

for $i \in \mathbb{N}_{[0,N-M]}$ then gives

$$y_{i|k+M} \in (\mathcal{Y} \ominus \mathcal{F}_i^1) = (\mathcal{Y} \oplus \mathcal{E}_i^1) \ominus \mathcal{F}_i^1, \tag{F.4}$$

using the fact that $\mathcal{E}_i^1 = \{0\}$ due to (17a). For $i = 0$ this already implies that $y_{i|k+M} \in \mathcal{Y}_i^1$. For $i \in \mathbb{N}_{[1,N-M]}$ we further infer

$$\begin{aligned} y_{i|k+M} &\in ((\mathcal{Y} \ominus \mathcal{F}_{i-1}^1) \oplus \mathcal{E}_i^1) \ominus (A+BK)^{i-1}\mathcal{F}_1^1 \\ &= \mathcal{Y}_i^1, \end{aligned} \tag{F.5}$$

where the first line follows from Remark 8. Define further for all $i \in \mathbb{N}_{[N-M+1,N]}$

$$\begin{aligned} y_{i|k+M} &:= (A+BK)^{i-N+M} y_{N-M|k+M} \\ &= (A+BK)^{i-N+M} y_{N|k} + (A+BK)^i g_M \\ &\quad - (A+BK)^{i-N+M} e_N. \end{aligned} \tag{F.6}$$

It follows that

$$\begin{aligned} y_{i|k+M} &\overset{(F.4)}{\in} (A+BK)^{i-N+M}(\mathcal{Y} \ominus \mathcal{F}_{N-M}^1) \\ &\overset{(19)}{\subseteq} (\mathcal{Y} \ominus \mathcal{F}_i^1) \\ &= ((\mathcal{Y} \ominus \mathcal{F}_{i-1}^1) \oplus \mathcal{E}_i^1) \ominus (A+BK)^{i-1}\mathcal{F}_1^1 \\ &= \mathcal{Y}_i^1, \end{aligned} \tag{F.7}$$

for all $i \in \mathbb{N}_{[N-M+1,N]}$. The above implies that (26a) is satisfied in $\mathcal{D}_N^1(x_{k+M})$ for our choice of $y_{i|k+M}$. Finally, define

$$v_{0|k+M} = K y_{0|k+M} \in K\mathcal{Y} = \mathcal{V}_0^1, \tag{F.8}$$

such that $\mathbf{d}_{k+M}^1 \in \mathcal{D}_N^1(x_{k+M})$ is satisfied for $\mathbf{d}_{k+M}^1 := ((x_{0|k+M}, \dots, x_{N|k+M}), (u_{0|k+M}, \dots, u_{N-1|k+M}), (y_{0|k+M}, \dots, y_{N|k+M}), v_{0|k+M})$. Note that by (C.4) for all $i \in \mathbb{N}_{[0,N-1]}$ it holds that

$$u_{i|k+M} = u_{i+M|k} + K(A+BK)^i g_M \tag{F.9a}$$

$$\overset{(F.3)}{=} u_{i+M|k} + K y_{i|k+M} - K y_{i+M|k} + K e_{i+M}, \tag{F.9b}$$

where $u_{N+i|k} := K(A+BK)^i x_{N|k}$, $i \in \mathbb{N}_{[0,M-1]}$, such that

$$\begin{aligned} v_{0|k+M} &\overset{(F.3)}{=} K y_{M|k} + K g_M - K e_M \\ &\overset{(F.9a)}{=} K y_{M|k} + u_{0|k+M} - u_{M|k} - K e_M. \end{aligned} \tag{F.10}$$

Using the fact that $\mathcal{E}_i^1 = \{0\}$ due to (F.4) for all $i \in \mathbb{N}_{[0,N]}$, it holds that $\ell_i^1 = \ell$ and $V_f^1 = V_f$, and hence

$$\begin{aligned} J_N^1(\mathbf{d}_{k+M}^1) &= \frac{1}{\beta}\ell(x_{0|k+M} - y_{0|k+M}, u_{0|k+M} - v_{0|k+M}) \\ &+ \sum_{i=1}^{N-1} \ell(x_{i|k+M} - y_{i|k+M}, u_{i|k+M} - K y_{i|k+M}) \\ &+ V_f(x_{N|k+M} - y_{N|k+M}) \\ &\overset{\beta \ge 1, (F.8)}{\le} \sum_{i=0}^{N-M-1} \ell(x_{i|k+M} - y_{i|k+M}, u_{i|k+M} - K y_{i|k+M}) \\ &+ \sum_{i=N-M}^{N-1} \ell(x_{i|k+M} - y_{i|k+M}, u_{i|k+M} - K y_{i|k+M}) \\ &+ V_f(x_{N|k+M} - y_{N|k+M}). \end{aligned} \tag{F.11}$$

From (F.1) and (F.3) it follows that $x_{i|k+M} - y_{i|k+M} = x_{i+M|k} - y_{i+M|k} + e_{i+M}$, $i \in \mathbb{N}_{[0,N-M-1]}$, and from (F.9b) it follows that $u_{i|k+M} - K y_{i|k+M} = u_{i+M|k} - K y_{i+M|k} + K e_{i+M}$, $i \in \mathbb{N}_{[0,N-M]}$. Further, it holds that

$$x_{i|k+M} \overset{(C.1),(C.3)}{=} (A+BK)^{i-N+M} x_{N|k} + (A+BK)^i g_M \tag{F.12a}$$

$$u_{i|k+M} \overset{(C.2),(C.4)}{=} K(A+BK)^{i-N+M} x_{N|k} + K(A+BK)^i g_M \tag{F.12b}$$

for $i \in \mathbb{N}_{[N-M,N]}$ and $i \in \mathbb{N}_{[N-M,N-1]}$, respectively, such that with (F.6), it follows that

$$\begin{aligned} J_N^1(\mathbf{d}_{k+M}^1) \le &\sum_{i=0}^{N-M-1} \ell(x_{i+M|k} - y_{i+M|k} + e_{i+M}, \\ &u_{i+M|k} - K y_{i+M|k} + K e_{i+M}) \\ &+ \sum_{i=N-M}^{N-1} \ell\left((A+BK)^{i-N+M}(x_{N|k} - y_{N|k} + e_N),\right. \\ &\left. K(A+BK)^{i-N+M}(x_{N|k} - y_{N|k} + e_N)\right) \\ &+ V_f\left((A+BK)^M(x_{N|k} - y_{N|k} + e_N)\right). \end{aligned} \tag{F.13}$$

With Lemma 18 we obtain

$$\begin{aligned} J_N^1(\mathbf{d}_{k+M}^1) \le &\sum_{i=0}^{N-M-1} \ell(x_{i+M|k} - y_{i+M|k} + e_{i+M}, \\ &u_{i+M|k} - K y_{i+M|k} + K e_{i+M}) \\ &+ V_f(x_{N|k} - y_{N|k} + e_N). \end{aligned} \tag{F.14}$$

Finally, for all $i \in \mathbb{N}_{[0,N-M]}$ it holds that $e_{i+M} \in \mathcal{E}_{i+M}^M$, and, hence,

$$\begin{aligned} J_N^1(\mathbf{d}_{k+M}^1) &\overset{(18)}{\le} \sum_{i=0}^{N-M-1} \ell_{i+M}^M(x_{i+M|k} - y_{i+M|k}, \\ &\quad u_{i+M|k} - K y_{i+M|k}) + V_f^M(x_{N|k} - y_{N|k}) \\ &= \sum_{i=M}^{N-1} \ell_i^M(x_{i|k} - y_{i|k}, u_{i|k} - K y_{i|k}) + V_f^M(x_{N|k} - y_{N|k}) \\ &= J_N^M(\mathbf{d}_k^M) - \frac{1}{\beta}\sum_{i=0}^{M-1} \ell_i^M(x_{i|k} - y_{i|k}, u_{i|k} - v_{i|k}), \end{aligned} \tag{F.15}$$

thereby completing the proof. ■

Appendix G. Proof of Lemma 22

It holds that $\hat{\mathcal{X}}_N \subseteq \mathcal{X}$, such that $\hat{\mathcal{X}}_N$ is bounded. As additionally all sets involved in the definition of $\mathcal{D}_N^1$ are closed, it follows that $\hat{\mathcal{X}}_N$ is compact, thereby completing the proof. ■

Appendix H. Proof of Lemma 23

Let $x \in \tilde{\mathcal{X}}_N$ and $z \in (A+BK)x \oplus \mathcal{W}$ arbitrary. Then it holds that

$$\begin{aligned} (A+BK)^N z &\in (A+BK)^{N+1} x \oplus (A+BK)^N \mathcal{W} \\ &\subseteq (A+BK)\mathcal{X}_f^1 \oplus (A+BK)^N \mathcal{F}_1^1 \\ &\overset{(12c)}{\subseteq} \mathcal{X}_f^1. \end{aligned} \tag{H.1}$$

Further, it holds that

$$\begin{aligned} (A+BK)^i z \oplus \mathcal{F}_i^1 &\subseteq (A+BK)^{i+1} x \oplus (A+BK)^i \mathcal{W} \oplus \mathcal{F}_i^1 \\ &= (A+BK)^{i+1} x \oplus \mathcal{F}_{i+1}^1 \end{aligned} \tag{H.2}$$

for $i \in \mathbb{N}_{[0,N-1]}$. Hence, it holds that $(A+BK)^i z \oplus \mathcal{F}_i^1 \subseteq \mathcal{X}$ for all $i \in \mathbb{N}_{[0,N-2]}$. Moreover, it follows that

$$(A+BK)^{N-1} z \oplus \mathcal{F}_{N-1}^1 \tag{H.3}$$

$$
\begin{aligned}
&= (A+BK)^N x \oplus (A+BK)^{N-1}\mathcal{W} \oplus \mathcal{F}^1_{N-1} \\
&= (A+BK)^N x \oplus \mathcal{F}^1_N \\
&\subseteq \mathcal{X}^1_{\mathrm{f}} \oplus \mathcal{F}^1_N \overset{(12a)}{\subseteq} \mathcal{X}.
\end{aligned} \tag{H.4}
$$

Hence, it holds that $(A+BK)^i z \oplus \mathcal{F}^1_i \subseteq \mathcal{X}$ for all $i \in \mathbb{N}_{[0,N-1]}$. By the same arguments it follows that also $K(A+BK)^i z \oplus K\mathcal{F}^1_i \subseteq \mathcal{U}$ for all $i \in \mathbb{N}_{[0,N-1]}$. From all of the above it follows that $z \in \tilde{\mathcal{X}}_N$. ■

Appendix I. Proof of Theorem 27

The proof is obtained by extending the proofs of Theorems 2 and 3 in Barradas Berglind et al. (2012) to the stabilization of compact sets for disturbed systems. By the definition of the optimization problem in (32), for the closed-loop system (34), any $x_0 \in \hat{\mathcal{X}}_N$ and all $j \in \mathbb{N}$ it holds that $V_N^{M^\star(x_{k_j})}(x_{k_j}) \le V_N^1(x_{k_j})$, and hence

$$
\begin{aligned}
V_N^1(x_{k_{j+1}}) &\overset{\text{Lemma 21}}{\le} V_N^1(x_{k_j}) \\
&- \sum_{i=0}^{k_{j+1}-k_j-1} \frac{1}{\beta} \ell_i^{M^\star(x_{k_j})}(x^\star_{i|k_j}(x_{k_j}) - y^\star_{i|k_j}(x_{k_j}), \\
&u^\star_{i|k_j}(x_{k_j}) - v^\star_{i|k_j}(x_{k_j})).
\end{aligned} \tag{I.1}
$$

Further, by the reasoning in the proof of Lemma 21, specifically considering (F.1), for all $j \in \mathbb{N}$ and all $k \in \mathbb{N}_{[k_j,k_{j+1}-1]}$ there exists a $g_k \in \mathcal{F}_{k-k_j}^{M^\star(x_{k_j})}$ such that $x_k = x^\star_{k-k_j|k_j}(x_{k_j}) + g_k$. By (15a), there exist $\bar{y}_k \in \mathcal{Y}$ and $\bar{e}_k \in \mathcal{E}_{k-k_j}^{M^\star(x_{k_j})}$ such that $y^\star_{k-k_j|k_j}(x_{k_j}) + g_k = \bar{y}_k + \bar{e}_k$ and, hence, $x_k - \bar{y}_k = x^\star_{k-k_j|k_j}(x_{k_j}) - y^\star_{k-k_j|k_j}(x_{k_j}) + \bar{e}_k$. Further, consider that for all $j \in \mathbb{N}$ and all $k \in \mathbb{N}_{[k_j,k_{j+1}-1]}$ it holds that $u_k = u^\star_{i|k_j}(x_{k_j})$ and define $\bar{v}_k := v^\star_{k-k_j|k_j}(x_{k_j})$. Hence, for any $T \in \mathbb{N}$ it holds that

$$
\begin{aligned}
&\sum_{k=0}^{T-1} \min_{\substack{y_k\in\mathcal{Y}\\ v_k\in K\mathcal{Y}}} \ell(x_k - y_k, u_k - v_k) \\
&\le \sum_{k=0}^{k_T-1} \min_{\substack{y_k\in\mathcal{Y}\\ v_k\in K\mathcal{Y}}} \ell(x_k - y_k, u_k - v_k) \\
&\le \sum_{k=0}^{k_T-1} \ell(x_k - \bar{y}_k, u_k - \bar{v}_k) \\
&= \sum_{j=0}^{T-1} \sum_{k=k_j}^{k_{j+1}-1} \ell(x^\star_{k-k_j}(x_{k_j}) - y^\star_{k-k_j}(x_{k_j}) + \bar{e}_k, \\
&\quad u^\star_{k-k_j}(x_{k_j}) - v^\star_{k-k_j}(x_{k_j})) \\
&\le \sum_{j=0}^{T-1} \sum_{k=k_j}^{k_{j+1}-1} \max_{e_k\in\mathcal{E}_{k-k_j}^{M^\star(x_{k_j})}} \ell(x^\star_{k-k_j}(x_{k_j}) - y^\star_{k-k_j}(x_{k_j}) + e_k, \\
&\quad u^\star_{k-k_j}(x_{k_j}) - v^\star_{k-k_j}(x_{k_j})) \\
&= \sum_{j=0}^{T-1} \sum_{k=k_j}^{k_{j+1}-1} \ell_{k-k_j}^{M^\star(x_{k_j})}(x^\star_{k-k_j}(x_{k_j}) - y^\star_{k-k_j}(x_{k_j}), \\
&\quad u^\star_{k-k_j}(x_{k_j}) - v^\star_{k-k_j}(x_{k_j})) \\
&\overset{(I.1)}{\le} \sum_{j=0}^{T-1} \beta V_N^1(x_{k_j}) - \beta V_N^1(x_{k_{j+1}}) \\
&= \beta V_N^1(x_0) - \beta V_N^1(x_{k_T}) \le \beta V_N^1(x_0).
\end{aligned} \tag{I.2}
$$

As the inequality in (I.2) holds for every $T \in \mathbb{N}$, it also holds that

$$
\sum_{k=0}^{\infty} \min_{\substack{y_k\in\mathcal{Y}\\ v_k\in K\mathcal{Y}}} \ell(x_k - y_k, u_k - v_k) \le \beta V_N^1(x_0), \tag{I.3}
$$

thereby completing the proof. ■

Appendix J. Proof of Theorem 28

By the reasoning in the proof of Theorem 27, particularly by (I.3) and by considering (22a) in Assumption 17, it holds that

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \alpha_1(|x_k|_{\mathcal{Y}}) \\
&= \sum_{k=0}^{\infty} \alpha_1\left(\min_{y_k\in\mathcal{Y}} |x_k - y_k|\right) = \sum_{k=0}^{\infty} \min_{y_k\in\mathcal{Y}} \alpha_1(|x_k - y_k|) \\
&\overset{(22a)}{\le} \sum_{k=0}^{\infty} \min_{\substack{y_k\in\mathcal{Y}\\ v_k\in K\mathcal{Y}}} \ell(x_k - y_k, u_k - v_k) \overset{(I.3)}{\le} \beta V_N^1(x_0).
\end{aligned} \tag{J.1}
$$

As $V_N^1(x_0)$ is finite for any $x_0 \in \hat{\mathcal{X}}_N$, it follows that $\lim_{k\to\infty} |x_k|_{\mathcal{Y}} = 0$ for any $x_0 \in \hat{\mathcal{X}}_N$ and any disturbances with $w_k \in \mathcal{W}$, $k \in \mathbb{N}$, thereby completing the proof. ■

Appendix K. Proof of Theorem 29

The proof of Theorem 28, particularly (J.1), gives that for all $x_0 \in \hat{\mathcal{X}}_N$ and all $k \in \mathbb{N}$ that $|x_k|_{\mathcal{Y}} \le \alpha_1^{-1}(\beta V_N^1(x_0)) \le \alpha_1^{-1}(\beta\alpha_3(|x_0|_{\mathcal{Y}})) = \gamma(|x_0|_{\mathcal{Y}})$, where the first inequality follows from Lemma 24 and γ is a $\mathcal{K}$-function defined by $\gamma(s) = \alpha_1^{-1}(\beta\alpha_3(s))$ for all $s \ge 0$. As by assumption there exists an $\eta > 0$ such that $\eta\mathcal{B} \oplus \mathcal{Y} \subseteq \tilde{\mathcal{X}}_N$ and the fact that $\tilde{\mathcal{X}}_N \subseteq \hat{\mathcal{X}}_N$, it follows that the set $\mathcal{Y}$ is robustly Lyapunov stable for the closed-loop dynamics (34). In combination with Theorem 28 it follows that the set $\mathcal{Y}$ is robustly asymptotically stable for the closed-loop dynamics (34) and the set $\hat{\mathcal{X}}_N$ belongs to its region of attraction, thereby completing the proof. ■

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