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Communication Scheduling in Robust Self-Triggered MPC for Linear Discrete-Time Systems ⋆

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Abstract: We consider a networked control system consisting of a physical plant, an actuator, a sensor, and a controller that is connected to the actuator and sensor via a communication network. The plant is described by a linear discrete-time system subject to additive disturbances. In order to reduce the required number of communications in the system, we propose a robust self-triggered model predictive controller based on rollout techniques that robustly asymptotically stabilizes a certain periodic sequence of sets in the state space while guaranteeing robust satisfaction of hard state and input constraints. At periodically occurring scheduling times, the self-triggered model predictive control algorithm determines the times at which the control input and plant measurement are updated in the time span until the next scheduling time. We establish a certain upper bound on the average sampling rate in the closed-loop system. Moreover, we show how increasing the asymptotic bound on the system state, which is a design parameter in the control scheme, can be used to further reduce the average number of communications in the system.

Keywords: predictive control, self-triggered control, robustness

1. INTRODUCTION

If the cost of communication in a networked control system (NCS) can not be neglected, the amount of communication between the different components must be included in the performance specification of any controller design. The cost of communication is especially high if the communication takes part over a wireless network. Please refer to Hespanha et al. (2007) for a discussion of these (and other) issues in NCSs. It has been found that a considerable reduction in communication between sensors, actuators, and controllers can be achieved, without sacrificing much performance, by employing aperiodic sampling techniques. Instead of sampling the plant outputs and updating the control inputs at a priori determined points in time, the sampling and update times are determined online depending, among other things, on the state of the plant. Two approaches to aperiodic sampling have received particular attention in recent years, that is event-triggered control and self-triggered control. The former method relies on continuous or periodic monitoring of the plant output, where control updates are only triggered if certain conditions on the plant output are met. In the latter method, given a time when the plant output is sampled and the control inputs are updated, the next sampling instant is computed as an explicit function of the current state of the system. Self-triggered control methods have the advantage of not requiring as many measurements of the plant output as event-triggered control methods, allowing additional energy to be saved by completely shutting down the sensors and the communication system between sampling instances. Please refer to Heemels et al. (2012) for an overview of event- and self-triggered control.

For setups with hard constraints on the input and state of the system, model predictive control (MPC) is a suitable choice for computing the control signals. In MPC, the input at a given sampling instant is defined as the first part of a solution to a finite horizon optimal control problem, parameterized by the state at the given sampling instant. Constraints are simply handled by inclusion in the optimal control problem. Please refer to Rawlings and Mayne (2009) for an overview of MPC. As such, in case hard constraints are present in the system *and* the cost of communication cannot be neglected, it is of interest to consider self-triggered MPC schemes.

In self-triggered control, if the goal of reducing the amount of communication is achieved, it necessarily follows that there are significant time-spans in which the system is controlled in an open-loop fashion. This fact requires particular attention to robustness considerations, especially if the system to be controlled is open-loop unstable. In the context of MPC, constraint tightening methods such as Tube MPC offer robustness guarantees against uncer-

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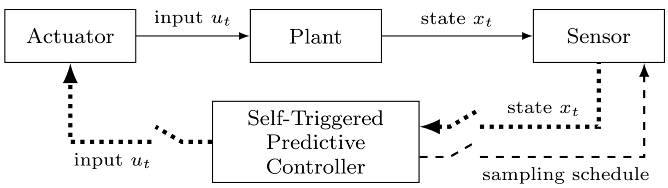

Fig. 1. Communication structure in the control system.

tainties and disturbances, see for example Chisci et al. (2001) and Langson et al. (2004). However, standard Tube MPC approaches rely on the assumption that feedback is possible at every point in time, an assumption that is intentionally not satisfied in self-triggered control approaches. However, it is possible to adapt these robust methods to the case when feedback is possible at *some* points in time, as we will show in the present paper.

The setup considered in this paper is illustrated in Figure 1. Our goal is to reduce the amount of communication between the sensor, the actuator and the controller, while guaranteeing robust constraint satisfaction and a certain asymptotic bound on the system state. For this, we propose a robust self-triggered model predictive controller inspired by the rollout techniques presented in Antunes and Heemels (2014) for the control of unconstrained systems. At periodically occurring scheduling times, based on the current state of the plant, the algorithm decides the time instances in the time span until the next scheduling time at which the control input is to be updated and the system state has to be sampled. This schedule is determined by the solution of an optimal control problem, where a certain base schedule is designed to be a feasible solution that upper bounds the number of sampling instances until the next scheduling time. This is the reason for the "rollout" terminology, see Bertsekas (2005), Antunes and Heemels (2014) and the references therein. We employ constraint tightening methods extending those in Chisci et al. (2001) in order to guarantee robust constraint satisfaction. Further, we propose a method to design the relevant optimization problems such that an a priori known asymptotic bound on the system state is satisfied, allowing a trade-off between disturbance rejection and communication in the closed-loop system.

Other results on robust self-triggered control can for example be found in Aydiner et al. (2015), Brunner et al. (2014), Eqtami (2013), and in the references therein, although without any stated guarantees on the average sampling rate. In Kögel and Findeisen (2014), a robust self-triggered controller guaranteeing upper bounds on the average sampling rate is proposed, without taking stability into consideration. MPC schemes considering scheduling of communication channels were recently proposed for example in Lješnjanin et al. (2014) and Zou et al. (2014).

The remainder of the paper is structured as follows. The notation and some preliminary statements are introduced in Section 2. The problem setup is presented in Section 3. The MPC scheme is described in Section 4 and its main properties are given in Section 5. Some issues regarding implementation of the scheme and its complexity are briefly discussed in Section 6. Section 7 contains a numerical example illustrating the results and Section 8 concludes the paper with an outlook on open questions.

For the sake of brevity, the proofs for the statements in the paper have been omitted.

2. NOTATION AND PRELIMINARIES

Notation: Let $\mathbb{N}$ denote the set of non-negative integers. For $q, s \in \mathbb{N}$, let $\mathbb{N}_{[q,s]}$ denote the set $\{r \in \mathbb{N} \mid q \leq r \leq s\}$ and $\mathbb{N}_{\geq q}$ the set $\{r \in \mathbb{N} \mid r \geq q\}$. For a given real number $a \in \mathbb{R}$, we use $\mathbb{R}_{\geq a}$ and $\mathbb{R}_{>a}$ to denote the set of real numbers greater than a, or greater than or equal to a, respectively. We use I_n to denote the n-dimensional identity matrix and 0 to denote a zero matrix of appropriate dimension. Given sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$, a scalar α, and a matrix $A \in \mathbb{R}^{m\times n}$, we define $\alpha\mathcal{X} := \{\alpha x \mid x \in \mathcal{X}\}$ and $A\mathcal{X} := \{Ax \mid x \in \mathcal{X}\}$. The Minkowski set addition is defined by $\mathcal{X} \oplus \mathcal{Y} := \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$. Given a vector $x \in \mathbb{R}^n$ we define $\mathcal{X} \oplus x := x \oplus \mathcal{X} := \{x\} \oplus \mathcal{X}$. The Pontryagin set difference is defined by $\mathcal{X} \ominus \mathcal{Y} := \{z \in \mathbb{R}^n \mid z \oplus \mathcal{Y} \subseteq \mathcal{X}\}$, see Kolmanovsky and Gilbert (1995, 1998). Given a sequence of sets $\mathcal{X}_i$ for $i \in \mathbb{N}_{[a,b]}$ with $a, b \in \mathbb{N}$, we define $\bigoplus_{i=a}^{b} \mathcal{X}_i = \mathcal{X}_a \oplus \mathcal{X}_{a+1} \oplus \ldots \oplus \mathcal{X}_b$. By convention, the empty sum is equal to $\{0\}$. Similarly, for any vectors $v_i \in \mathbb{R}^n$, $i \in \mathbb{N}$, we define $\sum_{i=a}^{b} v_i = 0$ and for any matrices $A_i \in \mathbb{R}^{n\times n}$, we define $\prod_{i=a}^{b} A_i = I_n$ for any $a, b \in \mathbb{N}$ if $a > b$. We call a compact, convex set containing the origin a C-set. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. If additionally $\alpha(s) \to \infty$ as $s \to \infty$, α is said to belong to class $\mathcal{K}_\infty$. The Euclidean norm of a vector $v \in \mathbb{R}^n$ is denoted by $|v|$. Given any compact set $\mathcal{Y} \subseteq \mathbb{R}^n$, the distance between v and $\mathcal{Y}$ is defined by $|v|_\mathcal{Y} := \min_{s\in\mathcal{Y}} |v - s|$. Define finally the Euclidean unit ball by $\mathcal{B} := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

Definition 1. Let a dynamical system of the form

$$(x_{t+1}, \eta_{t+1}) = f(x_t, \eta_t, w_t, t) \tag{1}$$

be given, where $f : \mathbb{R}^n \times \mathbb{R}^M \times \mathcal{W} \times \mathbb{N} \to \mathbb{R}^n \times \mathbb{R}^M$, $x_0 \in \mathbb{R}^n$, $\eta_0 = g(x_0)$ for a $g : \mathbb{R}^n \to \mathbb{R}^M$, $w_t \in \mathcal{W}$, $t \in \mathbb{N}$, with a compact set $\mathcal{W} \subseteq \mathbb{R}^n$. A sequence of compact sets $\mathcal{Y}^t \subseteq \mathbb{R}^n$, $t \in \mathbb{N}$, is *robustly asymptotically stable with region of attraction* $\hat{\mathcal{X}} \subseteq \mathbb{R}^n$ for this system, if there exists a class $\mathcal{K}$-function α, such that $|x_t|_{\mathcal{Y}^t} \leq \alpha(|x_0|_{\mathcal{Y}^0})$, $t \in \mathbb{N}$, and $\lim_{t\to\infty} |x_t|_{\mathcal{Y}^t} = 0$ for all $x_0 \in \hat{\mathcal{X}}$, $w_t \in \mathcal{W}$, $t \in \mathbb{N}$.

3. PROBLEM SETUP

We consider linear discrete-time systems of the form

$$x_{t+1} = Ax_t + Bu_t + w_t, \tag{2}$$

where $x_t \in \mathbb{R}^n$ is the state and $u_t \in \mathbb{R}^m$ is the control input at time $t \in \mathbb{N}$. The disturbance w_t is assumed to be time-varying, unknown, and to satisfy $w_t \in \mathcal{W} \subseteq \mathbb{R}^n$, $t \in \mathbb{N}$, where $\mathcal{W}$ is a known C-set. Further, hard constraints $x_t \in \mathcal{X}$, $u_t \in \mathcal{U}$, $t \in \mathbb{N}$, on the input and state are given, where $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are C-sets. We assume that the state x_t is available to the sensor as a measurement at any time step $t \in \mathbb{N}$, when needed. We want to reduce the amount of communication between the sensor and the controller and the controller and the actuator, compare Figure 1. In order to save communication, the input u_t will be determined by a self-triggered control scheme of the form

$$u_t = \begin{cases} \kappa(x_t, \mu_k, t - T_k) & \text{if } \mu_k^{t-T_k} = 1, \\ 0 & \text{else,} \end{cases} \tag{3a}$$

$$\mu_k = \Omega(x_{T_k}), \quad T_k = Mk, \tag{3b}$$

for all $t \in \mathbb{N}_{[T_k, T_{k+1}-1]}$, and $k \in \mathbb{N}$, with the controller $\kappa : \mathbb{R}^n \times \{0, 1\}^M \times \mathbb{N} \to \mathbb{R}^m$ and the scheduling function $\Omega : \mathbb{R}^n \to \{0, 1\}^M$. The number $M \in \mathbb{N}_{\geq 1}$ is fixed. Here,

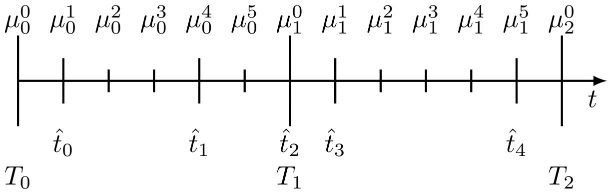

Fig. 2. Exemplary illustration of the different time scales in the control system. Communication of measurements are required at times $\hat{t}_j$ and T_k for $j, k \in \mathbb{N}$. Communications of control inputs occur at times $\hat{t}_j$ for $j \in \mathbb{N}$.

the *schedules* $\mu_k \in \{0,1\}^M$ are decided based on the state x_{T_k} at the *scheduling instances* T_k and determine the time when the control values can be non-zero, that is the *control update instants* $\hat{t} \in \hat{\mathcal{T}} \subseteq \mathbb{N}$, where $\hat{\mathcal{T}} := \{t \in \mathbb{N} \mid \exists k \in \mathbb{N} : \exists i \in \mathbb{N}_{[0,M-1]} : \mu_k^i = 1, t = T_k + i\}$. Here, μ_k^i refers to the ith position in μ_k for $i \in \mathbb{N}_{[0,M-1]}$. Clearly, $\hat{\mathcal{T}}$ depends on x_0 and the realization of the disturbances w_t, $t \in \mathbb{N}$. We define the sequence of control update instances by $\hat{t}_0 := \inf \hat{\mathcal{T}}$ and $\hat{t}_{j+1} := \inf\{\hat{t} \in \hat{\mathcal{T}} \mid \hat{t} > \hat{t}_j\}$, $j \in \mathbb{N}$. Please also refer to Figure 2 for an illustration of the different time scales for the case $M = 6$. Updates of the state measurements are only required at the time points $\hat{\mathcal{T}} \cup M\mathbb{N}$ and updates of the control inputs only at the time points $\hat{\mathcal{T}}$. The sensors and the respective communication links may be shut down at other times. Note that this setup requires the controller to be able to communicate the sampling times to the sensor, as shown in Figure 1.

Remark 1. The setup can easily be modified to the case where the input is kept constant between control update instants instead of being nonzero only at the control update instants by including the input as a state in the system description and interpreting the variable u_t as the difference between the current and the previous input at time t.

Our goal is to design the controller κ and the scheduling function Ω for the closed-loop system consisting of (2) and (3) such that (i) the constraints $x_t \in \mathcal{X}$, $u_t \in \mathcal{U}$, $t \in \mathbb{N}$, are robustly satisfied, (ii) a periodic sequence of C-sets $\mathcal{Y}^t \subseteq \mathbb{R}^n$ with $\mathcal{Y}^{t+M} = \mathcal{Y}^t$, $t \in \mathbb{N}$, is robustly asymptotically stable, and (iii) the amount of communication is minimized. For every schedule $\mu_k \in \{0,1\}^M$, $k \in \mathbb{N}$, we define $\|\mu_k\|_c$ as the number of required communications in the time span $[T_k, T_{k+1}-1]$, which is the sum of the number of scheduled control update instances and the number of measurements. The number of control update instances is equal to the number of nonzero entries in μ_k, and the number of measurements is equal to the number of nonzero entries in μ_k, where the first entry is set to one, due to a measurement being necessary for the scheduling occurring at the time instances T_k. We do not consider the resources required to communicate the schedule from the sensor to the controller, as this data only has to be communicated at times T_k and consists merely of ones and zeros. The efficiency with which this communication can take place depends on the protocol that is used, which is beyond the scope of the present paper.

4. ROBUST SELF-TRIGGERED MPC

We propose a solution to the problem described in the previous section based on Tube MPC methods. At every control update instant $\hat{t}_j \in \hat{\mathcal{T}} \cap [T_k, T_{k+1} - 1]$, $j, k \in \mathbb{N}$, a finite horizon optimal control problem, depending on the current schedule μ_k, is solved that determines the input $u_{\hat{t}_j}$. For all $t \in \mathbb{N} \setminus \hat{\mathcal{T}}$, the input u_t is set to zero. Furthermore, at every scheduling instant $T_k = Mk$, $k \in \mathbb{N}$, multiple finite horizon optimal control problems are solved in order to determine the schedule μ_k. As proposed in Chisci et al. (2001), the constraints in the optimal control problems are tightened in order to guarantee robust constraint satisfaction. As the uncertainty in the prediction depends on the assumed schedule, different schedules require different tightenings of constraints.

The control scheme relies on a periodic base schedule $\bar{\mu} \in \{0,1\}^M$, where we define $\bar{\mu}^{i+M} := \bar{\mu}^i$ for $i \in \mathbb{N}$, and an associated periodic feedback law defined by the matrices $K_i^{\bar{\mu}} \in \mathbb{R}^{m \times n}$, where $K_{i+M}^{\bar{\mu}} = K_i^{\bar{\mu}}$, $i \in \mathbb{N}$ and $\bar{\mu}^i = 0 \Rightarrow K_i^{\bar{\mu}} = 0$ for $i \in \mathbb{N}$. The following assumption is required to hold.

Assumption 1. The matrix $\prod_0^{M-1}(A + BK_{M-1-i}^{\bar{\mu}})$ is Schur.

4.1 Setup of the MPC scheme

For a given $x_t \in \mathbb{R}$ with $t \in \mathbb{N}_{[T_k, T_{k+1}-1]}$, $k \in \mathbb{N}$, and a schedule $\mu_k \in \{0,1\}^M$, the finite horizon optimal control problem is defined as follows. The decision variable of the optimization problem is

$$\mathbf{d}_t = ((x_{0|t}, \ldots, x_{N|t}), (u_{0|t}, \ldots, u_{N-1|t})) \in \mathbb{D}_N, \qquad (4)$$

where $\mathbb{D}_N = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$ and $N \in \mathbb{N}_{\geq M}$ is the prediction horizon. Let $l := t - T_k$. The constraints

$$x_{0|t} = x_t, \qquad (5a)$$

$$\forall i \in \mathbb{N}_{[0,N-1]},\ x_{i+1|t} = Ax_{i|t} + Bu_{i|t}, \qquad (5b)$$

$$\forall i \in \mathbb{N}_{[0,N-1]}, \quad x_{i|t} \in \mathcal{X}_i^{\mu_k,l}, \qquad (5c)$$

$$\forall i \in \mathbb{N}_{[0,N-1]}, \quad u_{i|t} \in \mathcal{U}_i^{\mu_k,l}, \qquad (5d)$$

$$\forall i \in \mathbb{N}_{[0,M-l-1]}, \quad u_{i|t} = 0,\ \text{if } \mu_k^{i+l} = 0, \qquad (5e)$$

$$\forall i \in \mathbb{N}_{[M-l,N-1]}, \quad u_{i|t} = 0,\ \text{if } \bar{\mu}^{i+l} = 0, \qquad (5f)$$

$$x_{N|t} \in \mathcal{X}_{\mathrm{f}}^{\mu_k,l} \qquad (5g)$$

are imposed on $\mathbf{d}_t$, where the variables $x_{i|t}$ represent a predicted trajectory for the undisturbed system generated by the inputs $u_{i|t}$ according to (5b). Following the ideas in Antunes and Heemels (2014), it is assumed in the predictions that the base schedule $\bar{\mu}$ is selected after the first $M - l$ steps for the remainder of the time. The sets $\mathcal{X}_i^{\mu_k,l}$ and $\mathcal{U}_i^{\mu_k,l}$, $i \in \mathbb{N}_{[0,N-1]}$, are tightened constraint sets, depending on the step i in the prediction. These sets are defined by

$$\mathcal{X}_i^{\mu_k,l} := \mathcal{X} \ominus \mathcal{F}_i^{\mu_k,l}, \quad i \in \mathbb{N}_{[0,N-1]}, \qquad (6a)$$

$$\mathcal{U}_i^{\mu_k,l} := \mathcal{U} \ominus K\mathcal{F}_i^{\mu_k,l},\ i \in \mathbb{N}_{[0,N-1]}, \qquad (6b)$$

where the sets $\mathcal{F}_i^{\mu_k,l} \subseteq \mathbb{R}^n$ are chosen in order to capture the worst-case uncertainty in the prediction. The set $\mathcal{X}_{\mathrm{f}}^{\mu_k,l}$ is a terminal set, which, as well as the sets $\mathcal{F}_i^{\mu_k,l}$, $i \in \mathbb{N}$, will be defined in Subsection 4.3. Define the set of all feasible decision variables for a given point $x_t \in \mathbb{R}^n$ with $t \in \mathbb{N}_{[T_k, T_{k+1}-1]}$, $k \in \mathbb{N}$, and a schedule $\mu_k \in \{0,1\}^M$, by

$$\mathcal{D}_N^{\mu_k,l}(x_t) = \{\mathbf{d}_t \in \mathbb{D}_N \mid \text{(5a) to (5g)}\}. \qquad (7)$$

We additionally define $\mathcal{D}_N^{\mu_k,M} := \mathcal{D}_N^{\bar{\mu},0}$ for all $\mu_k \in \{0,1\}^M$.

The cost function for the finite horizon optimal control problem is based on feedback matrices $K_i^{\mu_k}$, $i \in \mathbb{N}$, depending on the schedule μ_k, where $K_i^{\mu_k} = K_i^{\bar{\mu}}$, $i \in \mathbb{N}_{\geq M}$, which correspond to the feedback $u = K_i^{\mu_k} x$ satisfying $\mu_k^i = 0 \Rightarrow K_i^{\mu_k} = 0$ for $i \in \mathbb{N}_{[0,M-1]}$.

Remark 2. The matrices K_i^{μ}, $i \in \mathbb{N}_{[0,M-1]}$, may in principle be chosen arbitrarily, as long as they satisfy $\mu^i = 0 \Rightarrow K_i^{\mu} = 0$ for $i \in \mathbb{N}_{[0,M-1]}$. However, it makes sense to choose them in a way such that the disturbances are attenuated as much as possible.

For all $t \in \mathbb{N}_{[T_k,T_{k+1}]}$ and all $\mathbf{d}_t \in \mathbb{D}_N$ the cost function is defined by

$$J_N^{\mu_k,l}(\mathbf{d}_t) = \sum_{i=0}^{M-l-1} \ell(u_{i|t} - K_{i+l}^{\mu_k} x_{i|t}) + \sum_{i=M-l}^{N-1} \ell(u_{i|t} - K_{i+l}^{\bar{\mu}} x_{i|t}) \quad (8)$$

for a stage cost function $\ell : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$.

The finite horizon optimal control problem to be solved is defined for any $x_t \in \mathbb{R}$ with $t \in \mathbb{N}_{[T_k,T_{k+1}]}$, $k \in \mathbb{N}$, and any schedule $\mu_k \in \{0,1\}^M$ by

$$V_N^{\mu_k,l}(x_t) := \min_{\mathbf{d}_t \in \mathcal{D}_N^{\mu_k,l}(x_t)} J_N^{\mu_k,l}(\mathbf{d}_t), \quad (9a)$$

$$\mathbf{d}_t^{\mu_k,l}(x_t) := \arg\min_{\mathbf{d}_t \in \mathcal{D}_N^{\mu_k,l}(x_t)} J_N^{\mu_k,l}(\mathbf{d}_t). \quad (9b)$$

Remark 3. In the case of non-unique minimizers, it is assumed that $\mathbf{d}_t^{\mu_k,l}(x_t)$ is any solution to the optimization problem.

The set where the optimization problem in (9) is feasible is defined by $\hat{\mathcal{X}}_N^{\mu_k,l} := \{x \in \mathbb{R}^n \mid \mathcal{D}_N^{\mu_k,l}(x) \neq \emptyset\}$. Given any $\mathbf{d}_t^{\mu_k,l}(x_t) = ((x_{0|t}^{\mu_k,l}, \ldots, x_{N|t}^{\mu_k,l}), (u_{0|t}^{\mu_k,l}, \ldots, u_{N-1|t}^{\mu_k,l}))$ for a schedule $\mu_k \in \{0,1\}^M$ and an $x_t \in \mathbb{R}^n$, $t \in \mathbb{N}_{[T_k,T_{k+1}-1]}$, the control law is defined by $\kappa(x_t, \mu_k, t - T_k) = u_{0|t}^{\mu_k,l}$.

4.2 Scheduling Function

The asymptotic bound on the system state depends on the schedules that are applied, where schedules with less frequent updates typically lead to a larger asymptotic bound. For this reason, additional functions $V_{\mathrm{s}}^{\mu,l} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\mu \in \{0,1\}^M$, $l \in \mathbb{N}_{[0,M-1]}$, are introduced which put a cost on the attenuation of the disturbances in the closed-loop system guaranteed by the application of the schedule μ at a given point in the state space. In particular, let

$$V_{\mathrm{s}}^{\mu,l}(x) := \inf_{y \in \mathcal{Y}^{\mu,l}} J_{\mathrm{s}}^{\mu,l}(x - y) + c^{\mu,l}, \quad (10)$$

where $J_{\mathrm{s}}^{\mu,l} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\mathcal{Y}^{\mu,l} \subseteq \mathbb{R}^n$, and $c^{\mu,l} \in \mathbb{R}_{\geq 0}$, all of which will be defined later on. The sets $\mathcal{Y}^{\mu,0}$ are related to the sets of states which can be robustly controlled within M steps to the sequence of compact sets in the state space that is to be is stabilized for the closed-loop system under the overall scheme. A schedule μ with less frequent updates will typically [1] have a smaller set $\mathcal{Y}^{\mu,0}$ and hence in general a larger penalty $V_{\mathrm{s}}^{\mu,l}$ associated with it. This implies that this particular schedule will be chosen for a smaller set of states in the state space. Define also $V_{\mathrm{s}}^{\mu,M} := V_{\mathrm{s}}^{\bar{\mu},0}$ for all $\mu \in \{0,1\}^M$. Define further

$$V_{\Omega}^{\mu,l}(x) := V_{\mathrm{s}}^{\mu,l}(x) + V_N^{\mu,l}(x) \quad (11)$$

for all $x \in \hat{\mathcal{X}}_N^{\mu_k,l}$, $\mu \in \{0,1\}^M$ and $l \in \mathbb{N}_{[0,M]}$. Let $\mathcal{M} \subseteq \{0,1\}^M$ denote the set of all allowed schedules, where $\bar{\mu} \in \mathcal{M}$. For any given $x_{T_k} \in \mathbb{R}^n$, $T_k \in \mathbb{N}$, the scheduling function Ω is defined by the optimization problem

$$\Omega(x_{T_k}) = \arg\min_{\mu \in \mathcal{M}} \{\|\mu\|_c \mid \mathcal{D}_N^{\mu,0}(x_{T_K}) \neq \emptyset, \mathcal{D}_N^{\bar{\mu},0}(x_{T_K}) \neq \emptyset$$
$$V_{\Omega}^{\mu,0}(x_{T_K}) \leq V_{\Omega}^{\bar{\mu},0}(x_{T_K})\}, \quad (12)$$

which is adapted from Gommans and Heemels (2015). The intuition behind this definition of Ω is to select the schedule with the least number of required communications which guarantees (i) robust constraint satisfaction and (ii) a bound on the cost function which ensures the stabilization of a certain sequence of compact sets. With this, both components of the control scheme, that is the controller κ and the scheduling function Ω are defined, except for some of the cost functions and some of the sets involved in the optimization problem. These functions and sets are defined in the subsequent subsection.

The overall closed-loop system consisting of (2) and (3) is given by

$$x_{t+1} = Ax_t + Bu_t + w_t, \quad (13a)$$
$$u_t = \kappa(x_t, \mu_k, t - T_k), \text{ with } T_k = Mk \text{ and}$$
$$k = \min\{k' \in \mathbb{N} \mid Mk' \leq t\} \quad (13b)$$
$$\mu_k = \Omega(x_{T_k}), \quad (13c)$$

where $x_0 \in \mathbb{R}^n$, and $w_t \in \mathcal{W}$, $t \in \mathbb{N}$. Due to constraints (5e) and (5f), distinguishing several cases as in (3a) is not necessary. Note that this system can be written in the form of (1) by defining $\eta_t = \mu_k$ with $k = \min\{k' \in \mathbb{N} \mid Mk \leq t\}$ and $g = \Omega$.

4.3 Assumptions on the constraints and cost functions

The sets $\mathcal{F}_i^{\mu,l}$, $\mu \in \mathcal{M}$, $l \in \mathbb{N}_{[0,M-1]}$, $i \in \mathbb{N}$, are defined in the following recursive fashion.

$$\mathcal{F}_0^{\mu,l} := \{0\}, \ \mathcal{F}_{i+1}^{\mu,l} := (A + BK_{i+l}^{\mu})\mathcal{F}_i^{\mu,l} \oplus \mathcal{W}, \ i \in \mathbb{N}, \quad (14)$$

which, considering the definition of K_i^{μ}, is consistent with the assumption that the schedule μ is applied for the first $M - l$ time steps and the base schedule $\bar{\mu}$ is applied for all further time steps.

The following assumption is made on the terminal sets.

Assumption 2. It holds that $\mathcal{X}_{\mathrm{f}}^{\mu,l}$ are C-sets satisfying

$$\mathcal{X}_{\mathrm{f}}^{\mu,l} \subseteq \mathcal{X} \ominus \mathcal{F}_N^{\mu,l}, \quad (15a)$$
$$K_{l+N}^{\bar{\mu}} \mathcal{X}_{\mathrm{f}}^{\mu,l} \subseteq \mathcal{U} \ominus K_{l+N}^{\bar{\mu}} \mathcal{F}_N^{\mu,l}, \quad (15b)$$
$$(A + BK_{l+N}^{\bar{\mu}})\mathcal{X}_{\mathrm{f}}^{\mu,l} \oplus \prod_{i=0}^{N-1} (A + BK_{N+l-i}^{\mu})\mathcal{W} \subseteq \mathcal{X}_{\mathrm{f}}^{\mu,l+1}, \quad (15c)$$

for all $\mu \in \mathcal{M}$, $l \in \mathbb{N}_{[0,M-1]}$, where $\mathcal{X}_{\mathrm{f}}^{\mu,M} := \mathcal{X}_{\mathrm{f}}^{\bar{\mu},0}$.

Remark 4. Terminal sets satisfying Assumption 2 can be constructed by first computing $\mathcal{X}_{\mathrm{f}}^{\bar{\mu},l}$, $l \in \mathbb{N}_{[0,M-1]}$, as a robustly periodic invariant sequence of sets, see for example Gondhalekar and Jones (2011), and then computing $\mathcal{X}_{\mathrm{f}}^{\mu,l}$ for $\mu \in \mathcal{M} \setminus \{\bar{\mu}\}$, $l \in \mathbb{N}_{[0,M-1]}$, recursively backwards by intersecting the inclusions in (15), starting with $\mathcal{X}_{\mathrm{f}}^{\mu,M} := \mathcal{X}_{\mathrm{f}}^{\bar{\mu},0}$.

The sets $\mathcal{Y}^{\mu,l}$ used in the definition of the function $V_{\mathrm{s}}^{\mu,l}$ in (10), are required to satisfy the following assumption.

Assumption 3. The sets $\mathcal{Y}^{\bar{\mu},i}$ are C-sets for $\mu \in \mathcal{M}$, $i \in \mathbb{N}$ where

$$(A + BK_i^{\bar{\mu}})\mathcal{Y}^{\bar{\mu},i} \oplus \mathcal{W} \subseteq \mathcal{Y}^{\bar{\mu},i+1}, \quad (16a)$$
$$\mathcal{Y}^{\bar{\mu},i+M} = \mathcal{Y}^{\bar{\mu},i}. \quad (16b)$$

[1] assuming the feedback gains are chosen such that the disturbances are attenuated in comparison to the open-loop system

Further, the sets $\mathcal{Y}^{\mu,l}$ are C-sets and there exist C-sets $\mathcal{G}^{\mu,l}$, for $\mu \in \mathcal{M} \setminus \{\bar{\mu}\}$, $l \in \mathbb{N}_{[0,M-1]}$, such that

$$(A + BK_l^\mu)\mathcal{Y}^{\mu,l} \oplus \mathcal{W} \subseteq \mathcal{Y}^{\mu,l+1} \oplus \mathcal{G}^{\mu,l}, \quad (17a)$$

$$\mathcal{Y}^{\mu,l} \subseteq \mathcal{Y}^{\bar{\mu},l}, \quad (17b)$$

where $\mathcal{Y}^{\mu,M} = \mathcal{Y}^{\bar{\mu},0}$, $l \in \mathbb{N}_{[0,M-1]}$.

Remark 5. Remark 4 applies mutatis mutandis to the sets in Assumption 3, the main difference being that the sets in Assumption 2 are chosen as large as possible while the sets in Assumption 3 are usually chosen as small as possible, but see also the second paragraph in Section 6.

Finally, we make the following assumptions on the cost functions.

Assumption 4. The functions ℓ and $J_s^{\mu,l}$, $\mu \in \mathcal{M}$, $l \in \mathbb{N}_{[0,M-1]}$, are continuous and positive semi-definite. Further, there exist a positive definite function $q : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and nonnegative scalars $d^{\mu,l} \in \mathbb{R}_{\geq 0}$, $\mu \in \mathcal{M}$, $l \in \mathbb{N}_{[0,M-1]}$, such that $J_s^{\mu,l+1}((A+BK_l^\mu)z+Bv+g) \leq J_s^{\mu,l}(z)-q(z)+\ell(v)+d^{\mu,l}$, for all $z \in \mathbb{R}^n$, $v \in \mathbb{R}^n$, and $g \in \mathcal{G}^{\mu,l}$, where $J_s^{\mu,M} = J_s^{\bar{\mu},0}$, $\mu \in \mathcal{M}$ and $d^{\bar{\mu},l} = 0$, $l \in \mathbb{N}_{[0,M-1]}$. Finally, there exist $\mathcal{K}_\infty$-functions α_1 and α_2 such that $q(z) \geq \alpha_1(|z|)$ and $J_s^{\bar{\mu},0}(z) \leq \alpha_2(|z|)$ for all $z \in \mathbb{R}^n$.

The scalars $c^{\mu,l}$ are recursively defined by $c^{\mu,M} = 0$, $c^{\mu,l} = c^{\mu,l+1} + d^{\mu,l}$, $l \in \mathbb{N}_{[0,M-1]}$.

Remark 6. Quadratic functions satisfying Assumption 4 can be computed by solving appropriate linear matrix inequalities.

5. MAIN PROPERTIES OF THE MPC SCHEME

In this section, the main properties of the proposed self-triggered MPC scheme are presented, that is, well-definedness of the controller, robust constraint satisfaction, and asymptotic stability of a periodic sequence of compact sets for the closed-loop system (13).

First, we establish that the closed-loop system is well-defined in the sense that if the optimization problems in (9) and (12) are feasible at initialization, they remain feasible at the respective time instances they have to be solved in the closed-loop system (recursive feasibility).

Lemma 1. Let any $x_t \in \mathbb{R}^n$ with $t \in \mathbb{N}_{[T_k,T_{k+1}-1]} \cap (\hat{\mathcal{T}} \cup M\mathbb{N})$ be given for some $k \in \mathbb{N}$ and let $\mathbf{d}_t \in \mathcal{D}_N^{\mu_k,t-T_k}(x_t)$ for a $\mu_k \in \mathcal{M}$, where $\mathbf{d}_t = ((x_{0|t}, \ldots, x_{N|t}), (u_{0|t}, \ldots, u_{N-1|t}))$. Let further

$$x_{t+i+1} = Ax_{t+i} + Bu_{i|t} + w_{t+i} \quad (18)$$

with $w_{t+i} \in \mathcal{W}$ for $i \in \mathbb{N}_{[0,\bar{t}-t-1]}$, where $\bar{t} = \min\{t' \in \mathbb{N}_{[T_k+1,T_{k+1}]} \cap (\hat{\mathcal{T}} \cup M\mathbb{N}) \mid t' > t\}$. Then, it holds that $\mathcal{D}_N^{\mu_k,i+t-T_k}(x_{i+t}) \neq \emptyset$ for $i \in \mathbb{N}_{[1,\bar{t}-t]}$.

The following theorem guarantees robust satisfaction of constraints for the closed-loop system.

Theorem 1. Let $x_0 \in \hat{\mathcal{X}}_N^{\bar{\mu},0}$. Then for all $t \in \mathbb{N}$ and any realization of the disturbance sequence $w_t \in \mathcal{W}$, it holds that $x_t \in \mathcal{X}$ and $u_t \in \mathcal{U}$ for the closed-loop system (13).

Define the set $\bar{\mathcal{X}}_N^{\bar{\mu},0} \subseteq \mathbb{R}^n$ by $\bar{\mathcal{X}}_N^{\bar{\mu},0} := \{x \in \mathbb{R}^n \mid \prod_{j=0}^{i-1}(A+BK_{i-1-j}^{\bar{\mu}})x \in \mathcal{X}_i^{\bar{\mu},0}, K_i^{\bar{\mu}} \prod_{j=0}^{i-1}(A + BK_{i-1-j}^{\bar{\mu}})x \in \mathcal{U}_i^{\bar{\mu},0}, i \in \mathbb{N}_{[0,N-1]}, \prod_{j=0}^{N-1}(A + BK_{N-1-j}^{\bar{\mu}})x \in \mathcal{X}_f^{\bar{\mu},0}\}$, which is the set of all states for which the optimization problem in (9) is feasible at initialization with an input generated by the feedback matrices $K_i^{\bar{\mu}}$, $i \in \mathbb{N}$. We are now ready to state our main stability result.

Theorem 2. Let $x_0 \in \hat{\mathcal{X}}_N^{\bar{\mu},0}$. Then for any realization of the disturbance sequence $w_t \in \mathcal{W}$, $t \in \mathbb{N}$, it holds that $\lim_{t\to\infty} |x|_{\mathcal{Y}^{\bar{\mu},t}} = 0$ for the closed-loop system (13). Further, if there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $\mathcal{Y}^{\bar{\mu},0} \oplus \epsilon\mathcal{B} \subseteq \bar{\mathcal{X}}_N^{\bar{\mu},0}$, then the sequence $\mathcal{Y}^{\bar{\mu},t} \subseteq \mathbb{R}^n$, $t \in \mathbb{N}$, is robustly asymptotically stable with region of attraction $\hat{\mathcal{X}}_N^{\bar{\mu},0}$ for the closed-loop system (13).

Finally, we establish that the average number of updates for the closed loop system (13) are less than, or equal to the updates required by the base schedule

Lemma 2. Let $x_0 \in \hat{\mathcal{X}}_N^{\bar{\mu},0}$. Then, for all $j \in \mathbb{N}_{>1}$ it holds that $\frac{1}{j}\sum_{k=0}^{j-1} \|\mu_k\|_c \leq \|\bar{\mu}\|_c$ for the closed-loop system (13).

6. IMPLEMENTATION AND COMPLEXITY

For quadratic functions ℓ and $J_s^{\mu,0}$, and polytopic sets, the solutions to the optimization problems in (9) and (10) can be obtained by quadratic programming. It is important to note that the problem in (9) has the same level of complexity as a standard constrained MPC problem, as the tightening of polytopic constraints as in (6) does not increase their complexity, see Theorem 2.3 in Kolmanovsky and Gilbert (1998). The mixed-integer optimization problem in (12) can be solved by first solving (9) and (10) for every schedule in $\mathcal{M}$ and then comparing the respective values of the optimal cost function. This implies that the reduction in communication offered by the proposed scheme is paid for by an increased number of computations at the scheduling instants $T_k \in M\mathbb{N}$, compare Gommans and Heemels (2015). Note that in Gommans and Heemels (2015) these scheduling times are not periodic, i.e $M \in \mathbb{N}$ depends on the outcome of the self-triggered algorithm. Limiting the number of possible schedules in $\mathcal{M}$ also limits the required amount of computation, while still offering the possibility of a reduction in communication.

The actual schedules that are chosen by the scheduling function Ω strongly depend on values of the functions $V_s^{\mu,0}$, especially for system states x_t close to the origin. For states x_{T_k} contained in $\mathcal{Y}^{\bar{\mu},0}$ it holds that $V_s^{\bar{\mu},0}(x_{T_k}) = 0$, such that, for large times, schedules μ where $c_{\mu,0} > 0$ are not selected by Ω anymore. Enlarging the sets $\mathcal{Y}^{\bar{\mu},i}$, $i \in \mathbb{N}_{[0,M-1]}$, which are design parameters in the scheme, allows for more schedules to satisfy $c_{\mu,0} = 0$, compare Assumptions 3 and 4. This implies a trade-off between the size of the asymptotic bound on the system state and the number of communications for large times in the closed-loop system.

7. NUMERICAL EXAMPLES

Consider the system defined by [2]

$$x_{t+1} = \begin{bmatrix} 1.1 & 0.2 \\ 0 & 1.2 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t + w_t, \quad (19)$$

where $w_t \in \mathcal{W} = [-0.05, 0.05]^2$ for $t \in \mathbb{N}$ subject to the constraints $\mathcal{X} = [-10, 10]^2$ and $\mathcal{U} = [-2, 2]$. We chose $\bar{\mu} = (1, 0, 1, 0, 1, 0)$ as the base schedule and defined $\mathcal{M}$ as

[2] YALMIP (Löfberg (2004)), the Multi-Parametric Toolbox 3.0 (Herceg et al. (2013)) and IBM ILOG CPLEX Optimization Studio (IBM (2014)) were used in the simulations.

Table 1. Number of scheduled updates for different sizes of the guaranteed asymptotic bound. Percentage when compared to periodic implementation in brackets.

scaling factor γ of asymptotic bound	# control updates	# measurements
1.00	2458 (81.9%)	3000 (100.0%)
1.25	2254 (75.1%)	3000 (100.0%)
1.50	2100 (70.0%)	3000 (100.0%)
1.75	2037 (67.9%)	3000 (100.0%)
2.00	2012 (67.1%)	3000 (100.0%)
2.25	1967 (65.6%)	2398 (79.9%)
2.50	1063 (35.4%)	2000 (66.7%)
2.75	1030 (34.3%)	2000 (66.7%)
3.00	1017 (33.9%)	2000 (66.7%)

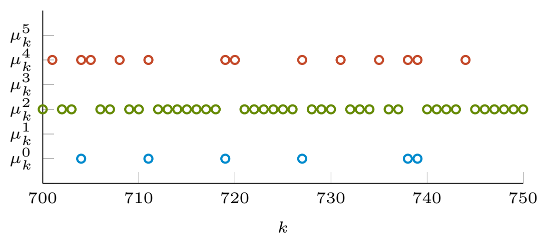

Fig. 3. Control update instances scheduled for $\gamma = 2.5$. Circles indicate nonzero entries in μ_k^i.

the set of all schedules of length 6 with 3 or less nonzero entries. The feedback gains $K_i^{\bar{\mu}}$, $i \in \mathbb{N}_{[0,5]}$, were computed by solving a periodic Riccati equation with the weighting matrices $Q = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$ and $R = 1$. Quadratic cost functions satisfying Assumption 4, as well as feedback gains for the schedules in $\mathcal{M} \setminus \{\bar{\mu}\}$, were computed by solving appropriate linear matrix inequalities. The sets $\mathcal{Y}^{\bar{\mu},i}$, $i \in \mathbb{N}_{[0,5]}$, were computed as an outer ϵ-approximation of the minimal robust periodic invariant sequence of sets, compare Raković (2007), with $\epsilon = 0.01$. The sets $\mathcal{G}^{\mu,l}$, $\mu \in \mathcal{M}$, $l \in \mathbb{N}_{[0,5]}$, were parameterized as $\mathcal{G}^{\mu,l} = \rho^{\mu,l}[-1,1]^2$ with $\rho^{\mu,l} \in \mathbb{R}_{\geq 0}$ chosen as small as possible. The prediction horizon was chosen to $N = 10$. The closed loop system was simulated with initial condition $x_0 = (0,0)^\top$ and disturbances w_t drawn from a uniform distribution on $\mathcal{W}$ for $t \in \mathbb{N}_{[0,5999]}$. The number of scheduled control update times was 2458 which corresponds to a reduction of 18.1% when compared to a periodic implementation using the base schedule. The number of required measurement updates was 3000 which is the same as for an periodic implementation using the base schedule. Note that the guaranteed asymptotic bound is (up to ϵ) the same as for an implementation using only the base schedule, such that a reduction of updates can be achieved without sacrificing guarantees on the worst-case asymptotic bound. Further simulations were performed where the sets $\mathcal{Y}^{\mu,i}$, $i \in \mathbb{N}_{[0,5]}$, were increased by a factor of γ ranging from 1.25 to 3.00. The resulting reduction in required updates, illustrating the trade-off between guaranteed asymptotic bound and required communication, is shown in Table 1. For $\gamma = 2.5$, the scheduled control update instances in the simulation are shown in Figure 3.

8. CONCLUSIONS AND OUTLOOK

We have proposed a self-triggered controller based on rollout techniques that allows a reduction of communication in a networked control system in comparison to a periodically triggered scheme while guaranteeing robust constraint satisfaction. It is further shown how the amount of communication can be traded-off with the guaranteed asymptotic bound on the system state. Future works include the output feedback case, independent scheduling of multiple input channels, and aperiodic scheduling times.

REFERENCES

Antunes, D. and Heemels, W.P.M.H. (2014). Rollout Event-Triggered Control : Beyond Periodic Control Performance. *IEEE Trans. Automat. Control*, 59(12), 3296–3311.

Aydiner, E., Brunner, F.D., Heemels, W.P.M.H., and Allgöwer, F. (2015). Robust Self-Triggered Model Predictive Control for Constrained Discrete-Time LTI Systems based on Homothetic Tubes. In *Proc. European Control Conf.*, 1581–1587.

Bertsekas, D.P. (2005). *Dynamic Programming and Optimal Control, vol. 1 and 2, 3rd ed.* Athena Scientific, Boston, MA, USA.

Brunner, F.D., Heemels, W.P.M.H., and Allgöwer, F. (2014). Robust Self-Triggered MPC for Constrained Linear Systems. In *Proc. European Control Conf.*, 472–477.

Chisci, L., Rossiter, J.A., and Zappa, G. (2001). Systems with persistent disturbances: predictive control with restricted constraints. *Automatica*, 37(7), 1019–1028.

Eqtami, A.M. (2013). *Event-Based Model Predictive Controllers.* Doctoral thesis, National Technical University of Athens.

Gommans, T. and Heemels, W. (2015). Resource-aware MPC for constrained nonlinear systems: A self-triggered control approach. *Systems & Control Letters*, 79, 59–67.

Gondhalekar, R. and Jones, C.N. (2011). MPC of constrained discrete-time linear periodic systems - A framework for asynchronous control: Strong feasibility, stability and optimality via periodic invariance. *Automatica*, 47(2), 326–333.

Heemels, W.P.M.H., Johansson, K., and Tabuada, P. (2012). An introduction to event-triggered and self-triggered control. In *Proc. 51st IEEE Conf. Decision and Control (CDC)*, 3270–3285.

Herceg, M., Kvasnica, M., Jones, C.N., and Morari, M. (2013). Multi-Parametric Toolbox 3.0. In *Proc. European Control Conf.*, 502–510. URL `control.ee.ethz.ch/~mpt` .

Hespanha, J., Naghshtabrizi, P., and Xu, Y. (2007). A Survey of Recent Results in Networked Control Systems. *Proceedings of the IEEE*, 95(1), 138–162.

IBM (2014). IBM ILOG CPLEX Optimization Studio 12.6. URL `www-01.ibm.com/software/integration/optimization/cplex-optimization-studio/` .

Kögel, M. and Findeisen, R. (2014). On self-triggered reduced-attention control for constrained systems. In *Proc. 53rd IEEE Conf. Decision and Control (CDC)*, 2795–2801.

Kolmanovsky, I. and Gilbert, E.G. (1995). Maximal Output Admissible Sets for Discrete-Time Systems with Disturbance Inputs. In *Proc. American Control Conf. (ACC)*, 1995–1999.

Kolmanovsky, I. and Gilbert, E.G. (1998). Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4(4), 317–367.

Langson, W., Chryssochoos, I., Raković, S.V., and Mayne, D.Q. (2004). Robust model predictive control using tubes. *Automatica*, 40(1), 125–133.

Lješnjanin, M., Quevedo, D.E., and Nešić, D. (2014). Packetized MPC with dynamic scheduling constraints and bounded packet dropouts. *Automatica*, 50(3), 784–797.

Löfberg, J. (2004). YALMIP : A toolbox for modeling and optimization in MATLAB. In *Proc. CACSD Conference*, 284–289. URL `users.isy.liu.se/johanl/yalmip` .

Raković, S.V. (2007). Minkowski Algebra and Banach Contraction Principle in Set Invariance for Linear Discrete Time Systems. In *Proc. 46th IEEE Conf. Decision and Control (CDC)*, 2169–2174.

Rawlings, J.B. and Mayne, D.Q. (2009). *Model Predictive Control: Theory and Design.* Nob Hill Publishing, Madison, WI, USA.

Zou, Y., Niu, Y., Li, S., and Li, D. (2014). Receding Horizon Control and Scheduling of Quantized Control Systems with Communication Constraints. In *Proc. 53rd IEEE Conf. Decision and Control (CDC)*, 3872–3877.