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Resource-aware set-valued estimation for discrete-time linear systems

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Abstract— **In this paper, we propose a self-triggered estimator for discrete-time linear systems subject to unknown but bounded disturbances affecting both the system states and outputs. The proposed self-triggered estimator is a set-valued estimator that employs rollout techniques to reduce the communication between the sensors and the estimator with respect to a periodic sampling strategy. Moreover, at each time instant it provides an estimate of the plant state and a guaranteed bound on the difference between the true plant state and the estimate.**

I. INTRODUCTION

In setups where the system state is estimated by an estimator that receives measurements from the system via a communication link, the cost of communication has to be weighed against the quality of the estimation. This cost is especially high in wireless networks. Measurements have to be sent over a communication link for example if it is not possible or not desired to include the observer directly in the sensor or, if multiple, spatially distributed sensors are connected to a single observer. A detailed survey of communication issues in networked control systems can be found in [1].

Several means of reducing the required communication in control and estimation have been proposed in recent years. Two methods in particular have emerged, that is event triggering, where the sensor decides, based on the current measurement, whether to send an update to the observer/controller or not, and self triggering, where the observer/controller decides, based on the current state (estimate), when the next sensor update will take place. For a recent overview on self- and event-triggered control, including event-triggered estimation, please refer to [2]. Interestingly, in self-triggered schemes, the sensors may be shut down between updates, which allows to save additional energy. This constitutes an advantage of self-triggered strategies over event-triggered strategies. On the other hand, more information is available to the estimator in event-triggered setups, as the sensors not sending measurement updates still allows conclusions to be drawn about the current measurements, based on the event conditions, see, e.g., [3], [4].

In this paper, we propose a self-triggered set-valued estimator which schedules its future measurements based on the current state estimate. In particular, at each point in time, the estimator computes an estimate of the system state and an upper bound on the estimation error. At periodically recurring scheduling time points, the estimator decides on the time points at which measurements are required from each sensor in the time span until the next scheduling time point. The selection of this measurement schedule is made based on the current known bound on the estimation error. The objective is to minimize the overall number of measurements subject to the constraint that the estimation error does not exceed a certain bound in the time span until the next scheduling time point. Thus, it is guaranteed that the actual estimation error always stays below this bound. The parameters of the set-valued estimator and the scheduling function are designed based on a certain well-chosen base schedule of measurements which is always a feasible choice for the scheduler, thereby guaranteeing a uniform upper bound on the average sampling frequency and on the estimation error. Additional constraints on the measurement schedules may be included such that for example at most one sensor can send its measurements over the communication system at a given point in time or that there exists a lower bound on the time between communications. The set-valued estimator used in this paper is based on linear programming, while the scheduling function requires the evaluation of scalar inequalities, allowing an efficient implementation of the proposed scheme.

The optimization over update patterns in self-triggered control and estimation was proposed in [5], [6], [7], where the disturbances and noise are assumed to be Gaussian distributed, that is, unbounded. Other recent results on self-triggered state estimation can for example be found in [8], [9], where input-to-state stability under self-triggered output-feedback control is shown. The estimators used in [8], [9] are of a type of Kalman filter and do not provide an online bound on the current estimation error. Hence, the current bound on the estimation error cannot be used to reduce future communications, as is proposed in the present paper. In [10], asymptotic stability of uncertain systems under self-triggered output-feedback *control* with continuous output *measurement* is shown. For an introduction to set-valued estimation see for example Chapter 10 of [11] and the references therein. Set-

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valued estimation based on self-triggered measurements is proposed in [12], where interval estimators are employed to estimate the state of a special class of nonlinear systems. In [13], a set-valued self-triggered estimator for linear parameter-varying systems is presented, using ellipsoidal bounding techniques. There, the self-triggered mechanism decides how much computations have to be performed at a given point in time, based on the current bound on the estimation error.

The remainder of the paper is organized as follows. After indicating the notational conventions used in this paper and providing an important lemma, Section II provides the problem formulation. A set-valued estimator is proposed in Section III, which forms the basis for developing the scheduler presented in Section IV. The effectiveness of the approach is demonstrated by means of numerical examples in Section V. Section VI presents a discussion on the implementation and computational aspects of the approach. Finally, Section VII presents the conclusions and an outlook.

Notation: The set of non-negative integers is denoted by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$. For $a, b \in \mathbb{R}$ we define $\mathbb{N}_{[a,b]} := \mathbb{N} \cap [a, b]$. By $\mathbb{N}_{>0}, \mathbb{R}_{>0}, \mathbb{R}_{\geq 0}$ we denote the positive integers, positive reals, and nonnegative reals, respectively. For $n \in \mathbb{N}$, the n-dimensional identity matrix is denoted by I_n and 1_n denotes a vector of length n, where each element is 1. A zero matrix of appropriate dimension is denoted by 0. The Kronecker product is denoted by $\otimes$. For a square matrix $X \in \mathbb{R}^{n\times n}$ we write $X \succ 0$ and $X \succeq 0$ if X is symmetric and, in addition, X is positive definite and positive semi-definite, respectively. For $X \in \mathbb{R}^{m\times n}$, inequalities of the type $X \geq 0$, should be interpreted element-wise, i.e., all entries of X are nonnegative. For a symmetric matrix $X = \begin{bmatrix} X_1 & X_2 \\ X_2^\top & X_3 \end{bmatrix}$, we use $\star$ to denote $X_2^\top$. For $A_i \in \mathbb{R}^{n\times n}$, and $a, b \in \mathbb{N}$, $a \leq b$, we denote $\Pi_{i=a}^{b} A_i = A_a \dots A_b$. For $a > b$, we define $\Pi_{i=a}^{b} A_i := I_n$. For a vector $\mu = [\mu^{(1)}, \mu^{(2)}, \dots]^\top$, $\mu^{(i)} \in \{0, 1\}$, $i = 1, 2, \dots$, we use $S = \mathrm{diag}(\mu)$ to denote a diagonal matrix with $\mu^{(1)}, \mu^{(2)}, \dots$, on its diagonal.

The following Farkas-type lemma will be used multiple times in the paper.

Lemma 1 ([14]): Let $G \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^m$, $F \in \mathbb{R}^{r\times n}$, $f \in \mathbb{R}^r$. If $\{x \in \mathbb{R}^n \mid Gx \leq g\} \neq \emptyset$, then the following two statements are equivalent.

(i) $\{x \in \mathbb{R}^n \mid Gx \leq g\} \subseteq \{x \in \mathbb{R}^n \mid Fx \leq f\}$

(ii) There exists a matrix $P \in \mathbb{R}^{r\times m}$, $P \geq 0$, such that $PG = F$ and $Pg \leq f$.

II. Problem Formulation

Consider the system

$$x_{t+1} = Ax_t + w_t, \tag{1a}$$

$$y_t = Cx_t + v_t, \tag{1b}$$

where $x_t \in \mathbb{R}^{n_x}$ is the state, $y_t \in \mathbb{R}^{n_y}$ is the output, and where $w_t \in \mathbb{W}$ and $v_t \in \mathbb{V}$ are the unknown but bounded disturbances affecting the system state and output, respectively, at discrete time $t \in \mathbb{N}$. Here the sets $\mathbb{W}$ and $\mathbb{V}$ contain the origin and are given by $\mathbb{W} := \{w \in \mathbb{R}^{n_x} \mid H_w w \leq h_w\}$ and $\mathbb{V} := \{v \in \mathbb{R}^{n_y} \mid H_v v \leq h_v\}$, for $H_w \in \mathbb{R}^{n_w\times n_x}$, $h_w \in \mathbb{R}^{n_w}$, $H_v \in \mathbb{R}^{n_v\times n_y}$, and $h_v \in \mathbb{R}^{n_v}$.

Remark 1: As the system is linear, the extension to systems with (known) inputs is straightforward and not considered here, compare [11].

In this paper, we consider the case where output measurements are not available to the estimator for all $t \in \mathbb{N}$. Consider the periodic scheduling times $t_l = lM$, $l \in \mathbb{N}$, for a fixed $M \in \mathbb{N}_{>0}$. At time t_l, $l \in \mathbb{N}$, let $\boldsymbol{\mu}_l = [\boldsymbol{\mu}_{l,1}, \dots, \boldsymbol{\mu}_{l,M}] \in \mathcal{M} \subseteq \{0, 1\}^{n_y\times M}$, $\boldsymbol{\mu}_{l,i} \subseteq \{0, 1\}^{n_y}$, $i \in \mathbb{N}_{[1,M]}$, denote the schedule determining which output measurements are available to the estimator at time instances $t = t_l + i$, $i \in \mathbb{N}_{[1,M]}$. Here $\mathcal{M}$ denotes the set of allowed measurement schedules. In particular, the information about the plant output that is available to the estimator at time $t = t_l + i$ is given by $[\bar{y}_{l,1}, \dots, \bar{y}_{l,i}]$, where $\bar{y}_{l,i} = S_{\boldsymbol{\mu}_{l,i}} y_{t_l+i}$ for $i \in \mathbb{N}_{[1,M]}$, where $S_{\boldsymbol{\mu}_{l,i}} = \mathrm{diag}(\boldsymbol{\mu}_{l,i})$. At every time instant $t \in \mathbb{N}$, the estimator should provide an estimate of the state $\hat{x}_t \in \mathbb{R}^{n_x}$ combined with a bound $a_t \in \mathbb{R}_{\geq 0}$, such that the estimation error $e_t := x_t - \hat{x}_t$ satisfies $e_t \in a_t\mathcal{G}$ for a fixed set $\mathcal{G} \subseteq \mathbb{R}^{n_x}$. The information available to the estimator at time t consists of the available output information in the time span $\mathbb{N}_{[t_l+1, t_l+i]}$, where $t = t_l + i$, the state estimate and the bound on the estimation error at time t_l, i.e., $\hat{x}_{t_l}$ and a_{t_l}. Here $\mathcal{G} = \{x \in \mathbb{R}^{n_x} \mid Gx \leq g\}$ with $G \in \mathbb{R}^{n_g\times n_x}$ and $g \in \mathbb{R}^{n_g}_{\geq 0}$ is a bounded polyhedron with the origin in its interior. The self-triggered set-valued estimator is defined by the functions $\Gamma_i : \{0, 1\}^{n_y\times M} \times \mathbb{R}^{n_x} \times \mathbb{R} \times \mathbb{R}^{n_y} \times \dots \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_x+1}$, $i \in \mathbb{N}_{[1,M]}$, and $\Omega : \mathbb{R} \to \{0, 1\}^{n_y\times M}$, such that for $l \in \mathbb{N}$, $i \in \mathbb{N}_{[1,M]}$,

$$[\hat{x}_{t_l+i}^\top, a_{t_l+i}]^\top = \Gamma_i(\boldsymbol{\mu}_l, \hat{x}_{t_l}, a_{t_l}, \bar{y}_{t_l+1}, \dots, \bar{y}_{t_l+i}), \tag{2a}$$

$$\boldsymbol{\mu}_l = \Omega(a_{t_l}), \tag{2b}$$

given $\hat{x}_0 \in \mathbb{R}^{n_x}$ and $a_0 \in \mathbb{R}_{\geq 0}$, such that $G(x_0 - \hat{x}_0) \leq a_0 g$. The goal of this paper is to design Γ_i, $i \in \mathbb{N}_{[1,M]}$, and Ω, such that the number of non-zero elements in $\boldsymbol{\mu}_l$ is minimized at times t_l, $l \in \mathbb{N}$, while guaranteeing $G(x_t - \hat{x}_t) \leq a_t g$ and $a_t \leq a_{\max}$, for all $t \in \mathbb{N}$, for a given $a_{\max} \in \mathbb{R}_{>0}$. We make the following technical assumption on the output disturbances.

Assumption 1: For all $v \in \mathbb{R}^{n_y}$ and all $\boldsymbol{\mu}_l \in \mathcal{M}$, $l \in \mathbb{N}$, $H_v v \leq h_v$ implies that $H_v S_{\boldsymbol{\mu}_{l,i}} v \leq h_v$, for all $i \in \mathbb{N}_{[1,M]}$.

Remark 2: This assumption is for example satisfied in the case where the output y_t is composed of n_y independent scalar measurements, i.e., $y_t^{(j)} = C^{(j)} x_t + v_t^{(j)}$, where $v^{(j)} \in [\underline{v}^{(j)}, \overline{v}^{(j)}]$, $\underline{v}^{(j)}, \overline{v}^{(j)} \in \mathbb{R}$, $0 \in [\underline{v}^{(j)}, \overline{v}^{(j)}]$, $j \in \mathbb{N}_{[1,n_y]}$, $t \in \mathbb{N}$. Then it can be assumed without loss of generality, that $H_v = I_{n_y} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \mathbb{R}^{n_v\times n_y}$ and $h_v \in \mathbb{R}^{n_v}_{\geq 0}$, where $n_v = 2n_y$. The structure of H_v and $S_{\boldsymbol{\mu}_{l,i}}$ then readily implies that the statement in the assumption holds.

III. Set-valued Estimator

In this section, we design the functions Γ_i, $i \in \mathbb{N}_{[1,M]}$, defining the set-valued estimator. Moreover, we provide a priori bounds on the future guaranteed bounds a_{t_l+i}, $i \in \mathbb{N}_{[1,M]}$, on the estimation error for a given schedule $\boldsymbol{\mu}_l$.

Given $\hat{x}_{t_l}$, a_{t_l} and a schedule $\boldsymbol{\mu}_l$, $l \in \mathbb{N}$, the information available to the set-valued estimator at times $t \in \mathbb{N}_{[t_l+1,t_l+M]}$ is given by

$$G(x_{t_l} - \hat{x}_{t_l}) \leq a_{t_l} g, \tag{3a}$$
$$H_w w_j \leq h_w, \quad j \in \mathbb{N}_{[t_l,t-1]}, \tag{3b}$$
$$H_v v_j \leq h_v, \quad j \in \mathbb{N}_{[t_l+1,t]}, \tag{3c}$$

$$\bar{y}_{l,j} = S_{\boldsymbol{\mu}_{l,j}}\left(C(A^j x_{t_l} + \sum_{k=0}^{j-1} A^{j-1-k} w_{t_l+k}) + v_{t_l+j}\right), \quad j \in \mathbb{N}_{[1,t-t_l]}, \tag{3d}$$

from which our set-valued estimator should obtain $\hat{x}_t$ and the smallest $a_t \geq 0$ such that

$$G(x_t - \hat{x}_t) \leq a_t g, \tag{4}$$

for all $t \in \mathbb{N}_{[t_l+1,t_l+M]}$.

A. Optimization problem for set-valued estimator

In the following, we will derive the optimization problem that defines the set-valued estimator. Using Assumption 1, from (3c) and (3d) it follows that $H_v \bar{y}_{l,j} - H_v S_{\boldsymbol{\mu}_{l,j}}\Big(C(A^j x_{t_l} + \sum_{k=0}^{j-1} A^{j-1-k} w_{t_l+k}\Big) \leq h_v$, $j \in \mathbb{N}_{[1,t-t_l]}$. Hence, the prior information given at times $t = t_l + i$, $i \in \mathbb{N}_{[1,M]}$, $l \in \mathbb{N}$, according to (3) can be expressed as

$$F_i^{\boldsymbol{\mu}_l,\text{prior}} \begin{bmatrix} x_{t_l} \\ w_{t_l} \\ \vdots \\ w_{t_l+i-1} \end{bmatrix} \leq f_i^{\text{prior}}(\hat{x}_{t_l}, a_{t_l}, \bar{y}_{l,1}, \dots, \bar{y}_{l,i}), \tag{5}$$

where

$$F_i^{\boldsymbol{\mu}_l,\text{prior}} = \begin{bmatrix} G & 0 & 0 & \cdots & 0 \\ 0 & H_w & 0 & \cdots & 0 \\ 0 & 0 & H_w & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & H_w \\ \Xi_{1,1} & \Xi_{1,0} & 0 & \cdots & 0 \\ \vdots & \vdots & & & \\ \Xi_{i,i} & \Xi_{i,i-1} & \Xi_{i,i-2} & \cdots & \Xi_{i,0} \end{bmatrix} \tag{6}$$

with $\Xi_{i,j} := -H_v S_{\boldsymbol{\mu}_{l,i}} C A^j$, and

$$f_i^{\text{prior}}(\hat{x}_{t_l}, a_{t_l}, \bar{y}_{l,1}, \dots, \bar{y}_{l,i}) = \begin{bmatrix} G\hat{x}_{t_l} + a_{t_l} g \\ h_w \\ h_w \\ \vdots \\ h_w \\ -H_v \bar{y}_{l,1} + h_v \\ \vdots \\ -H_v \bar{y}_{l,i} + h_v \end{bmatrix}. \tag{7}$$

Using the expression

$$x_{t_l+i} = A^i x_{t_l} + \sum_{j=0}^{i-1} A^{i-1-j} w_{t_l+j}, \tag{8}$$

for $i \in \mathbb{N}_{[1,M]}$, the posteriori information that the estimator should determine at time $t = t_l + i$ is $\hat{x}_{t_l+i}$ and a_{t_l+i} such that (4) is satisfied, which can be expressed as

$$F_i^{\text{post}} \begin{bmatrix} x_{t_l} \\ w_{t_l} \\ \vdots \\ w_{t_l+i-1} \end{bmatrix} \leq f_i^{\text{post}}(\hat{x}_{t_l+i}, a_{t_l+i}), \tag{9}$$

where $F_i^{\text{post}} = G\begin{bmatrix} A^i & A^{i-1} & \dots & I \end{bmatrix}$ and $f_i^{\text{post}}(\hat{x}_{t_l+i}, a_{t_l+i}) = G\hat{x}_{t_l+i} + a_{t_l+i} g$. By Lemma 1, (5) implies (9) if and only if there exists a matrix $P_i \in \mathbb{R}^{n_g \times n_g + i(n_w+n_v)}$ with $P_i \geq 0$, such that $P_i F_i^{\boldsymbol{\mu}_l,\text{prior}} = F_i^{\text{post}}$ and $P_i f_i^{\text{prior}}(\hat{x}_{t_l}, a_{t_l}, \bar{y}_{l,1}, \dots, \bar{y}_{l,i}) \leq f_i^{\text{post}}(\hat{x}_{t_l+i}, a_{t_l+i})$. Hence, the problem of finding an estimate $\hat{x}_{t_l+i}$ based on the given prior information such that the estimation error described by a_{t_l+i} is minimized, can be expressed as the following linear program.

Problem 1: Given $\hat{x}_{t_l}$, a_{t_l}, $\boldsymbol{\mu}_l$ and $\bar{y}_{l,1}, \dots, \bar{y}_{l,i}$, for $i \in \mathbb{N}_{[1,M]}$, $l \in \mathbb{N}$, solve[1]

$$\begin{bmatrix} \hat{x}_{t_l+i}(\boldsymbol{\mu}_l, \hat{x}_{t_l}, a_{t_l}, \bar{y}_{l,1}, \dots, \bar{y}_{l,i}) \\ a_{t_l+i}(\boldsymbol{\mu}_l, \hat{x}_{t_l}, a_{t_l}, \bar{y}_{l,1}, \dots, \bar{y}_{l,i}) \end{bmatrix} = \underset{[\hat{x}^\top, a]^\top \in \mathbb{R}^{n_x+1}}{\operatorname{argmin}} \min_{P_i \in \mathbb{R}^{n_g \times n_g + i(n_w+n_v)}} a \tag{10}$$
$$\text{s. t.} \quad a \geq 0, \quad P_i \geq 0, \quad P_i F_i^{\boldsymbol{\mu}_l,\text{prior}} = F_i^{\text{post}},$$
$$P_i f_i^{\text{prior}}(\hat{x}_{t_l}, a_{t_l}, \bar{y}_{l,1}, \dots, \bar{y}_{l,i}) \leq f_i^{\text{post}}(\hat{x}, a).$$

This optimization problem defines the functions Γ_i, $i \in \mathbb{N}_{[1,M]}$, in (2a). The following lemma states that the proposed estimator provides a valid bound on the estimation error.

Lemma 2: Let $\hat{x}_{t_l} \in \mathbb{R}^{n_x}$, $a_{t_l} \in \mathbb{R}_{\geq 0}$, $w_j \in \mathbb{R}^{n_x}$, $j \in \mathbb{N}_{[t_l,t-1]}$, $v_j \in \mathbb{R}^{n_y}$, $j \in \mathbb{N}_{[t_l+1,t]}$ and $\bar{y}_{l,j} \in \mathbb{R}^{n_y}$, $j \in \mathbb{N}_{[1,t-t_l]}$ be given for some $t \in \mathbb{N}_{[t_l+1,t_l+M]}$ and $l \in \mathbb{N}$ such that the constraints in (3) are satisfied. Let further $i = t - t_l$, let $\hat{x} \in \mathbb{R}^{n_x}$ and $a \in \mathbb{R}$ satisfy the constraints in Problem 1, and let x_t satisfy (8). Then it holds that $G(x_t - \hat{x}) \leq ag$.

Proof: The statement follows from the derivations leading up to Problem 1. ∎

B. A priori bounds on the estimation error

At times t_l, $l \in \mathbb{N}$, the scheduling function Ω proposed in Section IV provides a schedule $\boldsymbol{\mu}_l$ that minimizes the required communication in the time span $\mathbb{N}_{[t_l+1,t_l+M]}$ while guaranteeing bounds on a_t, $t \in \mathbb{N}_{[t_l+1,t_l+M]}$. We want these bounds to be explicit functions of a_{t_l}, such that they can be evaluated efficiently online for a large number of schedules. In the following, we derive *affine* functions based on the assumption that the optimization in Problem 1 only takes place over a_{t_l+i}, and that $\hat{x}_{t_l+i}$ is chosen (suboptimally) as the output of some linear estimator[2]. This is formalized in the following assumption.

Assumption 2: For any schedule $\boldsymbol{\mu}_l \in \mathcal{M}$, $l \in \mathbb{N}$, there exist Luenberger-like estimator gains $L_i^{\boldsymbol{\mu}_l} \in \mathbb{R}^{n_x \times n_y}$, $\lambda_i^{\boldsymbol{\mu}_l} \in \mathbb{R}_{\geq 0}$ and $\rho_i^{\boldsymbol{\mu}_l} \in \mathbb{R}_{\geq 0}$, $i \in \mathbb{N}_{[1,M]}$, such that for any

[1] In the case of non-unique minimizers, argmin is chosen to be any minimizer of the problem.

[2] Note that Problem 1 is feasible for any fixed $\hat{x} \in \mathbb{R}^{n_x}$ if $a \in \mathbb{R}_{\geq 0}$ is chosen sufficiently large.

w_{t_l+i-1}, v_{t_l+i}, $i \in \mathbb{N}_{[1,M]}$, satisfying $H_w w_{t_l+i-1} \le h_w$ and $H_v v_{t_l+i} \le h_v$, $i \in \mathbb{N}_{[1,M]}$, and any x_{t_l}, $\hat{x}_{t_l}$ and a_{t_l} with $G(x_{t_l} - \hat{x}_{t_l}) \le a_{t_l} g$, the inequality

$$G(x_{t_l+i} - \hat{x}_{t_l+i}) \le (\lambda_i^{\boldsymbol{\mu}_l} a_{t_l} + \rho_i^{\boldsymbol{\mu}_l}) g \tag{11}$$

is satisfied for all $i \in \mathbb{N}_{[1,M]}$, where

$$x_{t_l+i} = A x_{t_l+i-1} + w_{t_l+i-1}, \tag{12a}$$

$$\hat{x}_{t_l+i} = A\hat{x}_{t_l+i-1} + L_i^{\boldsymbol{\mu}_l}\left(\bar{y}_{l,i} - S_{\boldsymbol{\mu}_{l,i}} C A \hat{x}_{t_l+i-1}\right) \tag{12b}$$

with $\bar{y}_{l,i} = S_{\boldsymbol{\mu}_{l,i}}(C x_{t_l+i} + v_{t_l+i})$.

Remark 3: Less conservative a priori bounds on a_t, $t \in \mathbb{N}_{[t_l+1,t_l+M]}$, based on a_{t_l} for $l \in \mathbb{N}$, can be achieved by employing computation-heavy min-max optimization. This matter is subject to future research.

In the following, we will show how, for a given schedule $\boldsymbol{\mu}_l$ and given estimator gains $L_i^{\boldsymbol{\mu}_l}$, $i \in \mathbb{N}_{[1,M]}$, minimal $\lambda_i^{\boldsymbol{\mu}_l}, \rho_i^{\boldsymbol{\mu}_l}$, $i \in \mathbb{N}_{[1,M]}$, satisfying Assumption 2, can be obtained by linear programming. The inequalities in (3a) to (3c) can be written as

$$F_i^{\mathrm{LP,prior}} z_{t_l,i} \le f_i^{\mathrm{LP,prior}}, \tag{13}$$

where

$$z_{t_l,i} := [x_{t_l}^\top - \hat{x}_{t_l}^\top, a_{t_l}, w_{t_l}^\top, \ldots, w_{t_l+i}^\top, v_{t_l+1}^\top, \ldots, v_{t_l+i+1}^\top]^\top,$$

$$F_i^{\mathrm{LP,prior}} := \begin{bmatrix} G & -g & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & I_i \otimes H_w & 0 \\ 0 & 0 & 0 & I_i \otimes H_v \end{bmatrix} \tag{14}$$

and

$$f_i^{\mathrm{LP,prior}} := \begin{bmatrix} 0 \\ 0 \\ 1_i \otimes h_w \\ 1_i \otimes h_v \end{bmatrix}. \tag{15}$$

Using the recursive procedure in (12b) to obtain the estimates $\hat{x}_{t_l+i}$ leads to the error dynamics

$$\begin{aligned}(x_{t_l+i} - \hat{x}_{t_l+i}) &= (A - L_i^{\boldsymbol{\mu}_l} S_{\boldsymbol{\mu}_{l,i}} C A)(x_{t_l+i-1} - \hat{x}_{t_l+i-1}) \\ &\quad + (I_{n_x} - L_i^{\boldsymbol{\mu}_l} S_{\boldsymbol{\mu}_{l,i}} C) w_{t_l+i-1} - L_i^{\boldsymbol{\mu}_l} S_{\boldsymbol{\mu}_{l,i}} v_{t_l+i},\end{aligned} \tag{16}$$

for $i \in \mathbb{N}_{[1,M]}$, or, equivalently,

$$\begin{aligned}(x_{t_l+i} - \hat{x}_{t_l+i}) &= \underbrace{\left(\prod_{j=0}^{i-1} (A - L_{i-j}^{\boldsymbol{\mu}_l} S_{\boldsymbol{\mu}_{l,i-j}} C A)\right)}_{=:\Theta_i} (x_{t_l} - \hat{x}_{t_l}) \\ &+ \sum_{j=0}^{i-1} \underbrace{\left(\prod_{k=0}^{i-2-j} (A - L_{i-k}^{\boldsymbol{\mu}_l} S_{\boldsymbol{\mu}_{l,i-k}} C A)\right)(I_{n_x} - L_{j+1}^{\boldsymbol{\mu}_l} S_{\boldsymbol{\mu}_{l,j+1}} C)}_{=:\Phi_{i,j}} w_{t_l+j} \\ &+ \sum_{j=0}^{i-1} \underbrace{\left(\prod_{k=0}^{i-2-j} (A - L_{i-k}^{\boldsymbol{\mu}_l} S_{\boldsymbol{\mu}_{l,i-k}} C A)\right)(-L_{j+1}^{\boldsymbol{\mu}_l} S_{\boldsymbol{\mu}_{l,j+1}})}_{=:\Psi_{i,j}} v_{t_l+j+1}\end{aligned} \tag{17}$$

for $i \in \mathbb{N}_{[1,M]}$. Hence, the inequality in (11) can be written as

$$F_i^{\mathrm{LP,post}}(\lambda_i^{\boldsymbol{\mu}_l}) z_{t_l,i} \le f_i^{\mathrm{LP,post}}(\rho_i^{\boldsymbol{\mu}_l}), \tag{18}$$

where $F_i^{\mathrm{LP,post}}(\lambda_i^{\boldsymbol{\mu}_l}) :=$

$$\begin{bmatrix} G\Theta_i & -\lambda_i^{\boldsymbol{\mu}_l} g & G\Phi_{i,0} \ldots G\Phi_{i,i-1} & G\Psi_{i,0} \ldots G\Psi_{i,i-1} \end{bmatrix}$$

and $f_i^{\mathrm{LP,post}}(\rho_i^{\boldsymbol{\mu}_l}) := \rho_i^{\boldsymbol{\mu}_l} g$. By Lemma 1, if there exists a $z_{t_l,i} \in \mathbb{R}^{n_x+1+i(n_w+n_v)}$ satisfying (13), then (13) implies (18) if and only if there exists a matrix $P_i \in \mathbb{R}^{n_g \times n_g+1+i(n_w+n_v)}$ with $P_i \ge 0$, such that $P_i F_i^{\mathrm{LP,prior}} = F_i^{\mathrm{LP,post}}(\lambda_i^{\boldsymbol{\mu}_l})$ and $P_i f_i^{\mathrm{LP,prior}} \le f_i^{\mathrm{LP,post}}(\rho_i^{\boldsymbol{\mu}_l})$. Hence, given estimator gains $L_i^{\boldsymbol{\mu}_l}$, $i \in \mathbb{N}_{[1,M]}$, the following linear program can be used to obtain minimal $\lambda_i^{\boldsymbol{\mu}_l}, \rho_i^{\boldsymbol{\mu}_l}$, $i \in \mathbb{N}_{[1,M]}$, satisfying Assumption 2.

Problem 2:

$$\begin{bmatrix} \lambda_i^{\boldsymbol{\mu}_l} \\ \rho_i^{\boldsymbol{\mu}_l} \end{bmatrix} = \underset{[\lambda,\rho]^\top \in \mathbb{R}^2}{\operatorname{argmin}} \min_{P \in \mathbb{R}^{n_g \times n_g+1+i(n_w+n_v)}} \lambda + \rho \tag{19}$$

$$\text{s.t. } \lambda \ge 0,\ \rho \ge 0,\ P \ge 0,\ P F_i^{\mathrm{LP,prior}} = F_i^{\mathrm{LP,post}}(\lambda), \quad P f_i^{\mathrm{LP,prior}} \le f_i^{\mathrm{LP,post}}(\rho).$$

Note that due to the block-structure of $F_i^{\mathrm{LP,prior}}$ and the zero entries in $f_i^{\mathrm{LP,prior}}$, there is no interdependence between the variables λ and ρ in the optimization problem, such that they could be minimized using independent optimization problems yielding the same results.

The following lemma establishes that $\lambda_i^{\boldsymbol{\mu}_l}$ and $\rho_i^{\boldsymbol{\mu}_l}$ computed according to Problem 2 actually provide a bound on a_{t_l+i}.

Lemma 3: Let Assumption 2 hold true and let (5) be satisfied. Then, the constraints of Problem 1 are feasible for $a_{t_l+i} = \lambda_i^{\boldsymbol{\mu}_l} a_{t_l} + \rho_i^{\boldsymbol{\mu}_l}$ for some $i \in \mathbb{N}_{[1,M]}$.

Proof: By noting that $\bar{y}_{l,i} = S_{\boldsymbol{\mu}_{l,i}} y_{t_l+i}$ and that $S_{\boldsymbol{\mu}_{l,i}} S_{\boldsymbol{\mu}_{l,i}} = S_{\boldsymbol{\mu}_{l,i}}$, $i \in \mathbb{N}_{[1,M]}$, it follows that satisfaction of (5) implies the existence of $v_{t_l+j} \in \mathbb{R}^{n_y}$ for $j \in \mathbb{N}_{[1,i]}$, with $H_v v_{t_l+j} \le h_v$ such that $\bar{y}_{l,j}$ satisfies (3d) for $j \in \mathbb{N}_{[1,i]}$. From this it follows that (13) is satisfied. By Assumption 2, (13) implies that $G(x_{t_l+i} - \hat{x}_{t_l+i}) \le (\lambda_i^{\boldsymbol{\mu}_l} a_{t_l} + \rho_i^{\boldsymbol{\mu}_l}) g$ for the specific choice of $\hat{x}_{t_l+i}$ in (12b). The last inequality is, noting the definition of x_{t_l+i}, equivalent to (9) with $a_{t_l+i} = \lambda_i^{\boldsymbol{\mu}_l} a_{t_l} + \rho_i^{\boldsymbol{\mu}_l}$. As x_{t_l} and w_{t_l+j-1} for $j \in \mathbb{N}_{[1,i]}$ were arbitrary, we can conclude that (5) implies (9) with $a_{t_l+i} = \lambda_i^{\boldsymbol{\mu}_l} a_{t_l} + \rho_i^{\boldsymbol{\mu}_l}$ (and the specific choice of $\hat{x}_{t_l+i}$ in (12b)). By Lemma 1, this implies the existence of a matrix $P_i \in \mathbb{R}^{n_g \times n_g+i(n_w+n_v)}$ satisfying the constraints in Problem 1, thereby completing the proof. ■

The scheduling function Ω proposed in Section IV is defined by a minimization problem with the number of communications in the schedule as its cost function. By guaranteeing that a certain base schedule $\bar{\boldsymbol{\mu}} := (\bar{\mu}_1, \ldots, \bar{\mu}_M) \in \mathcal{M}$ is always a feasible solution of this optimization problem, it is ensured that the actual number of communications in the selected schedule is less than or equal to the number of communications required for the base schedule. The requirement that $a_t \le a_{\max}$, $t \in \mathbb{N}$, is enforced through constraints in this optimization problem. Hence, it is necessary that there exists some estimator using the base schedule $\bar{\boldsymbol{\mu}}$, that has certain convergence properties. In particular, we assume the existence of a linear estimator using the base schedule such that the set $\mathcal{G}$ is contractive (see, e.g., [11]) under the associated estimation error dynamics in the absence of disturbances. That is, there exists a $\bar{\lambda} \in [0,1)$ such that

$$Gx \le g \;\Rightarrow\; G \prod_{j=0}^{M-1} (A - L^{\bar{\boldsymbol{\mu}}_l}_{M-j} S_{\bar{\mu}_{i-j}} CA)x \le \bar{\lambda} g, \quad (20)$$

where $L^{\bar{\boldsymbol{\mu}}}_i$, $i \in \mathbb{N}_{[1,M]}$ are the Luenberger-like gains in Assumption 2, associated with the base schedule $\bar{\boldsymbol{\mu}}$. The following assumption provides a concise characterization of the required properties of $\mathcal{G}$ and $\bar{\boldsymbol{\mu}}$.

Assumption 3: It holds that $\lambda^{\bar{\boldsymbol{\mu}}}_M < 1$.

The question remains how to compute the Luenberger-like gains. For the base schedule we assume that these gains are computed using the periodic Riccati equation [15] with weighting matrices $Q_i \in \mathbb{R}^{n_x \times n_x}$, $R_i \in \mathbb{R}^{n_y \times n_y}$, where $Q_i, R_i \succ 0$, $i \in \mathbb{N}_{[1,M]}$, for the periodic time-varying system

$$x_{t+1} = Ax_t, \qquad y_t = C_t x_t, \quad (21)$$

where $C_{t+M} = C_t$, $t \in \mathbb{N}$ and $C_i = S_{\bar{\mu}_i} CA$, $i \in \mathbb{N}_{[1,M]}$. This approach, while guaranteeing a converging estimation error, also allows including knowledge about the disturbance distributions in the form of the weighting matrices Q_i and R_i. The gains $L^{\boldsymbol{\mu}}_i$ for schedules $\boldsymbol{\mu}$ other than the base schedule $\bar{\boldsymbol{\mu}}$, can in principle be chosen arbitrarily. However, it makes sense to choose them in way such that the estimation error remains small when using these particular gains in a Luenberger-like estimator, even if it can not be guaranteed that the estimation error converges. In particular, appropriate semidefinite programs may be used to compute the gains $L^{\boldsymbol{\mu}}_i$.

The necessary offline computations are summarized in the following algorithm.

Algorithm 1 Offline computations

1: Solve periodic Riccati equations for the base schedule $\bar{\boldsymbol{\mu}}$ to obtain the Luenberger-like gains $L^{\bar{\boldsymbol{\mu}}}_i$, $i \in \mathbb{N}_{[1,M]}$.
2: Compute a contractive polytope $\mathcal{G} := \{x \in \mathbb{R}^{n_x} \mid Gx \le g\}$ containing the origin in its interior satisfying (20).
3: For all schedules $\boldsymbol{\mu} \in \mathcal{M} \setminus \{\bar{\boldsymbol{\mu}}\}$, obtain the Luenberger-like gains $L^{\boldsymbol{\mu}}_i$, $i \in \mathbb{N}_{[1,M]}$.
4: For all schedules $\boldsymbol{\mu} \in \mathcal{M}$, solve Problem 2 and obtain $\lambda^{\boldsymbol{\mu}}_i, \rho^{\boldsymbol{\mu}}_i$, $i \in \mathbb{N}_{[1,M]}$.

IV. Self-triggered set-valued estimation

In this section, we will propose a scheduling function that decides online, based on the current bound on the estimation error a_{t_l}, which measurement schedule $\boldsymbol{\mu}_l$ to use in the time span $\mathbb{N}_{[t_l+1,t_l+M]}$. For any $\boldsymbol{\mu}_l \in \mathcal{M}$, with $\boldsymbol{\mu}_{l,i} = [\boldsymbol{\mu}^{(1)}_{l,i}, \dots, \boldsymbol{\mu}^{(n_y)}_{l,i}]^\top$, $i \in \mathbb{N}_{[1,M]}$, we define $s(\boldsymbol{\mu}_l) := \sum_{i=1}^{M} \sum_{j=1}^{n_y} \boldsymbol{\mu}^{(j)}_{l,i}$ indicating the total number of sensor measurements communicated during the time span $\mathbb{N}_{[t_l+1,t_l+M]}$. At time t_l, $l \in \mathbb{N}$, given an estimate $\hat{x}_{t_l}$, and an error bound a_{t_l}, the measurement schedule for the times $t \in \mathbb{N}_{[t_l+1,t_l+M]}$ is given as the solution of the following optimization problem, which defines the function Ω in (2b).

Problem 3: At time t_l, given $\hat{x}_{t_l}$ and a_{t_l},

$$\boldsymbol{\mu}_l(a_{t_l}) = \arg\min_{\boldsymbol{\mu} \in \mathcal{M}} \{ s(\boldsymbol{\mu}) \mid \lambda^{\boldsymbol{\mu}}_M a_{t_l} + \rho^{\boldsymbol{\mu}}_M \le a_{\min}, \\ \lambda^{\boldsymbol{\mu}}_i a_{t_l} + \rho^{\boldsymbol{\mu}}_i \le a_{\max}, i \in \mathbb{N}_{[1,M-1]} \},$$

with the additional parameter $a_{\min} \in \mathbb{R}_{>0}$, $a_{\min} \le a_{\max}$. That is, the schedule with the least number of required measurements is selected which still guarantees the upper bound $a_t \le a_{\max}$ for $t \in \mathbb{N}_{[t_l+1,t_l+M]}$. The inclusion of the additional parameter $a_{\min}$ and its associated constraint in Problem 3 endows the optimization problem with the property of recursive feasibility, meaning that if the problem is feasible at t_0, it will remain feasible for all t_l, $l \in \mathbb{N}$. In particular, the following assumption ensures that the base schedule $\bar{\boldsymbol{\mu}}$ is always a feasible (but not necessarily optimal) solution to Problem 3.

Assumption 4: It holds that

1) $a_{\min} \ge \frac{\rho^{\bar{\boldsymbol{\mu}}}_M}{1-\lambda^{\bar{\boldsymbol{\mu}}}_M}$ and
2) $a_{\max} \ge \max\{a_{\min}, \lambda^{\bar{\boldsymbol{\mu}}}_i a_{\min} + \rho^{\bar{\boldsymbol{\mu}}}_i \mid i \in \mathbb{N}_{[1,M]}\}$,

where the scalars $\lambda^{\bar{\boldsymbol{\mu}}}_i$ and $\rho^{\bar{\boldsymbol{\mu}}}_i$ satisfy Assumption 2 for the base schedule.

Lemma 4: Let $\hat{x}_{t_l}$ and a_{t_l} be given such that $G(x_{t_l} - \hat{x}_{t_l}) \le a_{t_l} g$, where $a_{t_l} \le a_{\min}$ for some $l \in \mathbb{N}$. Then $\bar{\boldsymbol{\mu}}$ is a feasible solution to Problem 3. Moreover, if $\boldsymbol{\mu}_l$ is computed by Problem 3 and a_t is computed by Problem 1 using schedule $\boldsymbol{\mu}_l$ for all $t \in \mathbb{N}_{[t_l+1,t_l+M]}$, then it holds that $a_t \le a_{\max}$ for all $t \in \mathbb{N}_{[t_l+1,t_l+M]}$ and Problem 3 is feasible for a_{t_l+M}. Moreover, it holds that $G(x_{t_l+M} - \hat{x}_{t_l+M}) \le a_{t_l+M} g$, where $a_{t_l+M} \le a_{\min}$.

Proof: By Assumption 4, if $a_{t_l} \le a_{\min}$, then it holds that $\lambda^{\bar{\boldsymbol{\mu}}}_M a_{t_l} + \rho^{\bar{\boldsymbol{\mu}}}_M \le \lambda^{\bar{\boldsymbol{\mu}}}_M a_{\min} + (1-\lambda^{\bar{\boldsymbol{\mu}}}_M) a_{\min} = a_{\min} \le a_{\max}$ and $\lambda^{\bar{\boldsymbol{\mu}}}_i a_{t_l} + \rho^{\bar{\boldsymbol{\mu}}}_i \le \lambda^{\bar{\boldsymbol{\mu}}}_i a_{\min} + \rho^{\bar{\boldsymbol{\mu}}}_i \le a_{\max}$, $i \in \mathbb{N}_{[1,M-1]}$, such that the base schedule $\bar{\boldsymbol{\mu}}$ is a feasible, but not necessarily an optimal solution of Problem 3. By Lemma 3, it holds that $a_t \le \lambda^{\boldsymbol{\mu}_l}_{t-t_l} a_{t_l} + \rho^{\boldsymbol{\mu}_l}_{t-t_l}$ for all $t \in \mathbb{N}_{[t_l+1,t_l+M]}$ and all $\boldsymbol{\mu}_l \in \mathcal{M}$, $l \in \mathbb{N}$, such that the satisfaction of the constraints in Problem 3 implies that $a_t \le a_{\max}$ for all $t \in \mathbb{N}_{[t_l+1,t_l+M]}$ and $a_{t_l+M} \le a_{\min}$, which implies feasibility of Problem 3 for a_{t_l+M}. Finally, by Lemma 2, it holds that $G(x_{t_l+M} - \hat{x}_{t_l+M}) \le a_{t_l+M} g$, thereby completing the proof. ■

The following theorem states the main results about the self-triggered set-valued estimation scheme.

Theorem 1: Let $\hat{x}_{t_0}$ and a_{t_0} be given such that $G(x_{t_0} - \hat{x}_{t_0}) \le a_{t_0} g$, where $a_{t_0} \le a_{\min}$. Let $\boldsymbol{\mu}_l$ be decided by Problem 3 for all $l \in \mathbb{N}$ and let $\hat{x}_t$ and a_t be computed by Problem 1 using schedule $\boldsymbol{\mu}_l$ for all $t \in \mathbb{N}_{[t_l+1,t_l+M]}$ and all $l \in \mathbb{N}$. Then, it holds that $s(\boldsymbol{\mu}_l) \le s(\bar{\boldsymbol{\mu}})$ for all $l \in \mathbb{N}$. Furthermore, it holds that $G(x_t - \hat{x}_t) \le a_{\max} g$ for all $t \in \mathbb{N}$.

Proof: As the requirements in Lemma 4 are satisfied at initialization, it follows by induction that $\bar{\boldsymbol{\mu}}$ is a feasible solution to Problem 3 for all $l \in \mathbb{N}$. As the scheduler selects a schedule $\boldsymbol{\mu}_l$ with the least number of measurements, it follows that $s(\boldsymbol{\mu}_l) \le s(\bar{\boldsymbol{\mu}})$ for all $l \in \mathbb{N}$. Furthermore, by Lemma 2, it holds that $G(x_t - \hat{x}_t) \le a_t g$ for all $t \in \mathbb{N}$. By Lemma 4 it follows that $a_t \le a_{\max}$ for all $t \in \mathbb{N}_{[t_l+1,t_l+M]}$, $l \in \mathbb{N}$, and, hence, $G(x_t - \hat{x}_t) \le a_{\max} g$, $t \in \mathbb{N}$, thereby completing the proof. ■

Remark 4: The assumption that $G(x_{t_0} - \hat{x}_{t_0}) \le a_{t_0} g$ with $a_{t_0} \le a_{\min}$ can be satisfied if some bound on the initial estimation error is known by choosing $a_{\min}$ large enough.

The overall self-triggered estimation algorithm can be summarized as follows.

Algorithm 2 Self-triggered set-valued estimator

Initialize $\hat{x}_0$ and a_0 such that $G(x_0 - \hat{x}_0) \leq a_0 g$.
$t_0 := 0$.
for $l \in \mathbb{N}$ **do**
  Obtain $\boldsymbol{\mu}_l(a_{t_l})$ by solving Problem 3.
  for $i \in \mathbb{N}_{[1,M]}$ **do**
    Obtain measurement $\bar{y}_{l,i}$ according to t_l, i and $\boldsymbol{\mu}_l$.
    Compute $\hat{x}_{t_l+i}(\boldsymbol{\mu}_l, \hat{x}_{t_l}, a_{t_l}, \bar{y}_{l,1}, \ldots, \bar{y}_{l,i})$ and $a_{t_l+i}(\boldsymbol{\mu}_l, \hat{x}_{t_l}, a_{t_l}, \bar{y}_{l,1}, \ldots, \bar{y}_{l,i})$ by solving Problem 1.
  end for
  $t_{l+1} := t_l + M$.
end for

Remark 5: The properties of the self-triggered estimation scheme are determined by the design parameters $\mathcal{M}$ and $\bar{\boldsymbol{\mu}}$. The former contains all possible communication schedules, which allows any type of constraints on the communication to be imposed, such as restrictions on the number of sensors that can communicate simultaneously, an upper bound on the average number of measurement updates in a given time span, or a lower bound on the time between two measurement updates. Via Assumption 4, the latter provides an upper bound on the estimation error.

V. Numerical Examples

In this section we provide two numerical examples demonstrating the effectiveness of the proposed scheme[3].

A. Single-output system

In the first example, we consider a system as defined in Section II with the system matrices given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{22}$$

The bounds on the disturbances are given by $\mathbb{W} = [-0.01, 0.01]^2$ and $\mathbb{V} = [-0.5, 0.5]$. The base schedule was chosen as $\bar{\boldsymbol{\mu}} = [1\ 1\ 1\ 1\ 1\ 1]$. For the periodic Riccati equation and the LMIs in the offline computations in Algorithm 1, we chose $Q_i = I_2$ and $R_i = 0.546$, $i \in \mathbb{N}_{[1,6]}$. We chose $R_i = R$, $i \in \mathbb{N}_{[1,6]}$, and tuned R in order to minimize the minimally possible $a_{\max}$ in Assumption 4. We chose $\mathcal{M}$ to be the set of all possible schedules of length $M = 6$. The gains for the schedules in $\mathcal{M} \setminus \{\bar{\boldsymbol{\mu}}\}$ were computed with appropriate semidefinite programs. We chose $\mathcal{G} = \{e \in \mathbb{R}^2 \mid \|e\|_1 \leq 1\}$, which satisfies (20), such that $e_t \in a_t\mathcal{G} \Leftrightarrow \|e_t\|_1 \leq a_t$. The values for $a_{\min}$ and $a_{\max}$ in Assumption 4 were chosen as $a_{\min} = \frac{\rho_M^{\bar{\boldsymbol{\mu}}}}{1-\lambda_M^{\bar{\boldsymbol{\mu}}}} = 0.8869$ and $a_{\max} = \max\{a_{\min}, \lambda_i^{\bar{\boldsymbol{\mu}}} a_{\min} + \rho_i^{\bar{\boldsymbol{\mu}}} \mid i \in \mathbb{N}_{[1,M]}\} = 1.2638$. With this choice of parameters, the self-triggered set-valued estimator guarantees bounds on the estimation error which are comparable to those guaranteed by a well-tuned Luenberger estimator receiving measurements at every point in time. The initial conditions were set to $\hat{x}_0 = x_0 = [0\ 0]^\top$ and $a_0 = a_{\min}$. We ran 100 simulations for $t \in \mathbb{N}_{[0,59]}$ with uniformly independently distributed random disturbances $w_t \in \mathbb{W}$ and $v_t \in \mathbb{V}$. Plots of the guaranteed and actual estimation error for one of the simulations are shown in Figure 1, as are the times when measurements were transmitted. The average number of transmitted output measurements was 88% of that for the base schedule. This example shows that a self-triggered set-valued estimator is able to reduce the number of communicated measurements while guaranteeing the *same* (worst-case) bounds on the estimation error as a set-valued estimator that receives measurements at every point in time. Note that increasing the maximal bounds on the estimation error allows to further reduce the number of communicated measurements, as we show in the next example.

[3] YALMIP [16], Multi-Parametric Toolbox 3.0 [17] and IBM ILOG CPLEX Optimization Studio [18] were used for the simulations.

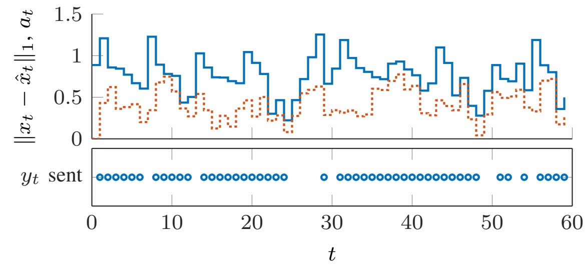

Fig. 1. The upper plot shows the evolution of the real (dashed) output error $\|x_t - \hat{x}_t\|_1$ and the upper bound a_t (solid) computed by the estimator. The lower plot shows the times at which measurements are sent.

B. Multiple-output system

In the second example, we show that our self-triggered set-valued estimation scheme can be employed to decide on which sensors to choose for future measurements under the constraint that only one sensor may send its measurements at a given point in time. Let $a^*_{\min} = \frac{\rho_M^{\bar{\boldsymbol{\mu}}}}{1-\lambda_M^{\bar{\boldsymbol{\mu}}}}$ and $a^*_{\max} = \max\{a^*_{\min}, \lambda_i^{\bar{\boldsymbol{\mu}}} a^*_{\min} + \rho_i^{\bar{\boldsymbol{\mu}}} \mid i \in \mathbb{N}_{[1,M]}\}$, i.e., $a^*_{\min}$ and $a^*_{\max}$ satisfy Assumption 4 with equality. In this example we will furthermore show that by selecting $a_{\min} > a^*_{\min}$ and $a_{\max} > a^*_{\max}$, our self-triggered set-based estimator is able to significantly reduce the required communication, at the price of obtaining a larger (but guaranteed) bound on the estimation error. We again consider a system as defined in Section II, the system matrices are given by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

Note that the system is not detectable if only one the two output channels of the system is used. The bounds on the disturbances are given by $\mathbb{W} = [-0.01, 0.01]^3$ and $\mathbb{V} = [-0.5, 0.5]^2$. The base schedule was chosen as $\bar{\boldsymbol{\mu}} = \left[\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right]$. For the periodic Riccati equation in Algorithm 1, we chose $Q_i = I_3$ and $R_i = I_2$, $i \in \mathbb{N}_{[1,4]}$. We chose $\mathcal{M}$ to be the set of all possible schedules of length $M = 4$, where at most one sensor is updated at any given point in time. We chose $\mathcal{G} = \{e \in \mathbb{R}^3 \mid \|e\|_\infty \leq 1\}$, which satisfies (20), such that $e_t \in a_t\mathcal{G} \Leftrightarrow \|e_t\|_\infty \leq a_t$. We obtained $a^*_{\min} = 1.0978$ and $a^*_{\max} = 2.6410$, and we chose $a_{\min} = 2$ and $a_{\max} = 4.5$. The initial conditions were set to $\hat{x}_0 = x_0 = [0\ 0\ 0]^\top$ and $a_0 = a_{\min}$. We ran 100 simulations for $t \in \mathbb{N}_{[0,59]}$ with uniformly independently distributed random

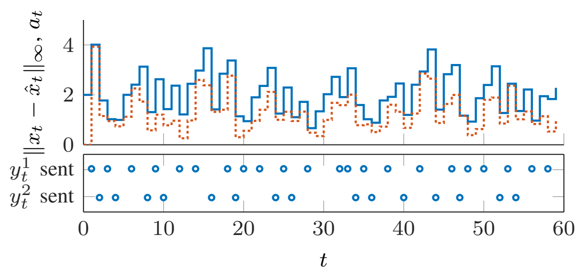

Fig. 2. The upper plot shows the evolution of the real (dashed) output error $\|x_t - \hat{x}_t\|_\infty$ and the upper bound a_t (solid) computed by the estimator. The lower plot shows the times at which measurements are sent over each of the two output channels.

disturbances $w_t \in \mathbb{W}$ and $v_t \in \mathbb{V}$. Plots of the guaranteed and actual estimation error for one of the simulations are shown in Figure 2, as are the times when measurements were transmitted. The average number of transmitted output measurements was 63% of that for the base schedule. In comparison, running a simulation for $a_{\min} = a^*_{\min} + \epsilon$, and $a_{\max} = \max\{a^*_{\min}, \lambda_i^{\bar{\boldsymbol{\mu}}} a^*_{\min} + \rho_i^{\bar{\boldsymbol{\mu}}} \mid i \in \mathbb{N}_{[1,M]}\} + \epsilon$, for $\epsilon = 0.5 \cdot 10^{-4}$, yielded an average number of transmitted output measurements which was 99.23% of that for the base schedule. Here, ϵ is included for numerical reasons.

VI. Discussion on Computational Complexity and Implementation

The offline computations in Algorithm 1 scale with the number of schedules. If $\mathcal{M} = \{0,1\}^{n_y \times M}$ the number of possible schedules scales exponentially with $n_y M$. The online computations require solving the LP related to the set-valued estimator once for each $t \in \mathbb{N}$, as given in Problem 1. These computations can be done efficiently. For the examples in Section V, the required computation times are in the order of milliseconds on a desktop computer. At times t_l, $l \in \mathbb{N}$, it is required to solve Problem 3 to obtain the next schedule. This schedule selection involves checking scalar inequalities, M for each schedule, which can be done in milliseconds, even when the number of schedules is in the order of 10^4. The measurement schedules that result from the self-triggered set-valued estimator have to be communicated to the sensors at times t_l, $l \in \mathbb{N}$, as these inform the sensors at which times they should communicate their measurements in the timespan $\mathbb{N}_{[t_l, t_l+M]}$. Communicating the schedules to the sensors will incur additional communication. As the schedules are sequences of binary values with length M, the amount of *information* that is to be communicated to the sensors is rather small. The efficiency with which this information can be communicated will strongly depend on the communication protocol that is used, which is beyond the scope of the present paper.

VII. Conclusions and Outlook

In this paper, we have proposed a self-triggered set-valued estimator with guaranteed bounds on the estimation error and the number of required measurement updates. As shown in the examples, the estimator is able to exploit its current knowledge of the size of the estimation error in order to decrease the average number of required measurements when compared to a periodic update schedule. The scheme is also able to schedule the measurements of multiple sensors.

Future work includes the improvement of the a priori bounds on the estimation error based on (offline) min-max optimization. Further, the prediction of the estimation error bounds may also be improved by more involved online computations based on solving linear programs similar to those of Problem 1, which then leads to a trade-off between computational demand and network usage. Finally, the set-valued estimator might benefit from an increased number of parameters describing the estimation error, when compared to the scalar a_t currently used. This also requires careful study in the future.

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