Event-Separation Properties of Event-Triggered Control Systems

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***Abstract*—In this paper, we study fundamental properties of minimum inter-event times for several event-triggered control architectures, both in the absence and presence of external disturbances and/or measurement noise. This analysis reveals, amongst others, that for several popular event-triggering mechanisms no positive minimum inter-event time can be guaranteed in the presence of arbitrary small external disturbances or measurement noise. This clearly shows that it is essential to include the effects of external disturbances and measurement noise in the analysis of the computation/communication properties of event-triggered control systems. In fact, this paper also identifies event-triggering mechanisms that do exhibit these important event-separation properties.**

***Index Terms*— Event-triggered control (ETC), minimum inter-event time (MIET).**

I. Introduction

EVENT-TRIGGERED control (ETC) is a new digital control paradigm that recently received a lot of attention [1]–[23]. In ETC the execution of the control tasks (the sampling of the plant's output and the updating of the control inputs) is triggered by specific conditions, involving actuator and (measured) output variables. Therefore, event-triggered control results in aperiodic execution of control tasks, as the time between two events (the inter-event time) is varying. This is in contrast to conventional time-triggered control schemes, in which the execution of the control tasks occurs periodically and the inter-event times are constant.

The recent interest in ETC is motivated by resource constraints in networked control systems, such as limited communication bandwidth and computational power, as well as restricted energy resources if battery-powered (wireless) devices are used. Due to these resource constraints it is desirable to only execute control tasks when this is really needed to guarantee the desired stability and performance properties of the system. This requires varying inter-event times in order for the control scheme to let the execution of the control tasks depend on how the system is operating. As such, ETC systems are much better equipped than time-triggered control systems to balance resource utilization and control performance. Several successful event-triggering mechanisms (ETMs) are proposed in [3]–[9], [11]–[18] and the references therein. See also [2] for a recent overview.

Just as time-triggered control systems, ETC systems should be robust to imperfections, such as external disturbances, modeling errors, measurement noise, and so on. However, for ETC systems it is not enough to only verify the robustness of the control properties (e.g., stability, convergence rates, $\mathcal{L}_2$/ISS-gains); also the robustness of the computation/communication properties need to be carefully examined. To make this more specific, observe first that in time-triggered control systems the event/execution times are not influenced by the presence of the mentioned imperfections. Since in ETC systems the triggering of tasks is state- or output-based, such imperfections affect the execution times, and as such the robustness in terms of a guaranteed (positive) minimum inter-event time (MIET) needs to be guaranteed. This is highly important, because when a nonzero MIET cannot be guaranteed, it might happen that an infinite number of events is generated in finite time (the so-called Zeno behaviour [24]), which makes the ETC scheme infeasible for practical implementation. Indeed, if an ETC system would operate properly in the absence of disturbances, e.g., large minimum and average inter-event times, but (small) disturbances can easily reduce the minimum and average inter-event times significantly, then the ETC system becomes rather ineffective, or even useless in practice.

Some results concerning robustness of computation/communication properties of ETC systems are provided in [3]–[7], [11]. In particular, in [3]–[5], the robustness of the minimum inter-event time with respect to time delays has been considered, and in [7] with respect to time delays and modeling uncertainties. These results exploit, amongst others, that when the system approaches the origin, the effect of the imperfections (time delays and modeling errors) vanishes. Non-vanishing imperfections, such as external disturbances and measurement noise, do not have this convenient property. In fact, the authors of [13] noticed that their ETC setup might not have a positive MIET in the presence of such disturbances, but did not study this phenomenon in detail. In this paper it will indeed be proved that for some classes of systems even if a positive MIET can be guaranteed in the absence of disturbances, still the MIET becomes zero for arbitrarily small disturbances. This indicates zero robustness of the MIET with respect to disturbances, which is clearly undesired. Despite

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the obvious need to study the robustness of the MIET of ETC systems with respect to (non-vanishing) external disturbances and measurement noise, there is a surprising lack of results in this area. Notable exceptions are [6], [23], in which a global positive MIET is guaranteed for a model-based state-feedback ETC scheme in the presence of bounded disturbances, and [11], in which a semi-global positive MIET is guaranteed for (decentralized) output-based ETC schemes in the presence of bounded disturbances.

In addition, it is good to mention various variations on ETC, such as the ETC schemes proposed in [8], [9], [14], periodic event-triggered control (PETC) laws [12], [14]–[18], [25] and self-triggered control (STC) systems [26]–[29]. These schemes all have a built-in lower bound on the MIET. Indeed, in STC, each next event time is pre-calculated using control state and/or output information at the current event time. As a consequence, in STC the system does not have to be monitored continuously, and unexpected disturbances do not lead to earlier generation of events. While this means that the computation/communication properties of STC systems are robust to disturbances, it also means that STC systems cannot react to disturbances as quickly as ETC systems, so the added robustness in terms of computation/communication properties comes at the price of less performance in terms of control properties.

In PETC the ETM is only evaluated periodically, and in the ETC schemes proposed in [8], [9], [14] the ETM is evaluated continuously after a certain time threshold has elapsed. So, in both types of systems, events are only generated after a predetermined lower time threshold has elapsed, automatically leading to robust positive MIETs. We will show that for these kind of ETMs it is still important to consider the influence of (non-vanishing) disturbances on the computation/communication properties of the system, as for arbitrarily small disturbances the ETMs proposed in these papers may generate many events when the system is operating satisfactorily, i.e., the system will operate almost as a time-triggered control system with sampling period equal to the lower threshold, thereby still wasting the scarce resources.

Motivated by the surprising lack of robustness analyses of minimum inter-event times in ETC systems, in this paper we will study the robustness properties of the MIET for well-known ETC schemes and introduce various new notions related to the existence of positive MIETs, that are called *event-separation properties*. These properties will be formulated for general impulsive systems [30], [31], as ETC systems can be well described through this hybrid formulation, see, e.g., [11], [32]. Next to formalizing the (robust) event-separation properties, we will also study if these properties hold globally, semi-globally or locally. In particular, we will investigate these important properties of closed-loop ETC systems for various classes of dynamical systems (linear, nonlinear), different controllers (state-feedback, output-feedback) and various ETMs (relative [3], absolute [6], [23], [33], mixed [11]). This leads to a categorization of ETC schemes which is valuable for ETC design in practice.

The outline of the paper is as follows. We present some necessary preliminaries in Section II and introduce the problem setting and the event-separation properties in Section III. In Section IV we investigate the event-separation properties for state-feedback systems that are triggered based on the states of the plant, and in Section V we extend our results to output-feedback systems that are triggered based on the measured output of the plant. Finally, in Section VI we illustrate our findings with several examples and provide conclusive remarks in Section VII.

A. Nomenclature

For a symmetric matrix $Z \in \mathbb{R}^{n\times n}$, we write $Z \succ 0$ if Z is positive definite and $Z \prec 0$ if Z is negative definite, and we denote by $\lambda_M(Z)$ and $\lambda_m(Z)$ its maximum and minimum eigenvalue, respectively. For vectors $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ we denote by $[x_1, x_2]$ the vector $[x_1^\top \; x_2^\top]^\top$. For a vector $x \in \mathbb{R}^{n_x}$, we denote by $\|x\| := \sqrt{x^\top x}$ its 2-norm, and for a matrix $A \in \mathbb{R}^{n\times m}$, we denote by $\|A\| := \sqrt{\lambda_M(A^\top A)}$ its induced 2-norm. By I we denote the identity matrix of appropriate size. By $\mathbb{N}$ we denote the set of natural numbers including zero, i.e., $\mathbb{N} := \{0, 1, 2, \ldots\}$. With $\mathcal{L}_\infty^n$ we denote the space of all essentially bounded functions of dimension n, and for a signal $w : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_w}$, $w \in \mathcal{L}_\infty^{n_w}$, we denote by $\|w\|_{\mathcal{L}_\infty} = \operatorname{ess\,sup}_{t\in\mathbb{R}_{\geqslant 0}} \|w(t)\|$ its $\mathcal{L}_\infty$-norm. A function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$, and a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and, in addition, $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{KL}$-function if for each fixed $t \geqslant 0$ the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function and for each fixed $s \geqslant 0$, $\beta(s, t)$ is decreasing in t and $\beta(s, t) \to 0$ as $t \to \infty$. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous on compacts if for every compact set $\mathcal{X} \subset \mathbb{R}^n$ there exists a constant $L > 0$ such that $\|f(x) - f(y)\| \leqslant L\|x - y\|$ for all $x, y \in \mathcal{X}$.

II. Preliminaries

Consider the system

$$\dot{\xi} = f(\xi, \omega) \tag{1}$$

in which $f : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\omega} \to \mathbb{R}^{n_\xi}$ is continuously differentiable and satisfies $f(0, 0) = 0$. The variable $\xi \in \mathbb{R}^{n_\xi}$ denotes the state, and $\omega \in \mathbb{R}^{n_\omega}$ is a disturbance. Given initial state $\xi_0 \in \mathbb{R}^{n_\xi}$ and disturbance $\omega \in \mathcal{L}_\infty^{n_\omega}$, we define $\xi(t, \xi_0, \omega)$ as the corresponding solution to (1) satisfying $\xi(0, \xi_0, \omega) = \xi_0$.

Definition II.1 ([34]): The system (1) is input-to-state practically stable (ISpS) if there exist a $\mathcal{KL}$-function β, a $\mathcal{K}$-function γ and a constant $d \in \mathbb{R}_{\geqslant 0}$ such that for each input $\omega \in \mathcal{L}_\infty^{n_\omega}$ and each $\xi_0 \in \mathbb{R}^{n_\xi}$ it holds that

$$\|\xi(t, \xi_0, \omega)\| \leqslant \beta(\|\xi_0\|, t) + \gamma(\|\omega\|_{\mathcal{L}_\infty}) + d \tag{2}$$

for each $t \in \mathbb{R}_{\geqslant 0}$. If the ISpS property holds with $d = 0$, then the system (1) is input-to-state stable (ISS).

Lemma II.1 ([34]): The system (1) is input-to-state practically stable if and only if there exists a continuously differentiable function $V : \mathbb{R}^{n_\xi} \to \mathbb{R}$ such that

$$\alpha_1(\|\xi\|) \leqslant V(\xi) \leqslant \alpha_2(\|\xi\|) \tag{3a}$$

$$\frac{\partial V(\xi)}{\partial \xi} f(\xi, \omega) \leqslant -W(\xi), \text{ when } \|\xi\| \geqslant \rho(\|\omega\|) + c \tag{3b}$$

for all $(\xi, \omega) \in \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\omega}$, where α_1, α_2 are $\mathcal{K}_\infty$-functions, ρ is a $\mathcal{K}$-function, W is a continuous positive definite function on $\mathbb{R}^{n_\xi}$, and c is a nonnegative constant. Moreover, if there exists a V satisfying all the above conditions with $c = 0$, then the system (1) is ISS.

III. Control Setup and Problem Statement

In this section we introduce a general state-feedback control system, which we will use to define the main objective of the paper, and define the event-separation properties. The event-separation properties for the class of state-feedback control systems described in this section will be studied in Section IV. In Section V we will extend the considered control system to output-feedback control systems with measurement noise, and also study the event-separation properties for this class of systems.

Consider a general state-feedback control system, shown in Fig. 1, for stabilizing a plant $\mathcal{P}$ in an appropriate sense. We assume that $\mathcal{P}$ can be described by

$$\dot{x} = f(x, u, w) \tag{4}$$

with $x \in \mathbb{R}^{n_x}$ the state of the plant, $u \in \mathbb{R}^{n_u}$ the control input and $w \in \mathbb{R}^{n_w}$ an external disturbance. Furthermore, we assume that the controller $\mathcal{C}$ is given by the state-feedback law

$$u = k(x^c) \tag{5}$$

where $x^c \in \mathbb{R}^{n_x}$ is the state information available to the controller, which is in general not equal to x, i.e., $x^c = x + e$, where $e \in \mathbb{R}^{n_x}$. We assume that k has been designed such that for the system

$$\dot{x} = f\left(x, k(x+e), w\right) \tag{6}$$

there exists a continuously differentiable function $V : \mathbb{R}^{n_x} \to \mathbb{R}$ satisfying

$$\alpha_1\left(\|x\|\right) \leqslant V(x) \leqslant \alpha_2\left(\|x\|\right) \tag{7a}$$

$$\frac{\partial V(x)}{\partial x} f\left(x, k(x+e), w\right) \leqslant -W(x), \quad \text{when } \|x\| \geqslant \rho_1\left(\|e\|\right) + \rho_2\left(\|w\|\right) \tag{7b}$$

for all $(x, e, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$, where α_1, α_2 are $\mathcal{K}_\infty$-functions, ρ_1 and ρ_2 are $\mathcal{K}$-functions, and W is a continuous positive definite function on $\mathbb{R}^{n_x}$. Obviously, according to Lemma II.1 this implies that (6) is ISS with respect to both measurement errors e and disturbances w. The problem of finding a controller k such that (7) is satisfied will not be discussed in this paper, readers interested in this problem are referred to [35] and the references therein.

At event time t_i determined by the event-triggering mechanism (ETM), $x^c(t_i)$, the state values known to the controller, are updated to the true state $x(t_i)$ of the plant, while between updates the input x^c is held constant in a zero-order-hold (ZOH) fashion, i.e.,

$$x^c(t) = x(t_i), \text{ for } t \in [t_i, t_{i+1}). \tag{8}$$

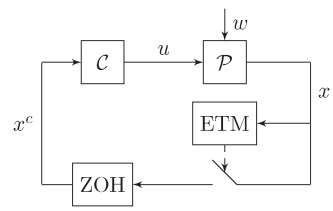

Fig. 1. Architecture I for event-triggered state-feedback control.

We assume that the first event is generated at the time the system is deployed, i.e.,

$$0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant \ldots. \tag{9}$$

Clearly, for the control architecture of Fig. 1, the error $e = x^c - x$ is given by

$$e(t) = x(t_i) - x(t), \text{ for } t \in [t_i, t_{i+1}). \tag{10}$$

Note that in conventional time-triggered control, we would have $t_{i+1} - t_i = h$ for all $i \in \mathbb{N}$, with h the sampling period. However, in ETC the inter-event times $t_{i+1} - t_i$ are varying and determined based on state information.

A common design practice, originating from [3], for finding ETMs that render the system (6), (10) ISpS with respect to w, is to make sure that the ETM enforces[1]

$$\rho_1\left(\|e\|\right) \leqslant \sigma\|x\| + \beta \tag{11}$$

with $0 \leqslant \sigma < 1$ and $\beta \geqslant 0$. This leads to the ETM

$$t_{i+1} = \inf\left\{t > t_i | \rho_1\left(\|e(t)\|\right) > \sigma\|x(t)\| + \beta\right\}. \tag{12}$$

Indeed, systems that satisfy (7), also satisfy

$$\alpha_1\left(\|x\|\right) \leqslant V(x) \leqslant \alpha_2\left(\|x\|\right) \tag{13a}$$

$$\frac{\partial V}{\partial x} f\left(x, k(x+e), w\right) \leqslant -W(x), \quad \text{when } \|x\| \geqslant \frac{1}{1-\sigma}\left(\rho_2\left(\|w\|\right) + \beta\right) \tag{13b}$$

as long as (11) holds, which guarantees the ISpS property according to Lemma II.1.

Combining (6), (10), and (12) leads to the closed-loop ETC system

$$\begin{cases} \dot{x}(t) = f\left(x(t), k\left(x(t) + e(t)\right), w(t)\right) \\ e(t) = x(t_i) - x(t), \text{ for } t \in [t_i, t_{i+1}) \\ t_{i+1} = \inf\left\{t > t_i \mid \rho_1\left(\|e(t)\|\right) > \sigma\|x(t)\| + \beta\right\} \end{cases} \tag{14}$$

which is ISpS[2] with respect to w if $\beta > 0$, and ISS with respect to w if $\beta = 0$, due to Lemma II.1 and (13).

[1]Note that the ISS characterizations used in this paper are slightly different from the ones used in [3], but are equivalent and result in essentially the same ETMs.

[2]When we say that (14) is ISpS, we mean that the solutions of (14) satisfy Definition II.1 with $\xi = x$ and $\omega = w$.

We will distinguish, based on (12), three classes of ETMs:

- If $0 < \sigma < 1$ and $\beta = 0$, which is the ETM as proposed in [3] and many follow-ups, including [4], [7]–[9], we will talk about a *relative ETM*.
- If $\sigma = 0$ and $\beta > 0$, as used in, e.g., [6], [33], we will talk about an *absolute ETM*.
- If $0 < \sigma < 1$ and $\beta > 0$, as proposed in [11], we will talk about a *mixed ETM*.

The main objective of this paper is to investigate under which conditions the ETC system (14) can be guaranteed to have inter-event times $t_{i+1} - t_i$, $i \in \mathbb{N}$, that are bounded from below by a positive constant, even in the presence of external disturbances w. In order to study this problem, we formally introduce the relevant *event-separation* properties in Section III-B. However, before doing so, we first introduce in Section III-A the special case where the functions f and k are linear. In Section IV we study the event-separation properties of a class of state-feedback ETC systems in the presence of disturbances w. As mentioned before, in Section V we extend the control setup to include also output-feedback systems, and study the event-separation properties of a class of output-feedback ETC systems in the presence of disturbances w and measurement noise.

A. The Linear Case

Consider the case in which plant $\mathcal{P}$ and controller $\mathcal{C}$ are linear. In this case (4) is given by $\dot{x} = Ax + Bu + w$, (5) by $u = Kx^c$, and the derivation above would replace (6) by

$$\dot{x} = (A + BK)x + BKe + w \tag{15}$$

in which we assume that $A + BK$ is Hurwitz, to satisfy the conditions in (7). If for this case we would like to design an ETM that results in an IS(p)S closed-loop system, we can use the following procedure, inspired by [3]. Since $A + BK$ is Hurwitz, we can find matrices $Z \succ 0$ and $Q \succ 0$, satisfying

$$(A + BK)^\top Z + Z(A + BK) = -Q. \tag{16}$$

From this we derive for $V(x) = x^\top Zx$ that

$$\dot{V} \leqslant -\lambda_m(Q)\|x\|^2 + 2\|ZBK\|\|x\|\|e\| + 2\|Z\|\|x\|\|w\|. \tag{17}$$

By selecting any $0 < \gamma < 1$, it follows that:

$$\dot{V} \leqslant (\gamma - 1)\lambda_m(Q)\|x\|^2, \quad \text{when } \|x\| \geqslant \frac{2\|ZBK\|}{\gamma\lambda_m(Q)}\|e\| + \frac{2\|Z\|}{\gamma\lambda_m(Q)}\|w\| \tag{18}$$

which is in the form of (7). In this case the ETM (12) is equal to

$$t_{i+1} = \inf\{t > t_i \mid \|e(t)\| > P\|x(t)\| + T\} \tag{19}$$

with

$$P = \sigma\gamma\frac{\lambda_m(Q)}{2\|ZBK\|} \tag{20}$$

and

$$T = \beta\gamma\frac{\lambda_m(Q)}{2\|ZBK\|}. \tag{21}$$

Combining (10), (15), and (19) leads to the closed-loop ETC system

$$\begin{cases} \dot{x}(t) = (A + BK)x(t) + BKe(t) + w(t) \\ e(t) = x(t_i) - x(t), \text{ for } t \in [t_i, t_{i+1}) \\ t_{i+1} = \inf\{t > t_i \mid \|e(t)\| > P\|x(t)\| + T\} \end{cases} \tag{22}$$

which is a special case of (14).

Remark III.1: Note that also in the nonlinear case we can consider an ETM of the form (19) when ρ_1 is Lipschitz continuous on compacts. By taking $P = \sigma/L_1$ and $T = \beta/L_1$, with $\rho_1(\|e\|) \leqslant L_1\|e\|$, it holds on compact sets that $\|e\| \leqslant P\|x\| + T$ implies $\rho_1(\|e\|) \leqslant \sigma\|x\| + \beta$, thus the IS(p)S properties of ETM (12) are preserved by ETM (19). Because ETM (12) cannot generate an event before ETM (19) does, we can use ETM (19) to find a lower bound on the inter-event times generated by ETM (12) in the closed-loop system (14).

B. Definitions

The resulting ETC systems can be written as impulsive systems (also called jump-flow systems), cf. [11], [27], of the form

$$\dot{\xi} = F(\xi, \omega), \text{ if } \xi \in \mathcal{F}(\omega) \tag{23a}$$
$$\xi^+ = G(\xi, \omega), \text{ if } \xi \in \mathcal{J}(\omega) \tag{23b}$$

where the flow set $\mathcal{F}$ and the jump set $\mathcal{J}$ are determined by the event-triggering mechanism, as we will see. Throughout this article we assume that $\mathcal{F}$, $\mathcal{J}$, F, and G are such that existence and uniqueness of solutions are guaranteed for each initial condition ξ_0 and each disturbance ω of interest. In addition, we assume that all solutions are complete in the sense of [30], i.e., loosely speaking, either the solution is defined for time $t \to \infty$, or the number of jumps $i \to \infty$, or both. See [30] for more details on the definitions of solutions and the hybrid model class (23).

To be precise, the ETC system (14), can be written in the form (23) by taking $\xi = [x, e] \in \mathbb{R}^{n_\xi}$ with $n_\xi = 2n_x$, $\omega = w$, and

$$F(\xi, \omega) = \begin{bmatrix} f(x, k(x+e), w) \\ -f(x, k(x+e), w) \end{bmatrix}, \quad G(\xi) = \begin{bmatrix} x \\ 0 \end{bmatrix}$$
$$\mathcal{F} = \{\xi \mid \rho_1(\|e\|) \leqslant \sigma\|x\| + \beta\}$$
$$\mathcal{J} = \{\xi \mid \rho_1(\|e\|) > \sigma\|x\| + \beta\}.$$

Note that in this case, G, $\mathcal{F}$, and $\mathcal{J}$ are independent of the disturbance ω.

Since we assume that for the ETC system (14) the first event is generated at the time the system is deployed, for a disturbance signal $\omega : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_\omega}$ the corresponding impulsive system (23) essentially starts in the set of initial states given by

$$\Xi_\omega := \{\xi_0 \in \mathbb{R}^{n_\xi} \mid \xi_0 = G(\xi, \omega(0)) \text{ for some } \xi \in \mathbb{R}^{n_\xi}\}. \tag{24}$$

We will sometimes use the notation $\Xi_0 = \Xi_\omega|_{\omega=0}$ to indicate the set of initial states in the absence of disturbances. Given disturbance ω and initial condition $\xi_0 \in \Xi_\omega$, the system (23) jumps according to (23b) at the jump times included in the set $\{t_i \mid i \in I(\xi_0, \omega)\}$, where $I(\xi_0, \omega)$ is an index set enumerating the jump times. Clearly, $I(\xi_0, \omega) = \mathbb{N}$ or $I(\xi_0, \omega) = \{0, 1, 2, \ldots, N\}$ for some $N \in \mathbb{N}$. To make the dependence of the jump/event times

on the initial condition $\xi_0 \in \Xi_\omega$, and on the disturbance signal ω explicit, we sometimes write $t_i = t_i(\xi_0, \omega)$, $i \in I(\xi_0, \omega)$. In the case that $I(\xi_0, \omega) = \{0, 1, 2, \ldots, N\}$ we define $t_{N+1} := \infty$ (as we know that solutions are complete).

All definitions below apply to the impulsive system (23).

Definition III.1: The i-th inter-event time $\tau_i(\xi_0, \omega)$ with $i \in I(\xi_0, \omega)$ corresponding to disturbance signal $\omega : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_\omega}$ and initial condition $\xi_0 \in \Xi_\omega$ is given as

$$\tau_i(\xi_0, \omega) := t_{i+1}(\xi_0, \omega) - t_i(\xi_0, \omega). \tag{25}$$

Definition III.2: The minimum inter-event time $\tau(\xi_0, \omega)$ for disturbance signal $\omega : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_\omega}$ and initial state $\xi_0 \in \Xi_\omega$ is defined as

$$\tau(\xi_0, \omega) := \inf_{i \in I(\xi_0, \omega)} t_{i+1}(\xi_0, \omega) - t_i(\xi_0, \omega). \tag{26}$$

Based on the above definitions on inter-event times, we introduce the following event-separation properties for system (23).

Definition III.3: The impulsive system (23) has the global event-separation property if

$$\inf_{\xi \in \Xi_0} \tau(\xi, 0) > 0. \tag{27}$$

Definition III.4: The impulsive system (23) has the semi-global event-separation property if for all compact subsets $\mathcal{X} \subset \mathbb{R}^{n_\xi}$

$$\inf_{\xi \in \mathcal{X} \cap \Xi_0} \tau(\xi, 0) > 0. \tag{28}$$

Definition III.5: The impulsive system (23) has the local event-separation property if for all $\xi \in \Xi_0$

$$\tau(\xi, 0) > 0. \tag{29}$$

In addition, we define their robust counterparts as follows.

Definition III.6: The impulsive system (23) has the robust global event-separation property if there exists $\varepsilon > 0$ such that

$$\inf_{\substack{\xi \in \Xi_\omega \\ \|\omega\|_{\mathcal{L}_\infty} \leqslant \varepsilon}} \tau(\xi, \omega) > 0. \tag{30}$$

Definition III.7: The impulsive system (23) has the robust semi-global event-separation property if there exists $\varepsilon > 0$ such that for all compact subsets $\mathcal{X} \subset \mathbb{R}^{n_\xi}$

$$\inf_{\substack{\xi \in \mathcal{X} \cap \Xi_\omega \\ \|\omega\|_{\mathcal{L}_\infty} \leqslant \varepsilon}} \tau(\xi, \omega) > 0. \tag{31}$$

Definition III.8: The impulsive system (23) has the robust local event-separation property if there exists $\varepsilon > 0$ such that for all $\omega \in \mathcal{L}_\infty$ satisfying $\|\omega\|_{\mathcal{L}_\infty} \leqslant \varepsilon$ and all $\xi \in \Xi_\omega$ it holds that

$$\tau(\xi, \omega) > 0. \tag{32}$$

A few comments on the above definitions are in order. If a system has the local event-separation property, then no Zeno behavior (see [24]) occurs in the impulsive system (23) when $\omega = 0$. If the system (23) has the local event-separation property, but does not have the semi-global event-separation property, then there exists $\xi^* \in \Xi_0$ such that $\liminf_{\xi \to \xi^*} \tau(\xi, 0) = 0$, while $\tau(\xi^*, 0) > 0$. This can happen, because $\tau(\xi, 0)$ is in general not continuous in ξ.

If a system possesses the semi-global event-separation property, but not the global property, then $\limsup_{\|\xi\| \to \infty} \tau(\xi, 0) = 0$. This might however not be a problem, as in a real systems the states have to remain bounded anyway. However, the larger the states, the smaller the MIET becomes.

Similar comments apply to the robust versions.

A system that has the robust global event-separation property has a lower bound on $\tau(\xi, \omega)$ that is uniform in ξ and all ω satisfying $\|\omega\|_{\mathcal{L}_\infty} \leqslant \varepsilon$, which guarantees that the system can work in practice for any arbitrary initial condition.

Note that the (robust) global event-separation property implies the (robust) semi-global event-separation property, and in turn, the (robust) semi-global event-separation property implies the (robust) local event-separation property.

The semi-global event-separation property is similar to the *semiglobal dwell time* property defined in [32, Definition 2].

IV. Main Results for the State-Feedback Case

In this section we study the event-separation properties for ETC systems (14) for the relative, mixed and absolute ETMs (12). Whenever we mention event-separation properties for ETC systems, we naturally mean the event-separation properties of the corresponding impulsive system (23).

A. Relative Triggering

From [3, Theorem III.1] it directly follows that the ETC system (14) using a relative ETM (12) has the semi-global event-separation property if the functions f, k, and ρ_1 are Lipschitz continuous on compacts, and the global event-separation property if the functions f, k, and ρ_1 are globally Lipschitz continuous. This is an interesting and valuable result, but this result does not extend to the case where arbitrarily small disturbances are present. Indeed, the system (14) using a relative ETM (12) in general does *not* have any *robust* event-separation properties. We prove this statement for the ETC system (22) with $P > 0$ and $T = 0$, i.e., focussing on the linear case.

Theorem IV.1: The closed-loop event-triggered control system (22) with $P > 0$ and $T = 0$ does not have the robust local event-separation property.

Proof: We prove the theorem by showing that for each $\varepsilon > 0$, there exist $x_0 \neq 0$ and w satisfying $\|w\|_{\mathcal{L}_\infty} \leqslant \varepsilon$, such that $\tau([x_0, 0], w) = 0$.

To do so, consider an arbitrary $x_0 \neq 0$ and note that for $t \in [0, t_1)$

$$\dot{x}(t) = Ax(t) + BKx_0 + w(t). \tag{33}$$

The disturbance

$$w(t) = -Ax(t) - BKx_0 - x_0 \tag{34}$$

enforces that $\dot{x} = -x_0$, thus $x(t) = (1 - t)x_0$ and $e(t) = tx_0$ as long as (34) describes the active disturbance. As an event is triggered as soon as $\|e(t)\| \leqslant P\|x(t)\|$ is violated, we have

that the first event time is $t_1 = P/(P+1) = 1 - 1/(P+1)$ for the disturbance in (34). Note that $t_1 < 1$ for any $P > 0$, and that the disturbance can also take the open-loop form $w(t) = -(I + (1-t)A + BK)x_0$, for $t \in [0, t_1)$, resulting in the same response.

At t_1, the error e is reset to zero, i.e., $e(t_1) = 0$, and $x(t_1) = (1 - t_1)x_0$. For $t \in [t_1, t_2)$ this leads to

$$\dot{x}(t) = Ax(t) + (1 - t_1)BKx_0 + w(t).$$

The disturbance

$$w(t) = -Ax(t) - (1 - t_1)BKx_0 - x_0, \text{ for } t \in [t_1, t_2)$$

enforces that $x(t) = (1-t)x_0$ and $e(t) = (t - t_1)x_0$ for $t \in [t_1, t_2)$. From this it follows that $t_2 = (P + t_1)/(P+1) = 1 - (P+1)^{-2}$. Repeating this argument leads to the conclusion that the disturbance

$$w(t) = ((t-1)A + (t_i - 1)BK - I)\, x_0, \text{ for } t \in [t_i, t_{i+1})$$

generates the event times

$$t_i\left([x_0, 0], w\right) = 1 - (P+1)^{-i} \tag{35}$$

which shows that there is an accumulation point at $t = 1$ and that $\tau([x_0, 0], w) = 0$. By taking some $x_0 \neq 0$ satisfying

$$\|x_0\| \leqslant \frac{\varepsilon}{1 + \|A\| + \|BK\|} \tag{36}$$

it is ensured that $\|w\|_{\mathcal{L}_\infty} \leqslant \varepsilon$, which completes the proof. ■

Since the ETC system (22) with $P > 0$ and $T = 0$ does not have the robust local event-separation property, obviously it also does not have the robust (semi-)global event-separation property.

B. Mixed Triggering

In the next theorem, which forms an extension to Theorem III.1 of [3] towards mixed ETMs, we state that ETC systems (14) satisfying (7), with f, k, and ρ_1 Lipschitz continuous on compacts and using mixed ETMs have the robust semi-global event-separation property.

Theorem IV.2: Consider the closed-loop event-triggered system (14) with $0 < \sigma < 1$ and $\beta > 0$. If

1) f, k, and ρ_1 are Lipschitz continuous on compacts;
2) there exists a continuously differentiable function V for the system satisfying (7),

then the system (14) is ISpS and has the robust semi-global event-separation property.

Proof: It is shown in Section III that the system (14) with $\beta > 0$ is ISpS. Thus, for all compact sets $\mathcal{S}$, all $\varepsilon \geqslant 0$ and all w satisfying $\|w\|_{\mathcal{L}_\infty} \leqslant \varepsilon$, there exists a compact set $\mathcal{S}'$, such that $x(t) \in \mathcal{S}'$ for all $t \geqslant 0$ whenever $x(0) \in \mathcal{S}$. Because f and k are Lipschitz continuous on compacts, we can find a constant L such that

$$\|f\left(x, k(x+e), w\right)\| \leqslant L\left(\|x\| + \|e\| + \|w\|\right) \tag{37}$$

for all $x \in \mathcal{S}'$, $\|w\| \leqslant \varepsilon$, and e satisfying $\rho_1(\|e\|) \leqslant \sigma\|x\| + \beta$.

To prove that for all $\varepsilon < \infty$ there exists $\tau^* > 0$ such that $\tau([x, 0], w) \geqslant \tau^*$ for all $x \in \mathcal{S}$ and all w satisfying $\|w\|_{\mathcal{L}_\infty} \leqslant \varepsilon$, we make use of Remark III.1, and consider ETM (19) instead of the actual ETM (12). We can now find $\tau^* > 0$ by investigating the dynamics of $\|e\|/(P\|x\| + T)$, as the inter-event time is equal to the time it takes for $\|e\|/(P\|x\| + T)$ to grow from 0 to 1.

Using (37) and $\dot{e} = -\dot{x}$, we find that for all trajectories starting in $\mathcal{S}$, it holds almost everywhere that

$$\begin{aligned}\frac{d}{dt}\frac{\|e\|}{P\|x\| + T} &= -\frac{e^\top \dot{x}}{\|e\|\,(P\|x\| + T)} - P\frac{x^\top \dot{x}}{(P\|x\| + T)^2}\frac{\|e\|}{\|x\|} \\ &\leqslant L\left(1 + P\frac{\|e\|}{P\|x\| + T}\right)\left(\frac{\|x\| + \varepsilon + \|e\|}{P\|x\| + T}\right) \\ &\leqslant L\left(1 + P\frac{\|e\|}{P\|x\| + T}\right)\left(\rho + \frac{\|e\|}{P\|x\| + T}\right)\end{aligned} \tag{38}$$

where $\rho = 1/P + \varepsilon/T \geqslant \|x\|/(P\|x\| + T) + \varepsilon/(P\|x\| + T)$. We conclude that $\|e(t_i + s)\|/(P\|x(t_i + s)\| + T) \leqslant \phi(s, 0)$ for $s \in [t_i, t_{i+1})$, where $\phi(t, \phi_0)$ is the solution to $\dot{\phi} = L(1 + P\phi)(\rho + \phi)$ satisfying $\phi(0, \phi_0) = \phi_0$. Since the ETM (12) generates larger inter-event times than the ETM (19), it follows that for the system (14) with $0 < \sigma < 1$ and $\beta > 0$, the inter-event times are lower bounded by $\tau([x, 0], w) \geqslant \tau^*$ for all $x \in \mathcal{S}$ and $\|w\|_{\mathcal{L}_\infty} \leqslant \varepsilon$, with τ^* such that $\phi(\tau^*, 0) = 1$, from which it immediately follows that the system has the robust semi-global event-separation property. ■

Corollary IV.3: Consider the closed-loop event-triggered system (14) with $0 < \sigma < 1$ and $\beta > 0$. If

1) f, k, and ρ_1 are globally Lipschitz continuous;
2) there exists a continuously differentiable function V for the system satisfying (7),

then the system (14) is ISpS and has the robust global event-separation property.

The proof of Corollary IV.3 can be directly derived from the proof of Theorem IV.2.

Remark IV.1: The robust event-separation property in Corollary IV.3 is global in x, but only semi-global in w, in the sense that for each ε there exists a lower bound τ^* on the MIET, but $\tau^* \to 0$ when $\varepsilon \to \infty$.

C. Absolute Triggering

Next we show that ETC systems (14) with f, k, and ρ_1 Lipschitz continuous on compacts, using absolute ETMs have the robust semi-global event-separation property.

Theorem IV.4: Consider the closed-loop event-triggered control system (14), with $\sigma = 0$ and $\beta > 0$. If

1) f, k, and ρ_1 are Lipschitz continuous on compacts;
2) there exists a continuously differentiable function V for the system satisfying (7),

then the system (14) is ISpS and has the robust semi-global event-separation property.

Proof: The proof can be obtained by adapting the proof of Theorem IV.2. We also use the same notation $\mathcal{S}$, $\mathcal{S}'$. In

particular, since $\sigma = 0$ it follows that $P = 0$, see Remark III.1. Using similar derivations that we used to find (38), we get

$$\frac{d}{dt}\frac{\|e\|}{T} \leqslant L\left(\rho + \frac{\|e\|}{T}\right) \tag{39}$$

with $\rho = (x_{\max} + \varepsilon)/T$ and $x_{\max}$ is such that $\|x\| \leqslant x_{\max}$ for all $x \in \mathcal{S}'$. Thus $\|e(t_i + s)\|/T \leqslant \phi(s, 0)$ for $s \in [t_i, t_{i+1})$, where $\phi(t, \phi_0)$ is the solution to $\dot{\phi} = L(\phi + \rho)$ satisfying $\phi(0, \phi_0) = \phi_0$. The inter-event times are lower bounded by $\tau([x, 0], w) \geqslant \tau^*$ for all $x \in \mathcal{S}$ and $\|w\|_{\mathcal{L}_\infty} \leqslant \varepsilon$, with τ^* such that $\phi(\tau^*, 0) = 1$, from which it immediately follows that the system has the robust semi-global event-separation property. Since $\phi(t, 0) = \rho e^{Lt} - \rho$, we find

$$\tau^* = \frac{1}{L}\ln\left(\frac{1+\rho}{\rho}\right). \tag{40}$$

■

The MIET τ^* is a function of $x_{\max}$, indicating that the semi-global result cannot be readily extended to a global result along the lines of the proof above. In fact, in the next theorem we state that ETC systems (22) using absolute ETM do not have the global event-separation property.

Theorem IV.5: The closed-loop event-triggered control system (22) with $A + BK \neq 0$, $P = 0$ and $T > 0$ does not have the global event-separation property.

Proof: From (22) with $w = 0$ and $P = 0$ it follows that:

$$t_1 = \inf\{t > 0 \mid \|e(t)\| > T\}. \tag{41}$$

For $t \in [0, t_1)$ it holds that

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{As}ds BKx_0, \\ &= x_0 + \int_0^t e^{As}ds(A + BK)x_0 \end{aligned} \tag{42}$$

and, using $e(t) = x_0 - x(t)$,

$$\|e(t)\| = \left\|\int_0^t e^{As}ds(A + BK)x_0\right\|. \tag{43}$$

We proceed now by contradiction. Suppose that the system would have the global event-separation property, then there exists $\tau^* > 0$ such that $t_1([x_0, 0], 0) \geqslant \tau^*$ for any $x_0 \in \mathbb{R}^{n_x}$. Since $A + BK \neq 0$, we can find for any $\tau^* > 0$ a vector v such that for some $0 < t^c \leqslant \tau^*$

$$\underbrace{\left\|\int_0^{t^c} e^{As}ds(A + BK)v\right\|}_{c} > 0. \tag{44}$$

Thus, by choosing $x_0 = \alpha v$ with $\alpha > T/c$ we find that $t_1([x_0, 0], 0) < t^c \leqslant \tau^*$. This clearly contradicts $t_1 \geqslant \tau^*$, from which it follows that the system does not have the global event-separation property. ■

TABLE I
EVENT-SEPARATION PROPERTIES FOR CLOSED-LOOP ETC SYSTEMS (22)

	event-separation property			robust event-separation property		
ETM	global	semi-global	local	global	semi-global	local
rel	✓	✓	✓	×	×	×
mix	✓	✓	✓	✓	✓	✓
abs	×	✓	✓	×	✓	✓

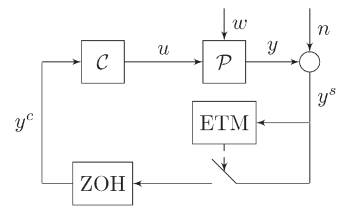

Fig. 2. Architecture II for output-feedback event-triggered control.

D. Overview of the Linear Case

Summarizing the above results for the linear case, the event-separation properties of closed-loop ETC systems (22) are shown in Table I for relative, mixed and absolute ETMs.

While the relative ETM has robustness issues at small $\|x\|$, the absolute ETM generates inter-event times that vanish as $\|x\| \to \infty$. The mixed ETM combines the good properties of the relative and absolute ETMs, i.e., a global MIET and no robustness issues near the origin. This will also be illustrated in Example 1 in Section VI.

For the linear case the above results still hold when the state measurements are contaminated with measurement noise, see [36] for the proofs.

V. MAIN RESULTS FOR THE OUTPUT-FEEDBACK CASE

In this section we extend Architecture I to output-based controllers, subject to measurement noise. This results in Architecture II, shown in Fig. 2. We assume that the plant $\mathcal{P}$ with state x can again be described by (4), that the output y is described by

$$y = h(x) \tag{45}$$

and that the controller $\mathcal{C}$ is given by

$$u = k(y^c). \tag{46}$$

In contrast to Architecture I, the outputs y are measured instead of the states x. Moreover, since sensors are imperfect, the measured values $y^s = y + n$ are contaminated with measurement noise n. As a consequence, y^c is given by

$$y^c(t) = y^s(t_i), \text{ for } t \in [t_i, t_{i+1}) \tag{47}$$

which is in line with (8).

If we now define the error e as $e = y^c - y$, or, stated differently

$$e(t) = y^s(t_i) - y(t), \text{ for } t \in [t_i, t_{i+1}) \tag{48}$$

we can write the system as

$$\dot{x} = f\left(x, k\left(h(x) + e\right), w\right). \tag{49}$$

Throughout this section, we will again assume that the controller $\mathcal{C}$ renders the system (49) ISS with respect to e and w.

Since we can only measure y^s, and not y, the ETM will generate events based on the measured signal $y^s(t)$, and the network-induced error $e^n(t)$, defined as

$$e^n(t) = y^c(t) - y^s(t) = e(t) - n(t). \tag{50}$$

Note that with this definition we have the identity

$$y + e = y^s + e^n = y^c. \tag{51}$$

How to design an ETM of the form (12) based on $y^s(t)$ and $e^n(t)$ which renders the closed-loop system IS(p)S with respect to disturbances w and measurement noise n is in general an open problem, but in the linear case the presented ETMs can be used with a slight modification. Therefore, we limit this section to the linear case.

We consider the linear system with $f(x,u,w) = Ax + Bu + w$, $h(x) = Cx$ and $k(y^c) = Ky^c$, with $A + BKC$ Hurwitz. The observability matrix O is given by

$$O = \begin{bmatrix} C^\top & (CA)^\top & \dots & (CA^{n_x-1})^\top \end{bmatrix}^\top. \tag{52}$$

To find ETMs that render the linear output-feedback system IS(p)S with respect to w and n, the following procedure can be used, as motivated also in Section III-A.

Since $A + BKC$ is Hurwitz, we can find matrices $Z \succ 0$ and $Q \succ 0$, satisfying

$$(A + BKC)^\top Z + Z(A + BKC) = -Q. \tag{53}$$

From this we derive for $V(x) = x^\top Zx$ that for any $0 < \gamma < 1$

$$\dot{V} \leqslant (\gamma - 1)\lambda_m(Q)\|x\|^2, \quad \text{when } \|x\| \geqslant \frac{2\|ZBK\|}{\gamma\lambda_m(Q)}\|e\| + \frac{2\|Z\|}{\gamma\lambda_m(Q)}\|w\|. \tag{54}$$

By making use of $\|y^s\| \leqslant \|C\|\|x\| + \|n\|$ and $\|e\| - \|n\| \leqslant \|e^n\|$, we can see that the ETM

$$t_{i+1} = \inf\{t > t_i \mid \|e^n(t)\| > P\|y^s(t)\| + T\} \tag{55}$$

enforces that

$$\|e\| \leqslant P\|C\|\|x\| + P\|n\| + \|n\| + T. \tag{56}$$

Then, by taking

$$P = \sigma\gamma\frac{\lambda_m(Q)}{2\|ZBK\|\|C\|} \tag{57}$$

and

$$T = \beta\gamma\frac{\lambda_m(Q)}{2\|ZBK\|} \tag{58}$$

we find, analogously to (13), that

$$\dot{V} \leqslant (\gamma - 1)\lambda_m(Q)\|x\|^2, \text{ when } (1-\sigma)\|x\| \geqslant \beta + \sigma\frac{\|n\|}{\|C\|} + \frac{2\|ZBK\|}{\gamma\lambda_m(Q)}\|n\| + \frac{2\|Z\|}{\gamma\lambda_m(Q)}\|w\| \tag{59}$$

whenever (54) and (56) hold.

In conclusion, we consider the ETC system

$$\begin{cases} \dot{x}(t) = Ax(t) + BK\left(y^s(t) + e^n(t)\right) + w(t) & \text{(60a)} \\ y^s(t) = Cx(t) + n(t) & \text{(60b)} \\ e^n(t) = y^s(t_i) - y^s(t), \text{ for } t \in [t_i, t_{i+1}) & \text{(60c)} \\ t_{i+1} = \inf\{t > t_i \mid \|e^n(t)\| > P\|y^s(t)\| + T\}, & \text{(60d)} \end{cases}$$

with $A + BKC$ Hurwitz.

Remark V.1: Equation (59) implies that the ETC system (60) with (57) and (58) is ISS if $0 \leqslant \sigma < 1$ and $\beta = 0$ and ISpS if $0 \leqslant \sigma < 1$ and $\beta > 0$.

Remark V.2: The ETM (55) with (57) and (58) might be more conservative than strictly necessary (in the sense that larger values of P may exist such that the IS(p)S property can be guaranteed). The conservatism is due to the use of small-gain arguments in (54). In fact, many systems will be IS(p)S for larger P. Less conservative ETMs might be found by using a hybrid modelling approach, see [11] for a discussion on this issue.

The ETC system (60) can be written in the form of (23) by taking $\xi = [x, e] \in \mathbb{R}^{n_x+n_y}$, $\omega = [w, n] \in \mathbb{R}^{n_x+n_y}$, and

$$F(\xi,\omega) = \begin{bmatrix} (A+BKC)x + BKe + w \\ -C(A+BKC)x - CBKe - Cw \end{bmatrix}$$

$$G(\xi,\omega) = [\,x^\top \quad n^\top\,]^\top$$

$$\mathcal{F}(\omega) = \left\{\xi \in \mathbb{R}^{n_x+n_y} \mid \|e - n\| \leqslant P\|Cx + n\| + T\right\}$$

$$\mathcal{J}(\omega) = \left\{\xi \in \mathbb{R}^{n_x+n_y} \mid \|e - n\| > P\|Cx + n\| + T\right\}.$$

Note that in contrast to the previous section, G, $\mathcal{F}$, and $\mathcal{J}$ depend on the disturbance ω.

In the remainder of this section, some theorems will be stated for the specific case with $P < 1$. The focus on this case is motivated by the next lemma in which we will show that the above common design procedure for ETMs will always result in $P < 1$ as long as A is not Hurwitz. The theorems formulated for $P < 1$ do not hold in general (i.e., they do not always hold for the case $P \geqslant 1$).

Lemma V.1: If A is not Hurwitz, then P in (57) satisfies $P < 1$.

Proof: Since $A + BKC$ is assumed to be Hurwitz, we can find $Z \succ 0$, $Q \succ 0$ satisfying (53). Since A is not Hurwitz, there exists no $Z' \succ 0$ such that $A^\top Z' + Z'A \prec 0$. Thus, $A^\top Z + ZA$ is not negative definite, and there exists $x \neq 0$ such that $x^\top(A^\top Z + ZA)x \geqslant 0$. For this specific x it follows from (53) that:

$$-x^\top(C^\top K^\top B^\top Z + ZBKC)x = x^\top(Q + A^\top Z + ZA)x$$

and thus

$$-x^\top(C^\top K^\top B^\top Z + ZBKC)x \geqslant x^\top Qx. \tag{61}$$

Since $Q \succ 0$, this leads to

$$\left\|-x^\top(C^\top K^\top B^\top Z + ZBKC)x\right\| \geqslant \left\|x^\top Qx\right\|. \tag{62}$$

Since

$$2\|ZBK\|\|C\|\|x\|^2 \geqslant \left\|x^\top(C^\top K^\top B^\top Z + ZBKC)x\right\| \tag{63}$$

and

$$\|x^\top Qx\| \geqslant \lambda_m(Q)\|x\|^2 \tag{64}$$

it follows by elimination of the term $\|x\|^2$ that $2\|ZBK\|\|C\| \geqslant \lambda_m(Q)$. Since, in addition, $\sigma < 1$ and $\gamma < 1$, it follows that $P < 1$. ■

As was indicated in [11] via a numerical example, not all linear systems using the relative triggering condition $\|e^n\| = P\|y^s\|$ have the local event-separation property. For ETC systems (60) with $P < 1, T = 0$ and $\mathrm{rank}(C) < \mathrm{rank}(O)$ this can be shown to hold in general, which we will do by proving the following theorem.

Theorem V.2: The ETC system (60) with $P < 1, T = 0$ and $\mathrm{rank}(C) < \mathrm{rank}(O)$ does not have the local event-separation property.

Proof: Consider the initial state $x_0 \in \ker C \setminus \ker O$, and note that in the absence of disturbances, i.e., $w = 0, n = 0$, the equality $\|e^n\| = P\|y^s\|$ reduces to $\|e\| = P\|y\|$, and that for $t \in [0, t_1)$ it holds that

$$\begin{aligned} y(t) &= Ce^{At}x_0 + C\int_0^t e^{As}dsBKCx_0 \\ &= Cx_0 + C\int_0^t e^{As}ds(A+BKC)x_0 \\ &= C\int_0^t e^{As}dsAx_0. \end{aligned} \tag{65}$$

Using $e(t) = Cx_0 - y(t)$ (as we took $n(t) = 0$), and $Cx_0 = 0$, we find that $\|e(t)\| = \|y(t)\|$ for all $t \in [0, t_1)$.

We again proceed by contradiction. Suppose that the system would have the local event-separation property, then for the initial state x_0, there exists $\tau^* > 0$ such that $t_1([x_0, 0], 0) \geqslant \tau^*$.

Since $x_0 \notin \ker O$, we can find for any $\tau^* > 0$ some t^c satisfying $0 < t^c \leqslant \tau^*$ such that

$$\left\|C\int_0^{t^c} e^{As}dsAx_0\right\| = \underbrace{\left\|\int_0^{t_c}\left(CA + sCA^2 + \frac{s^2CA^3}{2} + \dots\right)x_0ds\right\|}_{c} > 0. \tag{66}$$

Since t_1 satisfies

$$t_1 = \inf\{t > 0 \mid \|e(t)\| > P\|y(t)\|\} \tag{67}$$

with $P < 1$, obviously $t_1 < t^c \leqslant \tau^*$. This clearly contradicts $t_1 \geqslant \tau^*$, from which it follows that the system does not have the local event-separation property. ■

We will also demonstrate this in Example 2 in Section VI-B.

The theorem below states that when using mixed or absolute ETMs, the system (60) has the robust semi-global event-separation property.

Theorem V.3: Consider the closed-loop event-triggered control system (60) with $P \geqslant 0$ and $T > 0$. If there exists a continuously differentiable function V for the system (60a) satisfying (7), then the system (60) is ISpS and has the robust semi-global event-separation property with respect to w and n.

Proof: We prove the theorem for the absolute ETM with $P = 0$ and $T > 0$. Since the mixed ETM with $P > 0, T > 0$ generates events whenever $\|e^n\| \leqslant P\|y^s\| + T$ is violated and, since $P\|y^s\| + T \geqslant T$, the MIET generated by the mixed ETM is lower bounded by the MIET of the absolute ETM. The proof will be obtained by adapting the proof of Theorem IV.4, and we also use the sets $\mathcal{S}$ and $\mathcal{S}'$ as defined in the proof of Theorem IV.2. The ISpS property follows directly from Remark V.1.

To prove the robust semi-global event-separation property, consider that an event is triggered whenever $\|e - n\| \leqslant T$ is violated, and that $\|e - n\| \leqslant \|e\| + \|n\|$. Since $e^n(t_i)$ is reset to zero at event-times t_i, we have that $e(t_i) = n(t_i)$. If n satisfies $\|n\|_{\mathcal{L}_\infty} \leqslant \eta$ for some $\eta \in \mathbb{R}_{\geqslant 0}$, then the inter-event times are lower bounded by the time it takes for $\|e\|$ to grow from η to $T - \eta$, from which it immediately follows that to guarantee positive inter-event times, it is required that $\eta < T/2$.

Using (60) and (51), we find that

$$\dot{e} = -C(A + BKC)x - CBKe - Cw. \tag{68}$$

Since for all trajectories starting in $\mathcal{S}$, it holds almost everywhere that

$$\begin{aligned} \frac{d}{dt}\|e\| &= \frac{\dot{e}^\top e + e^\top \dot{e}}{2\sqrt{e^\top e}} \leqslant \frac{2\|\dot{e}\|\|e\|}{2\|e\|} = \|\dot{e}\| \\ &\leqslant \|C(A+BKC)\|x_{\max} + \|C\|\varepsilon + \|CBK\|\|e\| \end{aligned} \tag{69}$$

where $\|w\|_{\mathcal{L}_\infty} \leqslant \varepsilon$ and $\|x\| \leqslant x_{\max}$ for all $x \in \mathcal{S}'$, we conclude that $\|e(t_i + s)\| \leqslant \phi(s, \eta)$ for $s \in [t_i, t_{i+1})$, where $\phi(t, \phi_0)$ is the solution to $\dot{\phi} = \rho + L\phi$ satisfying $\phi(0, \phi_0) = \phi_0$. Here, $\rho = \|C(A + BKC)\|x_{\max} + \|C\|\varepsilon$ and $L = \|CBK\|$.

From this it follows that for the system (60) with $P \geqslant 0$, $T > 0$, $\varepsilon \geqslant 0$ and $T/2 > \eta \geqslant 0$, the inter-event times are lower bounded by $\tau([x, n(0)], [w, n]) \geqslant \tau^*$ for all $x \in \mathcal{S}$, all $\|w\|_{\mathcal{L}_\infty} \leqslant \varepsilon$ and all $\|n\|_{\mathcal{L}_\infty} \leqslant \eta$, with τ^* such that $\phi(\tau^*, \eta) = T - \eta$, from which it immediately follows that the system has the robust semi-global event-separation property. ■

Remark V.3: Note that to be able to guarantee a positive MIET, it is required that $T > 2\eta$. So, given a fixed T, the robust event-separation property is semi-global in x and w, but not semi-global in n, as for $\eta \geqslant T/2$ there exists $n \in \mathcal{L}_\infty^{n_y}$ with $\|n\|_{\mathcal{L}_\infty}$ such that $\tau([0, n(0)], [0, n]) = 0$.

Next we show that when using an absolute ETM or even a mixed ETM, the system (60) does not, in general, have the global event-separation property.

Theorem V.4: The ETC system (60) with $0 \leqslant P < 1, T > 0$ and $\mathrm{rank}(O) > \mathrm{rank}(C)$ does not have the global event-separation property.

Proof: Consider the case with $w = 0$ and $n = 0$. From (60) it follows that

$$t_1 = \inf\{t > 0 \mid \|e(t)\| > P\|y(t)\| + T\} \tag{70}$$

TABLE II
EVENT-SEPARATION PROPERTIES FOR CLOSED-LOOP ETC SYSTEMS (60), WITH $\mathrm{rank}(C) < \mathrm{rank}(O)$ AND $P < 1$

	event-separation property			robust event-separation property		
ETM	global	semi-global	local	global	semi-global	local
rel	×	×	×	×	×	×
mix	×	✓	✓	×	✓	✓
abs	×	✓	✓	×	✓	✓

and for $t \in [0, t_1)$ it holds that

$$\begin{aligned} y(t) &= Ce^{At}x_0 + C\int_0^t e^{As}ds BKCx_0 \\ &= Cx_0 + C\int_0^t e^{As}ds(A+BKC)x_0. \end{aligned} \tag{71}$$

Suppose that the system would have the global event-separation property, then there exists $\tau^* > 0$ such that $t_1([x_0, 0], 0) \geqslant \tau^*$ for any $x_0 \in \mathbb{R}^{n_x}$. Since $\mathrm{rank}(O) > \mathrm{rank}(C)$, there exists $v \in \ker(C) \setminus \ker(O)$, and for any $\tau^* > 0$ we can find some t^c satisfying $0 < t^c \leqslant \tau^*$ such that

$$\begin{aligned} &\left\| C\int_0^{t^c} e^{As}ds(A+BKC)v \right\| \\ &= \underbrace{\left\| \int_0^{t_c} \left(CA + sCA^2 + \frac{s^2CA^3}{2} + \ldots \right) vds \right\|}_{c} > 0. \end{aligned} \tag{72}$$

Using (71), (72), $e(t) = Cx_0 - y(t)$ and $Cv = 0$, we find that for $x_0 = \alpha v$,

$$\frac{\|e(t^c)\|}{P\|y(t^c)\| + T} = \frac{\alpha c}{P\alpha c + T} \tag{73}$$

from which follows that by choosing $\alpha > T/(c - Pc)$ we find that $t_1([\alpha v, 0], 0) < t^c \leqslant \tau^*$. This clearly contradicts $t_1 \geqslant \tau^*$, from which it follows that the system does not have the global event-separation property. ■

Summarizing the above results, the event-separation properties for closed-loop ETC systems (60) with $\mathrm{rank}(C) < \mathrm{rank}(O)$ and $P < 1$, using relative, mixed and absolute ETMs are shown in Table II.

VI. EXAMPLES

In this section we illustrate our findings above with three examples. Example 1 shows that for closed-loop ETC systems (22) using relative ETMs no MIET can be guaranteed in the presence of external disturbances, even though a global MIET is guaranteed in the absence of disturbances. Example 2 shows that for output-feedback ETC systems (60) a relative ETM can display Zeno behaviour even without disturbances, while a mixed ETM does not. Finally, Example 3 shows that ETC systems that enforce a positive lower bound on the inter-event times (as in, e.g., [8], [9], [12]) can also greatly benefit from using mixed ETMs instead of relative ETMs, in terms of the number of events that is generated.

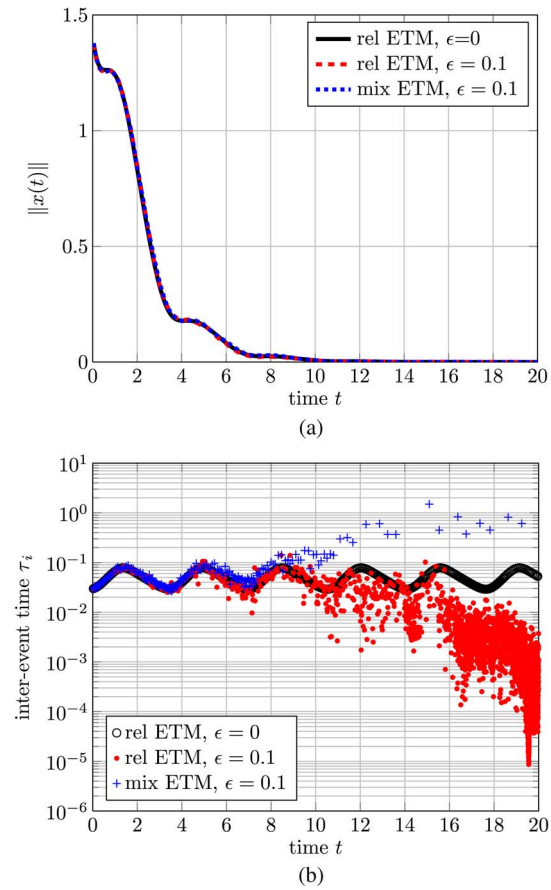

Fig. 3. Evolution of $\|x(t)\|$ (a) and inter-event times τ_i (b) for Example 1.

A. Example 1

As is shown above, closed-loop ETC systems (22), have no robust event-separation property when relative ETMs are used, but have the robust global event-separation property when a mixed ETM is used. We illustrate these findings by studying the example of [3] extended by the inclusion of disturbances w. This leads to the ETC system

$$\begin{cases} \dot{x} = Ax + BK(x+e) + w \\ e = x(t_i) - x, \text{ for } t \in [t_i, t_{i+1}) \\ t_{i+1} = \inf\{t > t_i \mid \|e(t)\| > P\|x(t)\| + T\} \end{cases} \tag{74}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K = [1 \quad -4] \tag{75}$$

and w is zero-mean white noise satisfying $\|w\|_{\mathcal{L}_\infty} \leqslant \varepsilon$. We compare two ETMs, the relative ETM with $P = 0.05$ and $T = 0$, and the mixed ETM with $P = 0.05$ and $T = 0.001$. The relative ETM renders the closed-loop ISS with respect to w, and the mixed ETM renders the closed-loop ISpS with respect to w.

The ETC system (74) is simulated using the relative ETM for the cases $\varepsilon = 0$ and $\varepsilon = 0.1$, and using the mixed ETM for the case $\varepsilon = 0.1$. Fig. 3 shows the evolution of $\|x(t)\|$ and the inter-event times τ_i. For all three cases, the control performance of the system is comparable. For the relative ETM we find,

using the results of [3], that $\inf_{x\in\mathbb{R}^{n_x}} \tau([x,0],0) = 0.028$. It can be seen in Fig. 3(b) that this lower bound is indeed satisfied when $\varepsilon = 0$. However, the inter-event times drop well below this bound for $\varepsilon = 0.1$, and many events are generated when $\|x\|$ becomes small, which is in accordance with Theorem IV.1. Clearly, in practice, where (small) disturbances are always present, the relative ETM does not save resources compared to a time-triggered approach. The mixed ETM, however, is robust to disturbances and generates only few events when the system is close to the origin, despite the presence of disturbances.

B. Example 2

For closed-loop ETC systems (60) with A not Hurwitz, $\mathrm{rank}(C) < \mathrm{rank}(O)$ and $P < 1$, we illustrate that when using relative ETMs even the local event-separation is lost, by considering the system

$$\begin{cases} \dot{x}(t) = (A + BKC)x(t) + BKe(t) \\ y(t) = Cx(t) \\ e(t) = y(t_i) - y(t), \text{ for } t \in [t_i, t_{i+1}) \end{cases} \tag{76}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^\top, \quad K = -3. \tag{77}$$

Additionally, we consider the relative ETM

$$t_{i+1} = \inf\{t > t_i \mid \|e(t)\| > P\|y(t)\|\} \tag{78}$$

and the mixed ETM

$$t_{i+1} = \inf\{t > t_i \mid \|e(t)\| > P\|y(t)\| + T\} \tag{79}$$

where $P = 0.5$ in both cases, and $T = 0.05$.

The evolution of y and the inter-event times τ_i are shown for both the relative and the mixed ETM in Fig. 4. As can be seen in this figure, when using the relative ETM there is an accumulation of events around $t = 2.2$ (when y passes through zero), while there are no problems when using the mixed ETM, which according to Theorem V.3 generates semi-globally separated events.

C. Example 3

As mentioned in the introduction, the robust global event-separation property can be easily attained by only evaluating the ETM after a predetermined lower time threshold has elapsed. These strategies fall into two categories. The strategies proposed in [8], [9], [14] only evaluate the ETM after a time $\tau_{\min}$ has elapsed after the previous execution time, which, combined with the ETMs proposed in this paper, leads to ETMs of the form

$$t_{i+1} = \inf\{t > t_i + \tau_{\min} \mid \|e^n(t)\| > P\|y^s(t)\| + T\}. \tag{80}$$

The strategies proposed in [12], [14]–[17] only evaluate the ETM at periodic intervals, which, combined with the ETMs proposed in this paper, leads to ETMs of the form

$$t_{i+1} = \inf\{t = t_i + kh \mid \|e^n(t)\| > P\|y^s(t)\| + T\} \tag{81}$$

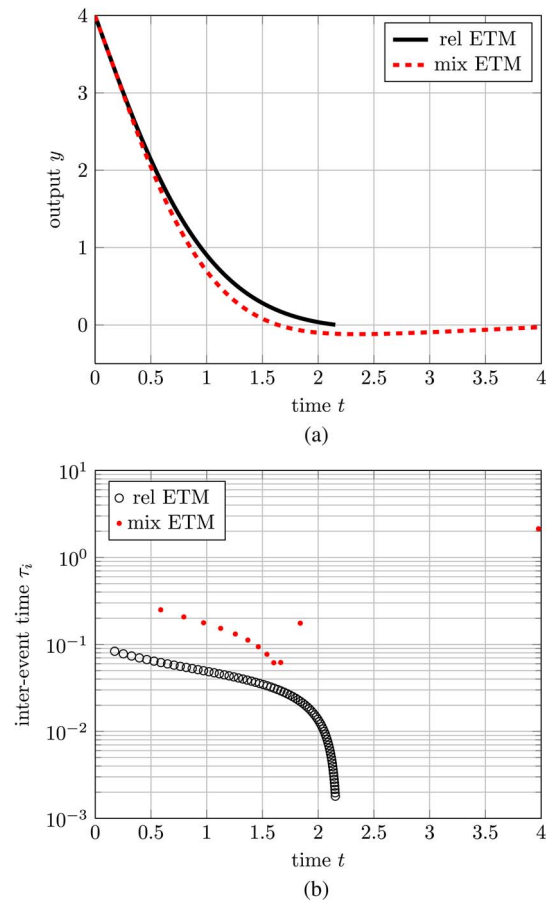

Fig. 4. Evolution of $y(t)$ (a) and inter-event times τ_i (b) for Example 2.

where $h > 0$ and $k \in \mathbb{N}_{>0}$. These strategies automatically guarantee a robust MIET of $\tau_{\min}$ and h, respectively, regardless of whether they are used in combination with a relative, absolute or mixed ETM. However, the proof of Theorem IV.1 indicates that a relative ETM generates very small inter-event times in the presence of disturbances w if w is large compared to x (or y when triggering is based on the output). For ETMs (80) and (81) this means that in the presence of arbitrarily small disturbances w (and/or measurement noise n), the system reduces to a time-triggered scheme when $T = 0$ and x approaches the origin. Thus, even for these types of ETMs, it can still make sense to choose a mixed ETM instead of a relative ETM.

We illustrate this by again studying the example of [3], now extended by the inclusion of both disturbances w and measurement noise n, and using an ETM of the form (80). The closed-loop ETC system then becomes

$$\begin{cases} \dot{x}(t) = Ax(t) + BK\left(y^s(t) + e^n(t)\right) + w(t) \\ y^s(t) = x(t) + n(t) \\ e^n(t) = y^s(t_i) - y^s(t), \text{ for } t \in [t_i, t_{i+1}) \\ t_{i+1} = \inf\{t > t_i + \tau_{\min} \mid \|e^n(t)\| > P\|y^s(t)\| + T\} \end{cases} \tag{82}$$

where we choose $\tau_{\min} = 0.025$, w is zero-mean white noise satisfying $\|w\|_{\mathcal{L}_\infty} \leqslant \varepsilon$, n is zero-mean white noise satisfying $\|n\|_{\mathcal{L}_\infty} \leqslant \eta$, and A, B and K are given by (75). Note that

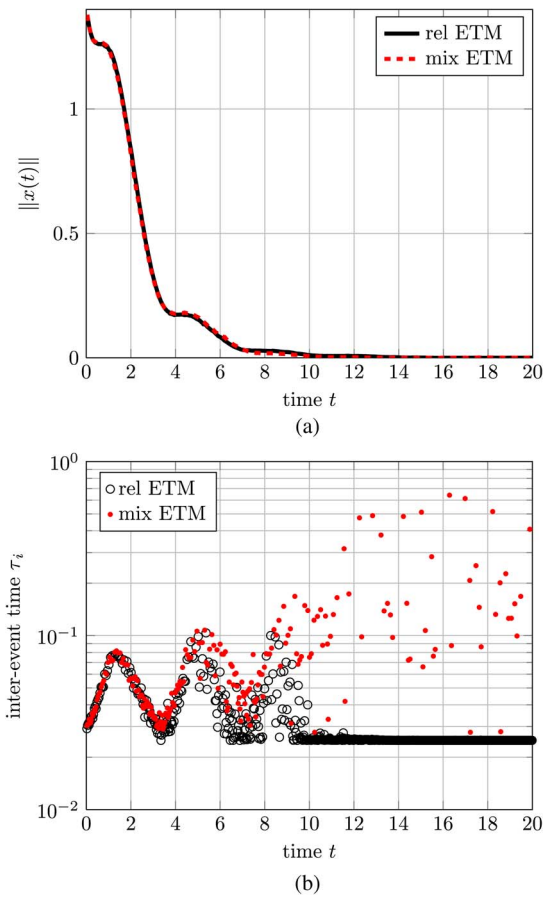

Fig. 5. Evolution of $\|x(t)\|$ (a) and inter-event times τ_i (b) for Example 3.

we still have a full state-feedback controller, but the measured states are contaminated with measurement noise n. We take $\varepsilon = 0.1$ and $\eta = 4 \cdot 10^{-4}$ and again compare two ETMs, the relative ETM with $P = 0.05$ and $T = 0$, and the mixed ETM with $P = 0.05$ and $T = 0.001$.

This example will show that a relative ETM generates many events (and acts almost like a time-triggered control system with sample time $\tau_{\min}$) when w and n are large compared to x, and that by using a mixed ETM similar performance can be achieved, while significantly reducing the number of events when the system is operating close to the origin (provided that $T > 2\eta$).

In Fig. 5 the evolution of $\|x(t)\|$ and the inter-event times τ_i are shown for the ETC system (82) using both the relative and mixed ETM. Here we can see that the ETC system using the mixed ETM performs comparable to the ETC system using the relative ETM. However, while the mixed ETM generated 230 events, the relative ETM generated 638 events and acts very much as a time-triggered system when x is close to the origin. Thus, in this case, the relative ETM uses more than twice as much bandwidth as the mixed ETM, while the closed-loop performance of both systems is very comparable.

Remark VI.1: Several other event-triggering strategies can be used to guarantee specific event-separation properties. For instance, combining a relative ETM with the strategies in [10], [14], [37], which generate no events while the state (or measured output) of the system lies in a compact set around the origin, could be one choice of interest, although even in this case it might still be useful to use a mixed ETM instead of a relative ETM.

VII. Conclusion

In this paper we have introduced the (robust) global, semi-global and local event-separation properties for impulsive systems, and have studied these event-separation properties for event-triggered closed-loop systems in the form of Architectures I and II, for relative, absolute and mixed event-triggering mechanisms. An overview of the results for linear systems was given in Tables I and II. It was found that even when a positive global minimum inter-event time can be guaranteed in the absence of disturbances and measurement noise, there might not exist a positive minimum inter-event time in the presence of arbitrarily small disturbances, indicating zero robustness of the computation/communication properties of particular ETC schemes. This observation issues a practical warning for the use of relative ETMs.

For the control systems and ETMs considered in this paper, mixed ETMs generally yield the most desirable event-separation properties. Relative ETMs generally generate many events in the presence of arbitrarily small disturbances when the system is operating close to the origin, and absolute ETMs, while generally robust to disturbances, generate many events when the system is operating far away from the origin. Mixed ETMs combine the advantages of both relative and absolute ETMs, i.e., robustness to disturbances, global minimum inter-event times in the state-feedback case and semi-global minimum inter-event times in the output-feedback case.

Clearly, there are many other ETC setups possible apart from Architectures I and II. However, because these architectures are relatively basic, they highlight the essence of the studied problems in a very clear manner. Besides, we believe that the results also hold for more complex architectures, including systems with dynamic feedback controllers. One such alternative (and more complex) ETC setup that was discussed in this paper, is to only evaluate the ETM after a predetermined lower time threshold has elapsed, which forms an obvious solution to attain the robust global event-separation property. As is shown in Example 3, such an ETC implementation might have little benefits over a time-triggered control system when using a relative ETM in the presence of (arbitrary small) disturbances or measurement noise. However, based on the insights provided in this paper, a new proposal for an output-based ETM for output feedback systems can be obtained. Indeed, a mixed ETM in combination with such a lower time threshold might be the best solution to reduce unnecessary use of computation and communication resources, while preserving the closed-loop control properties to a large extent.

Based on the findings in this paper, we conclude that, since every physical system is subject to external disturbances and measurement noise, the effect of these disturbances should not be disregarded in the analysis of both the control properties and the computation/communication properties of the system, as otherwise erroneous designs might be obtained.

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