Stability Analysis of Large-scale Networked Control Systems with Local Networks: A Hybrid Small-gain Approach

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ABSTRACT

In this paper we consider large-scale networked control systems (NCSs) with multiple communication networks connecting sensors, controllers and actuators. Using a recently developed small-gain theorem for general interconnections of hybrid systems, we are able to find to find a maximum allowable transmission interval (MATI) and a maximum allowable delay (MAD) for each individual network, such that input-to-state stability of the complete NCS is guaranteed.

Categories and Subject Descriptors

G.0 [**Mathematics of Computing**]: General; J.2 [**Physical Sciences and Engineering**]: Engineering

Keywords

Networked control systems; Hybrid systems; Small-gain theorem; Input-to-state stability; Protocols

1. INTRODUCTION

Networked control systems (NCSs) are control systems in which sensor and actuation data is transmitted via a shared (wired or wireless) communication network. This offers several advantages over conventional control systems, in which sensor and actuation data is transmitted using dedicated point-to-point wired links. These advantages include reduced installation costs, better maintainability and greater flexibility. However, next to these advantages, NCSs are adversely affected by several network effects, including varying transmission intervals and delays, quantization errors and packet losses. In addition, since the network is usually shared by multiple sensor, controller and actuator nodes, there is a need for a network protocol that governs the access of the nodes to the network in order to prevent packet losses. As a result, one needs to design the network and controller in such a way that the NCS is robust to these network-induced phenomena.

In most of the available literature on NCSs it is assumed that all sensor and actuation data is transmitted over *one* single communication network, see, e.g., [14–16, 24], and the required network performance ("quality-of-service", expressed in terms of, e.g., maximum allowable transmission interval (MATI), maximum allowable delay (MAD), required network reliability, etc.) to guarantee certain stability requirements is formulated globally.

However, it is not always reasonable to assume that there is one global communication network. For example, in the control of large-scale systems it is often more convenient and cost-efficient to use a local controller for each subsystem than one global controller for the whole system. Analogously, it is much more reasonable to close the local control loops over several local communication networks, instead of one global communication network. This leads to large-scale NCSs with local communication networks operating independently and asynchronously. Clearly, the required network parameters are to be formulated locally for each individual network.

Only very few papers consider NCSs in which sensors, controllers and actuators transmit via asynchronous communication links. Included in this small collection are [10] and [8].

In [10], we derive MATIs for each local network using a small-gain approach in order to guarantee closed-loop stability. In this paper we tackle a similar problem, but we take a different approach and make use of a small-gain theorem for general networks of interconnected (hybrid) systems which are input-to-state stable (ISS), developed in [3, 4], to derive network parameters for each local network, which guarantee ISS of the complete system. Since the stability analysis is based on the ISS gains of the interconnected systems, the analysis is based mostly on local information. This leads to a very systematic and modular setup, since when changing one subsystem, one only needs to find the ISS gains related to the changed subsystem in order to redo the stability analysis. Another advantage of the new approach is that it is much more general than the approach of [10], which makes it much easier to extend the results to more general control setups, to include other network-induced phenomena (e.g., quantization, transmission delays, packet losses, etc.), and even to include different types of communication networks,

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such as perhaps event-triggered networks. Already in this paper we consider a more general networked control setup than in [10], and, next to varying transmission intervals, we also consider varying transmission delays. Furthermore, the current analysis leads to a tradeoff between networks in terms of their required quality-of-service, and for each local network leads to a tradeoff between its network parameters. In this way, one can design an NCS on two levels. First, on a high level, one can tradeoff the quality-of-service requirements between networks that guarantee ISS of the complete NCS. Second, for each local communication network, one can tradeoff the various network parameters to arrive at the required network quality-of-service.

The recent paper [8] also bases its result on the general small-gain theorem of [3, 4] in order to guarantee stability of the complete system. However, the paper remains at a rather abstract mathematical level without connecting local quality-of-service parameters such as MATI and MAD to input-to-state stability (ISS) gains of the communication links in order to apply the small-gain theorems of [3, 4]. In this paper we derive these connections explicitly, which enables the use of these results in engineering practise. Besides, in [8] it is argued that extensions of the hybrid small-gain theorems of [3, 4] are needed, as data channels are assumed to be pre-globally stable (pre-GS). Here we show that for an important class of network protocols, and in the presence of varying delays and varying transmission intervals, such an extension is not needed and one can rely on more "classical" small-gain theorems as in [3, 4].

The paper is organized as follows. In Section 2 we present necessary preliminaries, and in Section 3 we describe the class of large-scale NCS considered in the paper and derive the resulting hybrid system models. We present the stability analysis in Section 4. How to find the important ISS gains is described explicitly in Section 5, and in Section 6 we demonstrate the usefulness and modularity of our results via a numerical example. Finally, possible extensions and future work are discussed in Section 7, and we end with our conclusions in Section 8.

1.1 Notation

For a vector $x \in \mathbb{R}^{n_x}$, we denote by $|x| := \sqrt{x^\top x}$ its 2-norm. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we denote by $\lambda_M(A)$ and $\lambda_m(A)$ its maximum and minimum eigenvalue, respectively. For a matrix $A \in \mathbb{R}^{n\times m}$, we denote by $|A| := \sqrt{\lambda_M(A^\top A)}$ its induced 2-norm. By I we denote the identity matrix of appropriate size. By $\mathbb{N}$ we denote the set of natural numbers excluding zero, i.e., $\mathbb{N} := \{1, 2, \dots\}$. For $N \in \mathbb{N}$ we define the set $\bar{N} := \{1, 2, \cdots, N\}$. With $\mathcal{L}_\infty^n$ we denote the space of all essentially bounded functions of dimension n, and for a signal $w : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_w}$, $w \in \mathcal{L}_\infty^{n_w}$, we denote by $\|w\| = \operatorname{ess\,sup}_{t\in\mathbb{R}_{\geqslant 0}} \|w(t)\|$ its $\mathcal{L}_\infty$-norm. A function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$, and a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and, in addition, $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{KL}$-function if for each fixed $t \geqslant 0$ the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function and for each fixed $s \geqslant 0$, $\beta(s, t)$ is decreasing in t and $\beta(s, t) \to 0$ as $t \to \infty$. For a square matrix Z we write $Z \succ 0$ if Z is positive definite, and $Z \prec 0$ if Z is negative definite. For vectors $x_i \in \mathbb{R}^{n_i}$, $i \in \bar{N}$, we denote by $(x_1, \cdots, x_N)$ the vector $\left[\begin{array}{c} x_1^\top \cdots x_N^\top \end{array}\right]^\top$. With $\bar{e}_i$ we denote the compound vector of all $e_j, j \in \bar{N} \setminus \{i\}$, i.e., $\bar{e}_i = (e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_N)$. For vectors $x, y \in \mathbb{R}^n$ we write $x \not\geq y$ if $x_i < y_i$ for at least one $i \in \bar{n}$. For a function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^n$, we use $f(t^+)$ to denote the limit $f(t^+) = \lim_{t\leftarrow s} f(s)$.

2. PRELIMINARIES

Consider a collection of N interconnected subsystems described by

$$\dot{x}_i = f_i(x, u), \; i \in \bar{N}, \tag{1}$$

where $x = (x_1, \cdots, x_N)$ is the state of the complete system, and u is an input. Note that the complete system of interconnected subsystems can be described by

$$\dot{x} = f(x, u), \tag{2}$$

where $f(x, u) = (f_1(x, u), f_2(x, u), \cdots, f_N(x, u))$.

Definition 2.1 ([18]). *The i-th subsystem of* (1) *is input-to-state stable (ISS) with respect to x_j, $j \in \bar{N} \setminus \{i\}$ and u, if there exist $\beta_i \in \mathcal{KL}$ and $\gamma_{ij}, \gamma_i^u \in \mathcal{K} \cup \{0\}$, with $\gamma_{ii} = 0$, such that for all $x_j \in \mathcal{L}_\infty^{n_{x_j}}$ and all $u \in \mathcal{L}_\infty^{n_u}$ the corresponding solution $x_i(t)$ starting in $x_i(0)$ satisfies*

$$|x_i(t)| \leqslant \beta_i(|x_i(0)|, t) + \sum_{j=1}^{N} \gamma_{ij}(\|x_j\|) + \gamma_i^u(\|u\|) \tag{3}$$

for all $x_i(0) \in \mathbb{R}^{n_{x_i}}$ and all $t \in \mathbb{R}_{\geqslant 0}$.

The functions γ_{ij}, γ_i^u are called (nonlinear) gains. Following [3], we can collect all γ_{ij} in a *gain matrix* $\Gamma = (\gamma_{ij})_{i,j=1}^N$, which defines the map $\Gamma : \mathbb{R}_{\geqslant 0}^N \to \mathbb{R}_{\geqslant 0}^N$ by

$$\Gamma(s) := \begin{bmatrix} \gamma_{11}(s_1) + \cdots + \gamma_{1N}(s_N) \\ \vdots \\ \gamma_{N1}(s_1) + \cdots + \gamma_{NN}(s_N) \end{bmatrix} \tag{4}$$

For functions $\alpha_i \in \mathcal{K}_\infty$, $i \in \bar{N}$, we define the diagonal operator $D : \mathbb{R}_{\geqslant 0}^N \to \mathbb{R}_{\geqslant 0}^N$ by

$$D(s) = \begin{bmatrix} (Id + \alpha_1)(s_1) \\ \vdots \\ (Id + \alpha_N)(s_N) \end{bmatrix}. \tag{5}$$

Definition 2.2 ([3]). *A gain matrix Γ satisfies the* strong small gain condition *if there exists an operator D as in* (5)*, such that for all $s \in \mathbb{R}_{\geqslant 0}^N, s \neq 0$ we have*

$$D \circ \Gamma(s) := D(\Gamma(s)) \not\geq s. \tag{6}$$

Lemma 2.1 ([3, Theorem 4.4]). *Consider the system* (1) *and suppose that each subsystem is ISS with respect to x_j, $j \in \bar{N} \setminus \{i\}$ and u, i.e.,* (3) *holds for all $i \in \bar{N}$. If Γ given by* (4) *satisfies the strong small gain condition* (6)*, then the system* (2) *is ISS with respect to u.*

Lemma 2.2 ([3, Corollary 4.9]). *Let $\Gamma \in (\mathcal{K} \cup \{0\})^{N\times N}$ be a gain matrix. If Γ is linear, i.e., $\Gamma(s) = Gs$, then Γ satisfies the strong small gain condition* (6) *if and only if the spectral radius of G is less than one.*

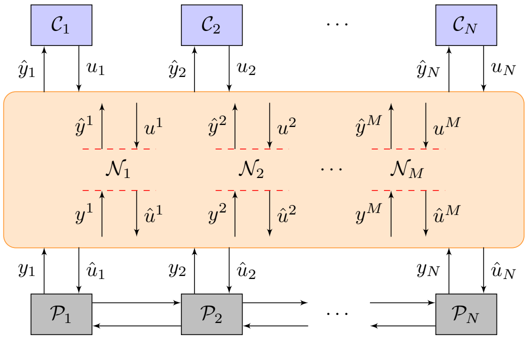


Figure 1: Networked control setup.

3. NETWORKED CONTROL SETUP

We consider the networked control setup shown in Figure 1, consisting of N physically coupled nonlinear continuous-time plants $\mathcal{P}_i$, $i \in \bar{N}$, controlled by N local controllers $\mathcal{C}_i$, $i \in \bar{N}$. The dynamics of the plants $\mathcal{P}_i$ are given by

$$\mathcal{P}_i : \begin{cases} \dot{x}_i^p &= f_i^p(x^p, \hat{u}_i, w), \\ y_i &= g_i^p(x_i^p, w), \end{cases} \tag{7}$$

where x_i^p is the state of plant $\mathcal{P}_i$, $x^p = (x_1^p, x_2^p, \cdots, x_N^p)$, y_i is the output of plant $\mathcal{P}_i$, $\hat{u}_i$ is the latest control input received by plant $\mathcal{P}_i$ and w is a disturbance signal. The dependence of f_i^p on the complete vector x^p describes the physical coupling of the plants. The controllers $\mathcal{C}_i$ are given by

$$\mathcal{C}_i : \begin{cases} \dot{x}_i^c &= f_i^c(x_i^c, \hat{y}_i, w), \\ u_i &= g_i^c(x_i^c, w), \end{cases} \tag{8}$$

where x_i^c is the state of controller $\mathcal{C}_i$, u_i its generated control input, and $\hat{y}_i$ is the latest output information received by controller $\mathcal{C}_i$, $i \in \bar{N}$.

The plant outputs y_i and control inputs u_i, $i \in \bar{N}$, are transmitted via M communication networks $\mathcal{N}_k$, $k \in \bar{M}$, which are operating independently from each other. We allow that multiple control loops are closed over one single network, or that a single control loop is closed over several different networks. This means that the output vector y_i might be transmitted via several networks. Each component in y_i and u_i, $i \in \bar{N}$, is transmitted via *exactly one* communication network, so redundant transmissions over multiple communication networks are not allowed.

We denote by y_i^k the part of y_i which is transmitted over network $\mathcal{N}_k$. Similarly, by u_i^k we denote the part of u_i which is transmitted over network $\mathcal{N}_k$. Thus, network $\mathcal{N}_k$, $k \in \bar{M}$, transmits the vector (y^k, u^k), where $y^k = (y_1^k, y_2^k, \cdots, y_N^k)$ and $u^k = (u_1^k, u_2^k, \cdots, u_N^k)$. We introduce the network-induced errors $e_k^y = \hat{y}^k - y^k$ and $e_k^u = \hat{u}^k - u^k$, which we group per network in $e_k = (e_k^y, e_k^u)$, $k \in \bar{M}$, and also into one vector $e = (e_1, e_2, \cdots, e_M)$. As a result, we can write

$$\hat{y}_i = y_i + Y_i e, \ i \in \bar{N}, \tag{9}$$

$$\hat{u}_i = u_i + U_i e, \ i \in \bar{N}, \tag{10}$$

where the matrices $Y_i \in \mathbb{R}^{n_{y_i} \times n_e}$ and $U_i \in \mathbb{R}^{n_{u_i} \times n_e}$ contain only ones and zeros. The interconnection of plants $\mathcal{P}_i$, $i \in \bar{N}$, and controllers $\mathcal{C}_i$, $i \in \bar{N}$, over the networks $\mathcal{N}_k$, $k \in \bar{M}$, leads to N controlled subsystems $\mathcal{G}_i$, given by the ordinary nonlinear dynamical systems

$$\mathcal{G}_i : \dot{x}_i = f_i\left(x, e, w\right), \ i \in \bar{N}, \tag{11}$$

where $x_i = (x_i^p, x_i^c)$, $x = (x_1, x_2, \cdots, x_N)$ and f_i is given by

$$f_i\left(x, e, w\right) = \left[\begin{array}{l} f_i^p(x^p, g_i^c(x_i^c, w) + U_i e, w) \\ f_i^c(x_i^c, g_i^p(x_i^p, w) + Y_i e, w) \end{array} \right]. \tag{12}$$

We assume that the controller $\mathcal{C}_i$ is designed such that the *isolated* subsystem $\mathcal{G}_i$ is stable in the absence of disturbances and network-induced errors, i.e., the system $\dot{x}_i = f_i\left(x_i, 0, 0\right)$ is stable in an appropriate sense.

Each network $\mathcal{N}_k$, $k \in \bar{M}$, consists of several nodes, with each node corresponding to one or more elements in $\left(y^k, u^k\right)$. At transmission times $t_{\kappa_k}^k$, the network protocol grants one of the nodes access to the network, where $\kappa_k \in \mathbb{N}$ is a counter keeping track of the transmission number of network $\mathcal{N}_k$, $k \in \bar{M}$. The node that is granted access collects the values of its corresponding entries in $\left(y^k(t_{\kappa_k}^k), u^k(t_{\kappa_k}^k)\right)$, which will be transmitted over the network. This information arrives after a delay of $\tau_{\kappa_k}^k$ time units at its destination, and results in an update of the corresponding values in $\left(\hat{y}^k(t_{\kappa_k}^k + \tau_{\kappa_k}^k), \hat{u}^k(t_{\kappa_k}^k + \tau_{\kappa_k}^k)\right)$, i.e., the updates of $\hat{y}^k$ and $\hat{u}^k$ satisfy

$$\hat{y}^k\left(\left(t_{\kappa_k}^k + \tau_{\kappa_k}^k\right)^+\right) = y^k\left(t_{\kappa_k}^k\right) + h_k^y\left(\kappa_k, e_k^y\left(t_{\kappa_k}^k\right)\right), \tag{13a}$$

$$\hat{u}^k\left(\left(t_{\kappa_k}^k + \tau_{\kappa_k}^k\right)^+\right) = u^k\left(t_{\kappa_k}^k\right) + h_k^u\left(\kappa_k, e_k^u\left(t_{\kappa_k}^k\right)\right). \tag{13b}$$

We will refer to $h_k = (h_k^y, h_k^u)$ as the network protocol corresponding to network $\mathcal{N}_k$, $k \in \bar{M}$.

Remark 3.1. *The communication networks $\mathcal{N}_k$, $k \in \bar{M}$, might be of a different nature, in the sense that they might employ different network protocols, or even different communication media. For example, some networks might be wired (using e.g., CAN or HART), while others might be wireless (using e.g., Zigbee, WirelessHART or WIFI).*

For each network $\mathcal{N}_k$, $k \in \bar{M}$, we assume that the time between two subsequent transmissions is upper bounded by a maximum allowable transmission interval (MATI) τ_{mati}^k, and that the delays are upper bounded by a maximum allowable delay (MAD) τ_{mad}^k.

Assumption 3.1. *The transmission times $t_{\kappa_k}^k$ satisfy*

$$0 \leqslant t_1^k \leqslant \tau_{mati}^k, \tag{14}$$

and

$$\delta_k \leqslant t_{\kappa_k+1}^k - t_{\kappa_k}^k \leqslant \tau_{mati}^k, \tag{15}$$

for all $\kappa_k \in \mathbb{N}$, where $\delta_k \in (0, \tau_{mati}^k]$, $k \in \bar{M}$, is arbitrary. Furthermore, the communication delays $\tau_{\kappa_k}^k$ satisfy

$$0 \leqslant \tau_{\kappa_k}^k \leqslant \max\{\tau_{mad}^k, t_{\kappa_k+1}^k - t_{\kappa_k}^k\}, \tag{16}$$

for all $\kappa_k \in \mathbb{N}$.

Furthermore, we assume that $\hat{y}^k$ and $\hat{u}^k$ are held constant in between updates, i.e., the networks operate in a zero-order hold (ZOH) fashion.

To model the network-induced error dynamics in a hybrid system formulation [9], we make use of the ideas in [14] and introduce the auxiliary variables $s_k \in \mathbb{R}^{n_{e_k}}$, $\kappa_k \in \mathbb{N}$, $\tau_k \in \mathbb{R}_{\geqslant 0}$ and $l_k \in \{0,1\}$, $k \in \bar{M}$. For each network $\mathcal{N}_k$, $k \in \bar{M}$, the variable s_k is used to store information, κ_k is a counter keeping track of the transmission number, τ_k is a timer to constrain both the transmission interval and the delay, and l_k is a Boolean variable keeping track of whether the next event is a transmission or an update. To be precise, when $l_k = 0$ the next event will be a transmission and when $l_k = 1$ the next event will be an update.

Further details on this modelling approach can be found in [14]. Because of space reasons we omit these details here and just state the resulting model.

By defining $\xi_k = (e_k, s_k, \tau_k, \kappa_k, l_k)$, the network-induced error dynamics for $\mathcal{N}_k$, $k \in \bar{M}$, is given by the hybrid model [9] consisting of the flow dynamics (17a) and the jump dynamics (17b),

$$\mathcal{E}_k : \begin{cases} \left.\begin{aligned} \dot{e}_k &= g_k(x,e,w) \\ \dot{s}_k &= 0 \\ \dot{\tau}_k &= 1 \\ \dot{\kappa}_k &= 0 \\ \dot{l}_k &= 0 \end{aligned}\right\} & \xi_k \in \mathcal{F}_k, \quad (17a) \\ \xi_k^+ = G_k(\xi_k), & \xi_k \in \mathcal{J}_k. \quad (17b) \end{cases}$$

In (17a) the flow set $\mathcal{F}_k$ is given by

$$\mathcal{F}_k := \Big\{ \xi_k \in \mathbb{R}^{2n_{e_k}+1} \times \mathbb{N} \times \{0,1\} \Big| (l_k = 0 \wedge \tau_k \in [0, \tau^k_{mati}]) \vee (l_k = 1 \wedge \tau_k \in [0, \tau^k_{mad}]) \Big\}, \quad (18)$$

and the jump set $\mathcal{J}_k$ in (17b) is given by

$$\mathcal{J}_k : \Big\{ \xi_k \in \mathbb{R}^{2n_{e_k}+1} \times \mathbb{N} \times \{0,1\} \Big| (l_k = 0 \wedge \tau_k \in [\delta_k, \tau^k_{mati}]) \vee (l_k = 1 \wedge \tau_k \in [\delta_k, \tau^k_{mad}]) \Big\}. \quad (19)$$

The reset equations in (17b) are given by the transmission resets (i.e., when $l_k = 0$)

$$G_k(e_k, s_k, \tau_k, \kappa_k, 0) = (e_k, h_k(\kappa_k, e_k) - e_k, 0, \kappa_k + 1, 1) \quad (20)$$

and the update resets (i.e., when $l_k = 1$)

$$G_k(e_k, s_k, \tau_k, \kappa_k, 1) = (s_k + e_k, -s_k - e_k, \tau_k, \kappa_k, 0). \quad (21)$$

Finally, $\delta_k \in (0, \tau^k_{mati}]$, $k \in \bar{M}$, is an arbitrary parameter used to prevent Zeno behavior. See [14] for more details on this modelling approach, including the definition of the function g_k.

The networked control setup of Figure 1 can thus be viewed as an interconnection of N controlled subsystems $\mathcal{G}_i$, $i \in \bar{N}$, given by the nonlinear model (11), and M network-induced error systems $\mathcal{E}_k$, $k \in \bar{M}$, given by the hybrid model (17), leading to the control setup shown in Figure 2.

Remark 3.2. *In this paper we could allow even more general networked control setups, but for the sake of clarity and readability we will not do so. Note that the presented control setup is still very general and contains as special cases the class of systems which employ only one global communication network, and the class of systems described in [10].*

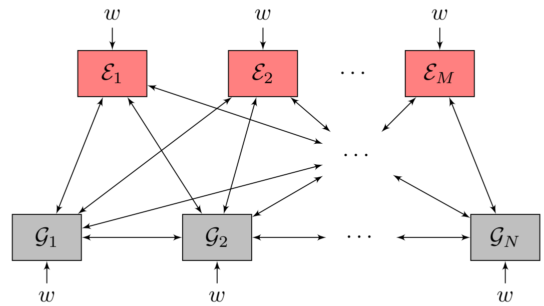


Figure 2: The networked control setup of Figure 1, viewed as an interconnection of N controlled subsystems $\mathcal{G}_i$, $i \in \bar{N}$, and M network-induced error systems $\mathcal{E}_k$, $k \in \bar{M}$.

Finally, to guarantee the existence of solutions of (11) and (17), we assume that for all $i \in \bar{N}$ and all $k \in \bar{M}$ it holds that f_i and g_k are continuous and h_k is locally bounded.

4. STABILITY ANALYSIS

By assuming ISS of all (closed-loop) subsystems (11) and all network-induced error systems (17), we can determine the stability of the complete system using the general small-gain theorem formulated in [3]. More specifically, we assume that each closed-loop subsystem (11) is ISS with respect to w, x_j, $j \in \bar{N} \setminus \{i\}$ and e_h, $h \in \bar{M}$, i.e., for each subsystem (11) the following condition holds.

Condition 4.1. *For all $x_j \in \mathcal{L}_\infty^{n_{x_j}}$, $j \in \bar{N} \setminus \{i\}$ all $e_h \in \mathcal{L}_\infty^{n_{e_h}}$, $h \in \bar{M}$, and all $w \in \mathcal{L}_\infty^{n_w}$, any corresponding solution to* (11) *satisfies*

$$|x_i(t)| \leqslant \beta_i(|x_i(0)|, t) + \sum_{j=1}^{N} \gamma_{ij}(\|x_j\|) + \sum_{h=1}^{M} \gamma^e_{ih}(\|e_h\|) + \gamma^w_i(\|w\|), \quad (22)$$

where $\beta_i \in \mathcal{KL}$ and $\gamma_{ij}, \gamma^e_{ih}, \gamma^w_i \in \mathcal{K} \cup \{0\}$, and $\gamma_{ii} \equiv 0$.

Additionally, we assume that each network-induced error system (17) is ISS with respect to w, x_j, $j \in \bar{N}$, and e_h, $h \in \bar{M} \setminus \{k\}$, and the corresponding gains can be scaled by a scalar ϕ_k, where ϕ_k is a function of the network parameters, i.e., for each network-induced error system (17) the following condition holds[1].

Condition 4.2. *Under Assumption 3.1 and for all $x_j \in \mathcal{L}_\infty^{n_{x_j}}$, $j \in \bar{N}$, all $e_h \in \mathcal{L}_\infty^{n_{e_h}}$, $h \in \bar{M} \setminus \{k\}$, and all $w \in \mathcal{L}_\infty^{n_w}$,*

[1]Strictly speaking, Condition 4.2 does not imply ISS of the error system (17), but rather (a strong version of) input-to-output stability. However, since we guarantee ISS bounds as in [3] for the state variables x and e, even though the auxiliary state variables $s_k, \tau_k, \kappa_k, l_k$, $k \in \bar{M}$, do not behave in an ISS manner, we still use the term ISS here with some abuse of terminology.

any corresponding solution to (17) *satisfies*

$$|e_k(t)| \leqslant \bar{\beta}_k(|e_k(0)|,t) + \phi_k \Bigg[\sum_{j=1}^{N} \bar{\gamma}_{kj}(\|x_j\|) + \sum_{h=1}^{M} \bar{\gamma}^e_{kh}(\|e_h\|) + \bar{\gamma}^w_k(\|w\|)\Bigg], \quad (23)$$

where ϕ_k *is a function of the network parameters* τ^k_{mati} *and* τ^k_{mad}, $\bar{\beta}_k \in \mathcal{KL}$, $\bar{\gamma}_{kj}, \bar{\gamma}^e_{kh}, \bar{\gamma}^w_k \in \mathcal{K} \cup \{0\}$ *and* $\bar{\gamma}^e_{kk} \equiv 0$.

The scaling functions ϕ_k, $k \in \bar{M}$, depend only on the network parameters τ^k_{mati} and τ^k_{mad}, and can therefore be regarded as a measure for the network performance (quality-of-service). How to find ϕ_k, $\bar{\beta}_k$, $\bar{\gamma}_{kj}$, $\bar{\gamma}^e_{kh}$ and $\bar{\gamma}^w_k$ will be discussed in detail in Section 5.2 for uniformly globally exponentially stable (UGES) protocols [15].

If Conditions 4.1 and 4.2 and Assumption 3.1 are satisfied, then given ϕ_k, $k \in \bar{M}$, we can verify the input-to-state stability of the complete system using the following lemma, which obviously follows from Lemma 2.1.

Lemma 4.1. *If Condition 4.1 holds for all* $i \in \bar{N}$, *Condition 4.2 holds for all* $k \in \bar{M}$, *and the gain matrix* Γ, *given by*

$$\Gamma = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1N} & \gamma^e_{11} & \cdots & \gamma^e_{1M} \\ \vdots & & \vdots & \vdots & & \vdots \\ \gamma_{N1} & \cdots & \gamma_{NN} & \gamma^e_{N1} & \cdots & \gamma^e_{NM} \\ \phi_1\bar{\gamma}_{11} & \cdots & \phi_1\bar{\gamma}_{1N} & \phi_1\bar{\gamma}^e_{11} & \cdots & \phi_1\bar{\gamma}^e_{1M} \\ \vdots & & \vdots & \vdots & & \vdots \\ \phi_M\bar{\gamma}_{M1} & \cdots & \phi_M\bar{\gamma}_{MN} & \phi_M\bar{\gamma}^e_{M1} & \cdots & \phi_M\bar{\gamma}^e_{MM} \end{bmatrix}$$

satisfies the strong small gain condition (6)*, then the complete system* (11)*,* (17) *is ISS with respect to* w.

Concluding, once we have all the ISS gains and scaling functions, we can check the input-to-state stability of the complete NCS. For linear systems Lemma 4.1 can be easily checked using Lemma 2.2. Using these tools, we can maximize the values of ϕ_k, $k \in \bar{M}$, in Pareto optimal sense, for which the condition $D \circ \Gamma(s) \not\geqslant s$ holds, see also [8]. Based on the values of ϕ_k, we can derive corresponding values for τ^k_{mati} and τ^k_{mad} using the results of Section 5, where the link between ϕ_k and τ^k_{mati} and τ^k_{mad} will be made explicit.

In short, one can find network parameters for each local network that guarantee ISS of the complete NCS by using the following procedure.

Procedure 4.1.

1. *Find ISS gains as in Condition 4.1 for each controlled subsystem* $\mathcal{G}_i$, $i \in \bar{N}$, *given by* (11).
2. *Determine the protocol for each network* $\mathcal{N}_k$, $k \in \bar{M}$.
3. *Find ISS gains as in Condition 4.2 for each network-induced error system* $\mathcal{E}_k$, $k \in \bar{M}$, *given by* (17).
4. *Using Lemma 4.1, find appropriate scaling functions* ϕ_k, $k \in \bar{M}$, *such that the complete NCS is ISS.*
5. *For each network* $\mathcal{N}_k$, $k \in \bar{M}$, *find appropriate network parameters (MATI and MAD), which lead to the required scaling* ϕ_k, *by using the results of Section 5.2.*

5. ISS GAINS

In this section we first show that Condition 4.1 holds for linear systems that are asymptotically stable in the absence of disturbances. Next, we derive the ISS gains and scaling factors of the network-induced error systems (17) for networks which employ UGES protocols.

5.1 Linear subsystems

Here we show that Condition 4.1 is a reasonable restriction for linear subsystems which are asymptotically stable in the absence of inputs x_j, $j \in \bar{N} \setminus \{i\}$, e_h, $h \in \bar{M}$ and w. To show this, consider the following linear subsystem $\mathcal{G}_i$

$$\dot{x}_i = A_i x_i + \sum_{\substack{j=1\\ j\neq i}}^{N} \Delta_{ij} x_j + \sum_{h=1}^{M} B_{ih} e_h + E_i w \qquad (24)$$

with A_i Hurwitz. Hence, we can find $P_i \succ 0$, $Q_i \succ 0$ such that $A_i^\top P_i + P_i A_i = -Q_i$ and $V_i(x_i) = x_i^\top P_i x_i$ satisfies

$$\underline{a}_i |x_i|^2 \leqslant V_i(x_i) \leqslant \bar{a}_i |x_i|^2$$

with $\underline{a}_i = \lambda_m(P_i)$ and $\bar{a}_i = \lambda_M(P_i)$. Moreover,

$$\dot{V}_i(x_i) \leqslant -c_i|x_i|^2 + 2|x_i|\Bigg(\sum_{\substack{j=1\\ j\neq i}}^{N} |P_i\Delta_{ij}||x_j| + \sum_{h=1}^{M} |P_i B_{ih}||e_h| + |P_i E_i||w|\Bigg)$$

with $c_i = \lambda_{min}(Q_i)$. So, $\dot{V}_i(x_i) \leqslant -\varepsilon c_i |x_i|^2$, $0<\varepsilon<1$, if

$$|x_i| \geqslant \left(\frac{2}{c_i(1-\varepsilon)}\right)\Bigg(\sum_{\substack{j=1\\ j\neq i}}^{N} |P_i\Delta_{ij}||x_j| + \sum_{h=1}^{M} |P_i B_{ih}||e_h| + |P_i E_i||w|\Bigg),$$

from which we derive that

$$|x_i(t)| \leqslant \beta_i(|x_i(0)|,t) + \sum_{j=1}^{N} \gamma_{ij}(\|x_j\|) + \sum_{h=1}^{M} \gamma^e_{ih}(\|e_h\|) + \gamma^w_i(\|w\|),$$

where

$$\beta(s,t) = \sqrt{\frac{\bar{a}_i}{\underline{a}_i}}\, e^{-\frac{\varepsilon c_i}{2\underline{a}_i}t} s,$$

$$\gamma^e_{ih}(s) = \left(\frac{2}{c_i(1-\varepsilon)}\right)|P_i B_{ih}| s,$$

$$\gamma_{ij}(s) = \left(\frac{2}{c_i(1-\varepsilon)}\right)|P_i \Delta_{ij}| s,\ j \neq i,$$

$$\gamma^w_i(s) = \left(\frac{2}{c_i(1-\varepsilon)}\right)|P_i E_i| s.$$

Thus, Assumption 4.1 is satisfied with linear gain functions $\gamma_{ij}, \gamma^e_{ih}, \gamma^w_i$.

5.2 Network-induced error systems with UGES protocols

In this section we derive the ISS gains and scaling factors in (23) of the network-induced error systems (17) under appropriate assumptions. We consider communication networks that suffer only from varying transmission intervals and varying delays, and which employ a uniformly globally exponentially stable (UGES) protocol [15]. This class of protocols includes the well-known Try-Once-Discard (TOD) and Round Robin (RR) protocols.

We need the following assumptions.

ASSUMPTION 5.1. *For each network-induced error system* (17)*, the protocol given by h_k is UGES, meaning that there exists a function $W_k : \mathbb{N} \times \mathbb{R}^{n_{e_k}} \to \mathbb{R}_{\geqslant 0}$ that is locally Lipschitz in its second argument such that for all $\kappa_k \in \mathbb{N}$ and all $e_k \in \mathbb{R}^{n_{e_k}}$*

$$\underline{a}_k^W |e_k| < W_k(\kappa_k, e_k) \leqslant \bar{a}_k^W |e_k| \tag{25}$$

$$W_k(\kappa_k + 1, h_k(\kappa_k, e_k)) \leqslant \lambda_k W_k(\kappa_k, e_k) \tag{26}$$

for constants $0 \leqslant \underline{a}_k^W \leqslant \bar{a}_k^W$ and $0 < \lambda_k < 1$.

ASSUMPTION 5.2. *For each network-induced error system* (17)*, the function W_k satisfies for all $\kappa_k \in \mathbb{N}$ and all $e_k \in \mathbb{R}^{n_{e_k}}$*

$$W_k(\kappa_k + 1, e_k) \leqslant \lambda_k^W W_k(\kappa_k, e_k) \tag{27}$$

for some constant $\lambda_k^W \geqslant 1$, and satisfies for almost all $e_k \in \mathbb{R}^{n_{e_k}}$ and all $\kappa_k \in \mathbb{N}$

$$\left| \frac{\partial W_k}{\partial e_k}(\kappa_k, e_k) \right| \leqslant M_k^W \tag{28}$$

for some $M_k^W > 0$. Additionally, for all $x_j \in \mathbb{R}^{n_{x_j}}$, $j \in \bar{N}$ and all $e_h \in \mathbb{R}^{n_{e_h}}$, $h \in \bar{M} \setminus \{k\}$, the growth of e_k is bounded during flow by

$$|g_k(x, e, w)| \leqslant M_k^e |e_k| + \sum_{j=1}^{N} \hat{\gamma}_{kj}(|x_j|) + \sum_{\substack{h=1\\h\neq k}}^{M} \hat{\gamma}_{kh}^e(|e_h|) + \hat{\gamma}_k^w(|w|), \tag{29}$$

where $M_k^e \geqslant 0$, and $\hat{\gamma}_{kj}, \hat{\gamma}_{kh}^e, \hat{\gamma}_k^w \in \mathcal{K} \cup \{0\}$ for all $j \in \bar{N}$ and all $h \in \bar{M} \setminus \{k\}$.

Note that for $x_j \in \mathcal{L}_\infty^{n_{x_j}}$, $j \in \bar{N}$, and $e_h \in \mathcal{L}_\infty^{n_{e_h}}$, $h \in \bar{M}$, the right-hand side of (29) is upper bounded by $M_k^e|e_k| + m_k(x, \bar{e}_k, w)$ with $m_k : \mathbb{R}_{\geqslant 0}^{N+M} \to \mathbb{R}_{\geqslant 0}$ given by

$$m_k(x, \bar{e}_k, w) := \sum_{j=1}^{N} \hat{\gamma}_{kj}(\|x_j\|) + \sum_{\substack{h=1\\h\neq k}}^{M} \hat{\gamma}_{kh}^e(\|e_h\|) + \hat{\gamma}_k^w(\|w\|). \tag{30}$$

From Assumptions 5.1 and 5.2 it follows that for almost all e_k and all κ_k

$$\left\langle \frac{\partial W_k(\kappa_k, e_k)}{\partial e_k}, g_k(x, e, w) \right\rangle \leqslant L_k W_k(\kappa_k, e_k) + M_k^W m_k(x, \bar{e}_k, w), \tag{31}$$

where

$$L_k = \frac{M_k^W M_k^e}{\underline{a}_k^W} \tag{32}$$

and $\langle \cdot, \cdot \rangle$ denotes the inner product. We are now ready to derive the scaling factors and ISS gains, which we will first do for the case without delays in Section 5.2.1, and then for the case with delays in Section 5.2.2.

5.2.1 *Varying transmission intervals*

First we state our results for communication networks without delays, i.e., $\tau_{\kappa_k}^k = 0$ for all $\kappa_k \in \mathbb{N}$. Since there are no delays, we do not need the variables s_k and l_k to fully describe the network-induced error dynamics, and (17) reduces to

$$\mathcal{E}_k : \begin{cases} \left.\begin{array}{l} \dot{e}_k = g_k(x, e, w) \\ \dot{\tau}_k = 1 \\ \dot{\kappa}_k = 0 \end{array}\right\}, & \tau_k \in [0, \tau_{mati}^k], \\ \left.\begin{array}{l} e_k^+ = h_k(\kappa_k, e_k) \\ {\tau_k}^+ = 0 \\ {\kappa_k}^+ = \kappa_k + 1 \end{array}\right\}, & \tau_k \in [\delta_k, \tau_{mati}^k]. \end{cases} \tag{33}$$

THEOREM 5.1. *If*

- *Assumptions 3.1, 5.1 and 5.2,*
- $\lambda_k e^{L_k \tau_{mati}^k} < 1$,
- $\tau_{\kappa_k}^k = 0$ *for all* $\kappa_k \in \mathbb{N}$,

hold, then e_k satisfying (33) *is ISS with respect to w, x_j, $j \in \bar{N}$, and e_h, $h \in \bar{M} \setminus \{k\}$, and Condition 4.2 holds with*

$$\phi_k(\tau_{mati}^k) = \frac{e^{L_k \tau_{mati}^k} - 1}{L_k(1 - \lambda_k e^{L_k \tau_{mati}^k})}, \tag{34}$$

and

$$\bar{\gamma}_k^w = \frac{M_k^W}{\underline{a}_k^W} \hat{\gamma}_k^w, \tag{35a}$$

$$\bar{\gamma}_{kj} = \frac{M_k^W}{\underline{a}_k^W} \hat{\gamma}_{kj}, \text{ for all } j \in \bar{N}, \tag{35b}$$

$$\bar{\gamma}_{kh}^e = \frac{M_k^W}{\underline{a}_k^W} \hat{\gamma}_{kh}^e, \text{ for all } h \in \bar{M} \setminus \{k\}. \tag{35c}$$

The proof can be obtained following similar arguments as in [15, 20] and can be found in [2].

5.2.2 *Varying transmission intervals and delays*

Next we also allow nonzero communication delays bounded by τ_{mad}^k and $t_{\kappa_k+1}^k - t_{\kappa_k}^k$ as in Assumption 3.1, next to time-varying transmission intervals. Thus the obtained network-induced error is described by (17). In order to guarantee ISS of the network-induced errors, we make the following assumption.

ASSUMPTION 5.3. *For each network-induced error system* (17) *there exist a function $\tilde{W}_k : \mathbb{N} \times \{0,1\} \times \mathbb{R}^{n_{e_k}} \times \mathbb{R}^{n_{e_k}} \to \mathbb{R}_{\geqslant 0}$ with $\tilde{W}_k(\kappa_k, l_k, \cdot, \cdot)$ locally Lipschitz for all $\kappa_k \in \mathbb{N}$ and $l_k \in \{0,1\}$, continuous functions $H_{l_k}^k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_e - n_{e_k}} \to \mathbb{R}_{\geqslant 0}$, functions $\underline{b}_k^W, \bar{b}_k^W \in \mathcal{K}_\infty$ and constants $0 \leqslant \lambda_k < 1$, $L_{l_k}^k \geqslant 0$ such that:*

- *for all $\kappa_k \in \mathbb{N}$ and all $s_k, e_k \in \mathbb{R}^{n_{e_k}}$ it holds that*

$$\tilde{W}_k(\kappa_k + 1, 1, e_k, h_k(\kappa_k, e_k) - e_k) \leqslant \lambda_k \tilde{W}_k(\kappa_k, 0, e_k, s_k) \tag{36}$$

$$\tilde{W}_k(\kappa_k, 0, s_k + e_k, -s_k - e_k) \leqslant \tilde{W}_k(\kappa_k, 1, e_k, s_k); \tag{37}$$

- *for all $\kappa_k \in \mathbb{N}$, $l_k \in \{0,1\}$ and all $s_k, e_k \in \mathbb{R}^{n_{e_k}}$ it holds that*

$$\underline{b}_k^W(|(e_k,s_k)|) \leqslant \tilde{W}_k(\kappa_k,l_k,e_k,s_k) \leqslant \bar{b}_k^W(|(e_k,s_k)|); \quad (38)$$

- *for all $\kappa_k \in \mathbb{N}$, $l_k \in \{0,1\}$, $s_k \in \mathbb{R}^{n_{e_k}}$, $x \in \mathbb{R}^{n_x}$ and almost all $e \in \mathbb{R}^{n_e}$ it holds that*

$$\left\langle \frac{\partial \tilde{W}_k(\kappa_k,l_k,e_k,s_k)}{\partial e_k}, g_k(x,e,w) \right\rangle \leqslant L_{l_k}^k \tilde{W}_k(\kappa_k,l_k,e_k,s_k) + H_{l_k}^k(x,w,\bar{e}_k). \quad (39)$$

Theorem 5.2. *If Assumptions 3.1, 5.1 and 5.2 hold, then the function $\tilde{W}_k$ given by*

$$\tilde{W}_k(\kappa_k,l_k,e_k,s_k) := \begin{cases} \max\Big\{W_k(\kappa_k,e_k), \\ \quad W_k(\kappa_k,e_k+s_k)\Big\}, & l_k=0, \\ \max\Big\{\frac{\lambda_k}{\lambda_k^W} W_k(\kappa_k,e_k), \\ \quad W_k(\kappa_k,e_k+s_k)\Big\}, & l_k=1, \end{cases} \quad (40)$$

satisfies Assumption 5.3 with

$$H_0^k(x,\bar{e}_k,w) = H_1^k(x,\bar{e}_k,w) = M_W^k m^k(x,\bar{e}_k,w), \quad (41)$$

$$L_0^k = \frac{M_k^W M_k^e}{\underline{a}_k^W} \text{ and } L_1^k = \frac{\lambda_k^W M_k^W M_k^e}{\lambda_k \underline{a}_k^W}, \quad (42)$$

$\underline{b}_k^W(r) = \underline{b}_k^W r$ and $\bar{b}_k^W(r) = \bar{b}_k^W r$, where $\underline{b}_k^W$ and $\bar{b}_k^W$ are some positive constants.

If, in addition, τ_{mati}^k and τ_{mad}^k are such that

$$\lambda_k e^{L_0^k(\tau_{mati}^k - \tau_{mad}^k)} e^{L_1^k \tau_{mad}^k} < 1, \quad (43)$$

then Condition 4.2 holds with

$$\bar{\gamma}_k^w = \frac{\lambda_k^W}{\lambda_k} \frac{M_k^W}{\underline{a}_k^W} \hat{\gamma}_k^w, \quad (44a)$$

$$\bar{\gamma}_{kj} = \frac{\lambda_k^W}{\lambda_k} \frac{M_k^W}{\underline{a}_k^W} \hat{\gamma}_{kj}, \text{ for all } j \in \bar{N}, \quad (44b)$$

$$\bar{\gamma}_{kh}^e = \frac{\lambda_k^W}{\lambda_k} \frac{M_k^W}{\underline{a}_k^W} \hat{\gamma}_{kh}^e, \text{ for all } h \in \bar{M} \setminus \{k\}, \quad (44c)$$

and

$$\phi_k(\tau_{mati}^k, \tau_{mad}^k) = \frac{L_0^k F_0^k F_1^k + (L_1^k - L_0^k) F_0^k - L_1^k}{L_0^k L_1^k \left(1 - \lambda_k F_0^k F_1^k\right)}, \quad (45)$$

where

$$F_0^k = e^{L_0^k(\tau_{mati}^k - \tau_{mad}^k)} \quad (46)$$

and

$$F_1^k = e^{L_1^k \tau_{mad}^k}. \quad (47)$$

The proof can be found in [2].

Remark 5.3. *Note that $\phi_k(\tau_{mati}^k, 0)$ in (45) is larger than $\phi_k(\tau_{mati}^k)$ in (34), and that the gains $\gamma_{kj}, \gamma_{kh}^e$ of (44) are larger than the gains $\gamma_{kj}, \gamma_{kh}^e$ of (35). In other words, by introducing transmission delays in the analysis, the estimates of the scaling factors and ISS gains of the network-induced errors immediately become more conservative.*

To show how $\phi_k(\tau_{mati}^k, \tau_{mad}^k)$ depends on τ_{mati}^k and τ_{mad}^k, we show in Figure 3 which combinations of τ_{mati}^k and τ_{mad}^k lead to a certain value of ϕ_k, for a communication network with $\lambda_k^W = 1$, $\lambda_k = \sqrt{1/2}$, $L_0^k = 50$ and $L_1^k = \lambda_k^W L_0^k / \lambda_k$. From this figure it is apparent that there is a tradeoff of MATI versus MAD, and that the size of the tradeoff depends on the value of ϕ_k.

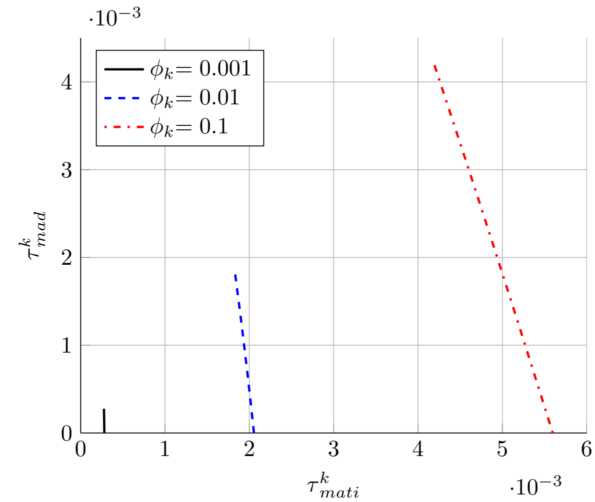


Figure 3: Tradeoff of MATI versus MAD for a network with $\lambda_k^W = 1$, $\lambda_k = \sqrt{1/2}$, $L_0^k = 50$, $L_1^k = \lambda_k^W L_0^k / \lambda_k$.

6. NUMERICAL EXAMPLE

To illustrate the findings of the paper, we consider the problem of stabilizing two coupled cart-pendulum systems $\mathcal{P}_i$, $i = 1,2$, with the pendula in their (unstable) upright equilibrium, see Figure 4. This is the same example as in [10]. Each subsystem consists of a moving support (cart) with mass M_i, a rigid massless beam of length l_i, and a point mass m_i attached to the end of the beam, $i = 1,2$. The pendula are coupled via a linear spring with stiffness k. The system is actuated via input forces $\hat{u}_i$, $i = 1,2$.

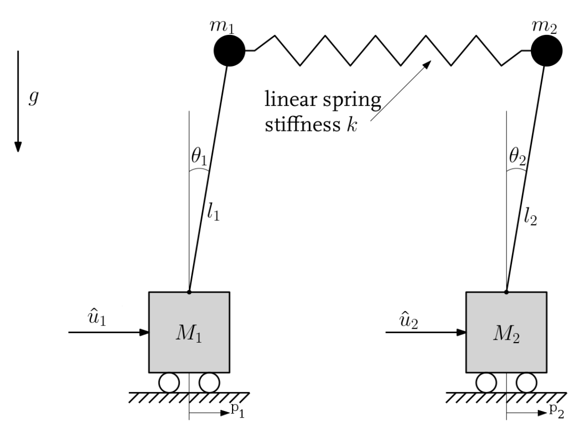


Figure 4: Schematic of the two coupled cart-pendulum systems.

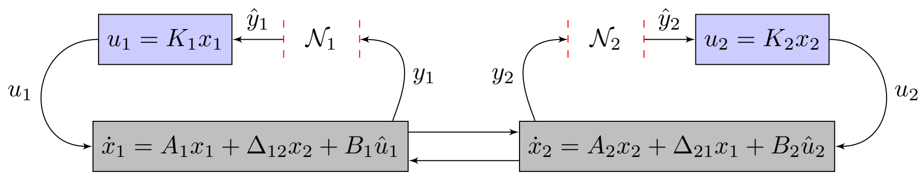


Figure 5: Networked control setup of the two coupled cart-pendulum systems.

Linearizing the pendula around their unstable upright equilibria, we find,

$$\dot{x}_1 = A_1x_1 + \Delta_{12}x_2 + B_1\hat{u}_1, \tag{48}$$
$$\dot{x}_2 = A_2x_2 + \Delta_{21}x_1 + B_2\hat{u}_2, \tag{49}$$

where $x_i = (p_i, \dot{p}_i, \theta_i, \dot{\theta}_i)$ is the state variable of subsystem $\mathcal{P}_i$, $i = 1, 2$. The numerical values of all parameters and matrices that are used in this example can be found in [10].

Each subsystem has its own decentralized controller collocated with the actuator (and thus $\hat{u}_i = u_i$, $i = 1, 2$), given by $u_i = K_i\hat{x}_i$, where K_i are such that the eigenvalues of $A_1 + B_1K_1$ are $\{-1, -2, -3, -4\}$, and the eigenvalues of $A_2 + B_2K_2$ are $\{-2, -3, -4, -5\}$.

Furthermore, each subsystem employs its own local network as in Figure 5, over which the state values $y_i = x_i$ are transmitted to the controller. The flow dynamics of the closed-loop hybrid model can then be written in the form

$$\dot{x}_1 = (A_1 + B_1K_1)x_1 + \Delta_{12}x_2 + B_1K_1e_1, \tag{50a}$$
$$\dot{x}_2 = (A_2 + B_2K_2)x_2 + \Delta_{21}x_1 + B_2K_2e_2, \tag{50b}$$
$$\dot{e}_1 = -\dot{x}_1, \tag{50c}$$
$$\dot{e}_2 = -\dot{x}_2 \tag{50d}$$

in which $e_k = \hat{x}_k - x_k$, $k = 1, 2$ are the network-induced (state) errors.

For subsystems (50a) and (50b) we derive ISS gains γ_{12}, γ^e_{11}, γ_{21} and γ^e_{22} satisfying Condition 4.1 by using the procedure described in Section 5.1 and local Lyapunov functions $V_i(x_i) = x_i^\top P_i x_i$ satisfying

$$(A_i + B_iK_i)^\top P_i + P_i(A_i + B_iK_i) = -3I. \tag{51}$$

We assume that the networks suffer from varying transmission intervals and varying delays, satisfying Assumption 3.1, but not from other network effects, such as packet dropouts and quantization. Furthermore, we assume that both networks employ the TOD protocol, which satisfies Assumption 5.1 with $W_k(\kappa_k, e_k) = |e_k|$, $\underline{a}^W_k = \bar{a}^W_k = 1$, and $\lambda_k = \sqrt{(n_k - 1)/n_k}$, where n_k is the number of nodes in network $\mathcal{N}_k$ [15]. Then, using (50), we find that Assumption 5.2 is satisfied with

$$\lambda^W_k = 1, \tag{52a}$$
$$M^W_k = 1, \tag{52b}$$
$$M^e_k = |B_kK_k|, \tag{52c}$$
$$\hat{\gamma}_{kk}(|x_k|) = |A_k + B_kK_k||x_k|, \tag{52d}$$
$$\hat{\gamma}_{kj}(|x_j|) = |\Delta_{kj}||x_j|,\ j \neq k, \tag{52e}$$
$$\hat{\gamma}^e_{kh}(|e_h|) = 0,\ h \neq k. \tag{52f}$$

The gain matrix now becomes

$$\Gamma = \begin{bmatrix} 0 & \gamma_{12} & \gamma^e_{11} & 0 \\ \gamma_{21} & 0 & 0 & \gamma^e_{22} \\ \phi_1\bar{\gamma}_{11} & \phi_1\bar{\gamma}_{12} & 0 & 0 \\ \phi_2\bar{\gamma}_{21} & \phi_2\bar{\gamma}_{22} & 0 & 0 \end{bmatrix}. \tag{53}$$

We can now use Procedure 4.1 and the results of Section 5.2 to find suitable values for $\tau^k_{mati}, \tau^k_{mad}$, $k = 1, 2$. Since in this case $\Gamma(s)$ is linear we use Lemma 2.2 to maximize ϕ_2 for each given scalar ϕ_1 thus finding Pareto optimal values for ϕ_1 and ϕ_2 such that $\Gamma(s)$ satisfies the strong small gain condition,

First we consider the delay-free case. Using Theorem 5.1, we find a tradeoff of τ^1_{mati} versus τ^2_{mati}, shown in Figure 6. From Figure 6 we can see that by decreasing τ^1_{mati} we can

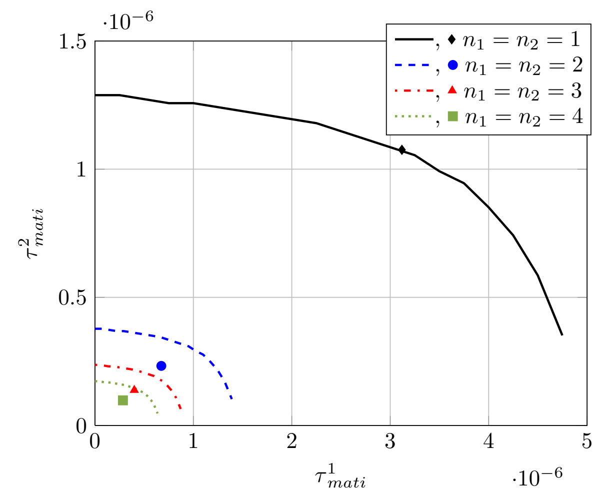
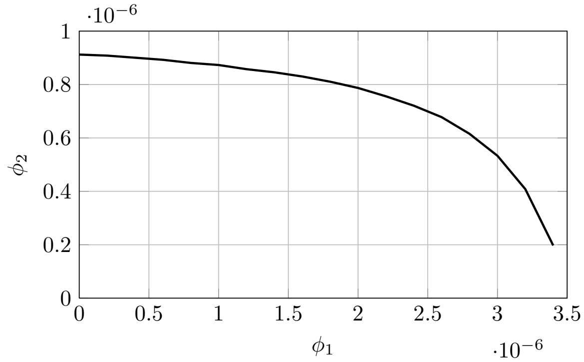


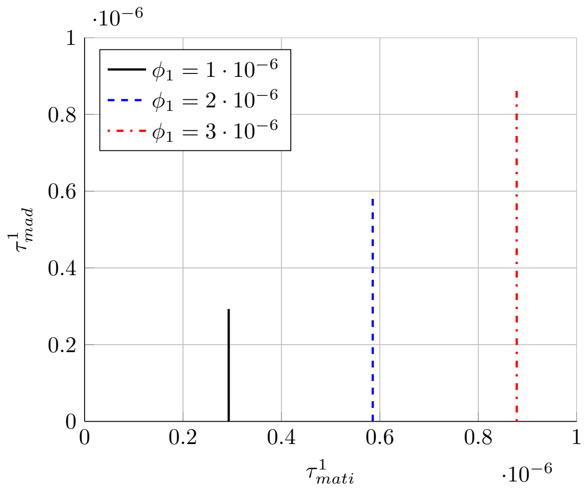
Figure 6: Tradeoff of τ^1_{mati} versus τ^2_{mati} for the delay-free case, for varying number of nodes per network n_i. The MATIs found in [10] are shown with marks for comparison.

allow a larger τ^2_{mati}, and vice versa. This is a very useful result, since it implies that we can decrease the required performance of one network by improving the other. The figure also shows the MATIs found in [10], and we can see that the approach of Section 5.2.1 is, at least for this example, no more conservative than the approach of [10], while we gain the freedom of trading MATIs.

Next we also allow delays, so we have to use the results of Section 5.2.2 for finding $\tau^k_{mati}, \tau^k_{mad}$, $i = 1, 2$. Again maximizing ϕ_2 for each ϕ_1, we find a tradeoff curve of ϕ_1 versus ϕ_2, shown in Figure 7(a) for $n_1 = n_2 = 2$. For each network we can now trade τ^i_{mati} versus τ^i_{mad} to arrive at a given value of ϕ_k, $k = 1, 2$. For network $\mathcal{N}_1$ this tradeoff is shown in Figure 7(b), again for $n_1 = 2$. From this figure we can



(a) Tradeoff ϕ_1 versus ϕ_2



(b) Tradeoff τ^1_{mati} vs τ^1_{mad} for given ϕ_1

Figure 7: The tradeoff curve of ϕ_1 versus ϕ_2, and the tradeoff of τ^i_{mati} versus τ^i_{mad} for network $\mathcal{N}_1$, for the case with delays and with $n_1 = n_2 = 2$.

see that for this example, the dependence of ϕ_1 on τ^1_{mad} is negligible, and that we can allow any delay bounded by the transmission interval, without any significant change in τ^1_{mati}.

7. POSSIBLE EXTENSIONS AND FUTURE WORK

The presented stability analysis is of course not limited to the communication networks described in this paper. An important observation from [8] is that ISS of the communication networks is sufficient, but not necessary to guarantee ISS of the plant and controller states of the interconnected system. For this, it is sufficient that the network-induced errors are globally stable (GS). Certainly, non-UGES protocols can be allowed, as long the resulting network-induced error systems are GS, see, e.g., [15].

Furthermore, instead of only considering time-triggered networks (characterized by MATIs), also event-triggered networks can be allowed. In an event-triggered network, the transmission times are not determined (or restricted) based on time, but based on the actual information content of the involved signals, see, e.g., [12] for a recent overview. This guarantees that transmissions only occur when necessary, reducing the load on the network. For such networks it is trivial to find the ISS gains of the network-induced error system, since these gains can be directly enforced by the choice of event-triggering mechanism (ETM), see, e.g., [21]. Interesting and useful results in this direction have already been published in [6], where decentralized ETMs are derived using the general small-gain theorem of [4]. Furthermore, in [17] event-triggered control is studied in the setting of networks using UGES protocols. The key problem in event-triggered control is preventing Zeno behavior, i.e., it is necessary that a nonzero minimum transmission interval is guaranteed, and that this minimum transmission interval is robust to disturbances [1]. Most works in the literature guaranteeing this property only consider setups in which events are generated based on the full state of the system, however, when only using output information it is in general an open question how to prevent Zeno behavior. Possible solutions would be to resort to periodic event-triggered control (PETC) systems [11], in which the event-triggering mechanism is not evaluated continuously, but rather periodically, or to evaluate the ETM only after a predetermined lower time threshold has elapsed [5, 13, 22, 23]. These approaches automatically guarantee a positive minimum transmission interval. Such event-triggered communication networks can then directly be incorporated in the above stability analysis, without any significant changes. Another option would be to add an absolute threshold to the triggering condition as in [7], for which a semi-global lower bound on the inter-event times can be guaranteed. This would lead to input-to-state practically stable (ISpS) systems, which also fit into the framework proposed in this paper.

Furthermore, we aim to extend the current results to packet losses (with [19] being an interesting starting point), delays greater than the transmission interval and quantization effects. Finally, we hope to reduce the conservatism of the current results, to get larger MATIs and MADs guaranteeing stability of the complete system.

8. CONCLUSIONS

We used a recently introduced small-gain theorem for general networks to analyze the stability of a large class of large-scale networked control systems with multiple communication networks.

The class of NCSs we considered can be modelled as an interconnection of ordinary nonlinear systems (the controlled subsystems) and hybrid systems (the communication networks). We assumed that all controlled subsystems are ISS, and we showed that networks that employ UGES protocols give rise to ISS error systems with ISS gains that can be scaled by a function ϕ, which is a function of the network parameters (MATI and MAD). Then, by using the general small-gain theorem, we can optimize the scaling functions ϕ_k corresponding to networks $\mathcal{N}_k$, $k \in \bar{M}$ in a Pareto optimal sense. This leads to a tradeoff between the scaling functions. Once the scaling function of a network is determined, one can tradeoff its MATI versus its MAD to meet the required value of ϕ.

Based on this analysis we find that by increasing the performance of one network (where the performance of a network is characterized by its scaling function ϕ), we can decrease the required performance of other networks. Furthermore, by decreasing the MATI of a network, we can allow a larger MAD, while not decreasing the network performance.

Since the stability analysis is only based on the ISS gains of the subsystems and the network-induced error systems, it is very easy to use different types of communication networks in one NCS. For example, it is possible to mix both time-triggered and event-triggered networks, and also to mix different UGES or even non-UGES protocols, as discussed in Section 7. Furthermore, when changing, removing or adding a communication network, one only needs to find the ISS gains related to the changed or added network in order to redo the stability analysis.

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10. REFERENCES

[1] D.P. Borgers and W.P.M.H. Heemels, *On minimum inter-event times in event-triggered control*, Decision and Control (CDC), 2013 52nd IEEE Conference on, 2013, to appear.

[2] ———, *Stability analysis of large-scale networked control systems with local networks: A hybrid small-gain approach*, Tech. report, Eindhoven University of Technology, 2014, Technical report, available online at http://www.heemels.tue.nl/content/papers/BorHee_TUE14a.pdf.

[3] S.N. Dashkovskiy, B.S. Rüffer, and F.R. Wirth, *An ISS small gain theorem for general networks*, Mathematics of Control, Signals, and Systems **19** (2007), no. 2, 93–122.

[4] ———, *Small gain theorems for large scale systems and construction of ISS lyapunov functions*, SIAM Journal on Control and Optimization **48** (2010), no. 6, 4089–4118.

[5] C. De Persis, R. Sailer, and F. Wirth, *On inter-sampling times for event-triggered large-scale linear systems*, Decision and Control (CDC), 2013 52nd IEEE Conference on, 2013, to appear.

[6] C. De Persis, R. Sailer, and H. Wirth, *Parsimonious event-triggered distributed control: A zeno free approach*, Automatica **49** (2013), no. 7, 2116–2124.

[7] M.C.F. Donkers and W.P.M.H. Heemels, *Output-based event-triggered control with guaranteed $\mathcal{L}_\infty$-gain and improved and decentralized event-triggering*, Automatic Control, IEEE Transactions on **57** (2012), no. 6, 1362–1376.

[8] R. Geiselhart and F.R. Wirth, *On maximal gains guaranteeing a small-gain condition*, Submitted for publication.

[9] R. Goebel, R.G. Sanfelice, and A.R. Teel, *Hybrid dynamical systems: Modeling, stability, and robustness*, Princeton University Press, 2012.

[10] W.P.M.H. Heemels, D.P. Borgers, N. Van de Wouw, D. Nešić, and A.R. Teel, *Stability analysis of nonlinear networked control systems with asynchronous communication: A small-gain approach*, Decision and Control (CDC), 2013 52nd IEEE Conference on, 2013, to appear, available online at http://www.dct.tue.nl/New/Heemels/Conferences/HeeBor_CDC_13.pdf.

[11] W.P.M.H. Heemels, M.C.F. Donkers, and A.R. Teel, *Periodic event-triggered control for linear systems*, Automatic Control, IEEE Transactions on **58** (2013), no. 4, 847–861.

[12] W.P.M.H. Heemels, K.H. Johansson, and P. Tabuada, *An introduction to event-triggered and self-triggered control*, Decision and Control (CDC), 2012 IEEE 51st Annual Conference on, 2012, pp. 3270–3285.

[13] W.P.M.H. Heemels, J.H. Sandee, and P.P.J. Van Den Bosch, *Analysis of event-driven controllers for linear systems*, International journal of control **81** (2008), no. 4, 571–590.

[14] W.P.M.H. Heemels, A.R. Teel, N. Van de Wouw, and D. Nešić, *Networked control systems with communication constraints: Tradeoffs between transmission intervals, delays and performance*, Automatic Control, IEEE Transactions on **55** (2010), no. 8, 1781–1796.

[15] D. Nešić and A.R. Teel, *Input-output stability properties of networked control systems*, Automatic Control, IEEE Transactions on **49** (2004), no. 10, 1650–1667.

[16] ———, *Input-to-state stability of networked control systems*, Automatica **40** (2004), no. 12, 2121–2128.

[17] R. Postoyan, P. Tabuada, D. Nešić, and A. Anta, *Event-triggered and self-triggered stabilization of distributed networked control systems*, Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on, 2011, pp. 2565–2570.

[18] E. D. Sontag and Y. Wang, *On characterizations of the input-to-state stability property*, Systems and Control Letters **24** (1995), no. 5, 351–359.

[19] M. Tabbara and D. Nešić, *Input-output stability of networked control systems with stochastic protocols and channels*, Automatic Control, IEEE Transactions on **53** (2008), no. 5, 1160–1175.

[20] ———, *Input-output stability with input-to-state stable protocols for quantized and networked control systems*, Decision and Control (CDC), 2008 47th IEEE Conference on, 2008, pp. 2680–2685.

[21] P. Tabuada, *Event-triggered real-time scheduling of stabilizing control tasks*, Automatic Control, IEEE Transactions on **52** (2007), no. 9, 1680–1685.

[22] P. Tallapragada and N. Chopra, *Event-triggered decentralized dynamic output feedback control for LTI systems*, IFAC Workshop on Distributed Estimation and Control in Networked Systems, 2012, pp. 31–36.

[23] ———, *Event-triggered dynamic output feedback control for LTI systems*, Decision and Control (CDC), 2012 IEEE 51st Annual Conference on, 2012, pp. 6597–6602.

[24] G.C. Walsh, O. Beldiman, and L.G. Bushnell, *Asymptotic behavior of nonlinear networked control systems*, Automatic Control, IEEE Transactions on **46** (2001), no. 7, 1093–1097.