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On minimum inter-event times in event-triggered control

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Abstract— **In this paper we study fundamental properties of minimum inter-event times in event-triggered control systems, both in the absence and presence of external disturbances. This analysis reveals, amongst others, that for several popular event-triggering mechanisms no positive minimum inter-event time can be guaranteed in the presence of arbitrary small external disturbances. This clearly shows that it is essential to include the effects of external disturbances in the analysis of the computation/communication properties of event-triggered control systems. In fact, this paper also identifies event-triggering mechanisms that do exhibit these important event-separation properties.**

I. INTRODUCTION

Event-triggered control (ETC) is a new digital control paradigm that recently received a lot of attention [1]–[10]. In ETC the execution of the control tasks (the sampling of the plant's output and the updating of the control inputs) is triggered by specific conditions, involving actuator and (measured) output variables. Clearly, event-triggered control results in aperiodic execution of control tasks, as the time between two events (the inter-event time) is varying. This is in contrast to conventional time-triggered control schemes, in which the execution of the control tasks occurs periodically and the inter-event times are constant.

The recent interest in ETC is motivated by resource constraints in networked control systems, such as limited communication bandwidth and computational power, as well as restricted energy resources to perform computations and transmit information if battery-powered (wireless) devices are used. Due to these resource constraints it is desirable to only execute control tasks when this is really needed to guarantee the desired stability and performance properties of the system. This requires varying inter-event times in order for the control scheme to let the execution of the control tasks depend on how the system is operating. As such, ETC systems are much better equipped than time-triggered control systems to balance resource utilization and control performance. Several succesful event-triggering mechanisms (ETMs) are proposed in [2]–[10]. See also [1] for a recent overview.

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Just as time-triggered control systems, ETC systems should be robust to imperfections, such as external disturbances, modeling errors, transmission delays, packet dropouts, and so on. However, for ETC systems it is not enough to only verify the robustness of the control properties (e.g., stability, convergence rates, $\mathcal{L}_2$/ISS-gains); also the robustness of the computation/communication properties needs to be carefully examined, while for time-triggered systems this would be trivial. Indeed, if an ETC system would operate properly in the absence of disturbances, e.g., large minimum and average inter-event times, but (small) disturbances can easily reduce the minimum and average inter-event times significantly, then the ETC system becomes rather ineffective in practical situations, in which disturbances are always present.

Some results concerning robustness of computation/communication properties of ETC systems are provided in [2]–[5], [8], [10]. In particular, in [2], [3], [10], the robustness of the minimum inter-event time (MIET) with respect to time delays has been considered, and in [5] with respect to time delays and modeling uncertainties. These results exploit, amongst others, that even while the imperfections (time delays and modeling errors) do not change in size, their *effect* on the system vanishes when the system approaches the origin. Non-vanishing imperfections such as external disturbances do not have this property. In fact, in this paper it is shown that for some systems even if a positive MIET can be guaranteed in the absence of disturbances, still the MIET becomes zero for arbitrarily small (non-vanishing) disturbances. This indicates *zero robustness* of the MIET with respect to disturbances and issues a strong warning that it is crucial to study the robustness of the MIET of ETC systems with respect to (non-vanishing) external disturbances. Surprisingly, there is a lack of results in this area. Notable exceptions are [4], in which a global positive MIET is guaranteed of a model-based state-feedback ETC scheme in the presence of bounded disturbances, and [8], in which a semi-global positive MIET is guaranteed for (decentralized) output-based ETC schemes in the presence of bounded disturbances.

In this paper we will study the effect of disturbances on the MIET for a well-known ETC scheme and introduce various new notions related to the existence of positive MIETs, that are called *event-separation properties*. These properties will be formulated for general impulsive systems [11], [12], as ETC systems can be well described through this hybrid formulation, see, e.g., [8], [13], [14]. Next to formalizing the (robust) event-separation properties, we will also study if these properties hold globally, semi-globally or locally.

In particular, we will investigate for various classes of dynamical systems (linear, nonlinear) and various ETMs (relative [2], absolute [4], [15], mixed [8]) these important properties of closed-loop ETC systems using state-feedback.

The outline of the paper is as follows. We present some necessary preliminaries in Section II and introduce the problem setting and the event-separation properties in Section III. The main results on event-separation properties for state-feedback controllers are presented in Section IV. Finally, in Section V we illustrate our findings with an example and provide conclusive remarks in Section VI.

A. Nomenclature

For a vector $x \in \mathbb{R}^{x_n}$, we denote by $\|x\| := \sqrt{x^\top x}$ its 2-norm. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we denote by $\lambda_M(A)$ and $\lambda_m(A)$ its maximum and minimum eigenvalue, respectively. For a matrix $A \in \mathbb{R}^{n\times m}$, we denote by $\|A\| := \sqrt{\lambda_M(A^\top A)}$ its induced 2-norm. By $\mathbb{N} := \{0, 1, 2, \dots\}$ we denote the set of natural numbers including zero. With $\mathcal{L}_\infty^n$ we denote the space of all essentially bounded functions of size n, and for a signal $w : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_w}$, $w \in \mathcal{L}_\infty^{n_w}$, we denote by $\|w\|_{\mathcal{L}_\infty} = \operatorname{ess\,sup}_{t\in\mathbb{R}_{\geqslant 0}} \|w(t)\|$ its $\mathcal{L}_\infty$-norm. A function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$, and a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and, in addition, $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{KL}$-function if for each fixed $t \geqslant 0$ the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function and for each fixed $s \geqslant 0$, $\beta(s, t)$ is decreasing in t and $\beta(s, t) \to 0$ as $t \to \infty$. For vectors $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ we denote by $[x_1, x_2]$ the vector $[x_1^\top\ x_2^\top]^\top \in \mathbb{R}^{n_1+n_2}$.

II. Preliminaries

We consider the system

$$\dot{\xi} = f(\xi, \omega), \tag{1}$$

in which $f : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\omega} \to \mathbb{R}^{n_\xi}$ is continuously differentiable and satisfies $f(0, 0) = 0$. The variable $\xi \in \mathbb{R}^{n_\xi}$ denotes the state, and $\omega \in \mathbb{R}^{n_\omega}$ is a disturbance. Given initial state $\xi_0 \in \mathbb{R}^{n_\xi}$ and $\omega \in \mathcal{L}_\infty^{n_\omega}$, we define $\xi(t, \xi_0, \omega)$ as the corresponding solution to (1) satisfying $\xi(0, \xi_0, \omega) = \xi_0$. We now recall some properties and results from [16] and [17].

Definition 2.1 ([16]): The system (1) is input-to-state stable (ISS) if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that for each input $\omega \in \mathcal{L}_\infty^{n_\omega}$ and each $\xi_0 \in \mathbb{R}^{n_\xi}$ it holds that

$$\|\xi(t, \xi_0, \omega)\| \leqslant \beta(\|\xi_0\|, t) + \gamma(\|\omega\|_{\mathcal{L}_\infty}) \tag{2}$$

for each $t \in \mathbb{R}_{\geqslant 0}$.

Lemma 2.1 ([16]): The system (1) is input-to-state stable if and only if there exists a continuously differentiable function $V : \mathbb{R}^{n_\xi} \to \mathbb{R}_{\geqslant 0}$ such that

$$\alpha_1(\|\xi\|) \leqslant V(\xi) \leqslant \alpha_2(\|\xi\|) \tag{3a}$$

$$\frac{\partial V(\xi)}{\partial \xi} f(\xi, \omega) \leqslant -W(\xi), \text{ when } \|\xi\| \geqslant \rho(\|\omega\|) \tag{3b}$$

for all $(\xi, \omega) \in \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\omega}$, where α_1, α_2 are $\mathcal{K}_\infty$-functions, ρ is a $\mathcal{K}$-function, and W is a continuous positive definite function on $\mathbb{R}^{n_\xi}$.

Definition 2.2 ([17]): The system (1) is input-to-state practically stable (ISpS) if there exist a $\mathcal{KL}$-function β, a $\mathcal{K}$-function γ and a constant $d \in \mathbb{R}_{\geqslant 0}$ such that for each input $\omega \in \mathcal{L}_\infty^{n_\omega}$ and each $\xi_0 \in \mathbb{R}^{n_\xi}$ it holds that

$$\|\xi(t, \xi_0, \omega)\| \leqslant \beta(\|\xi_0\|, t) + \gamma(\|\omega\|_{\mathcal{L}_\infty}) + d \tag{4}$$

for each $t \in \mathbb{R}_{\geqslant 0}$.

Lemma 2.2 ([17]): The system (1) is input-to-state practically stable if and only if there exists a continuously differentiable function $V : \mathbb{R}^{n_\xi} \to \mathbb{R}$ such that

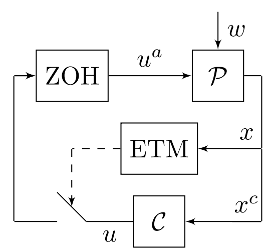
$$\alpha_1(\|\xi\|) \leqslant V(\xi) \leqslant \alpha_2(\|\xi\|) \tag{5a}$$

$$\frac{\partial V(\xi)}{\partial \xi} f(\xi, \omega) \leqslant -W(\xi), \text{ when } \|\xi\| \geqslant \rho(\|\omega\|) + c \tag{5b}$$

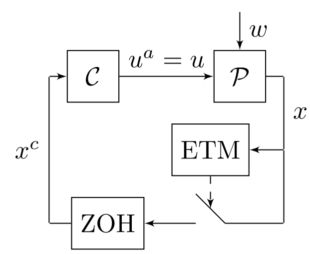
for all $(\xi, \omega) \in \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\omega}$, where α_1, α_2 are $\mathcal{K}_\infty$-functions, ρ is a $\mathcal{K}$-function, W is a continuous positive definite function on $\mathbb{R}^{n_\xi}$, and $c \in \mathbb{R}_{\geqslant 0}$ is a constant.

III. Control setup and problem statement

In this section we introduce two general control architectures, shown in Figure 1, for stabilizing a plant $\mathcal{P}$ in an appropriate sense. We assume that $\mathcal{P}$ can be described by



(a) Architecture I



(b) Architecture II

Fig. 1. The two architectures for event-triggered state-feedback control considered in this paper.

$$\dot{x} = f(x, u^a, w), \tag{6}$$

with $x \in \mathbb{R}^{n_x}$ the state of the plant, $u^a \in \mathbb{R}^{n_u}$ the actuator values and $w \in \mathbb{R}^{n_w}$ an external disturbance. Furthermore, we assume that the controller $\mathcal{C}$ is given by the state-feedback

$$u = k(x^c), \tag{7}$$

where $x^c \in \mathbb{R}^{n_x}$ is the state information available to the controller and $u \in \mathbb{R}^{n_u}$ is the controller output.

In Architecture I, the controller $\mathcal{C}$ is collocated with the sensors of the plant, therefore the state x^c available to the controller is equal to the true state x of the plant $\mathcal{P}$. At event time t_i determined by the event-triggering mechanism (ETM), the actuator values $u^a(t_i)$ are updated to the output $u(t_i)$ of the controller, while between updates the input u^a is held constant in a zero-order-hold (ZOH) fashion, i.e.,

$$u^a(t) = u(t_i), \text{ for } t \in [t_i, t_{i+1}). \tag{8}$$

In Architecture II, the controller is collocated with the actuators, thus $u^a = u$. At event time t_i the sampled state $x^c(t_i)$ is updated to $x(t_i)$. Between updates the sampled state x^c is held constant in a ZOH fashion, resulting similarly as in (8) in

$$x^c(t) = x(t_i), \text{ for } t \in [t_i, t_{i+1}). \tag{9}$$

In both architectures the ETM has full acces to the state x of the plant, and we assume that the first event is generated at the time the system is deployed, i.e.,

$$0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant \dots \tag{10}$$

Note that in conventional time-triggered control, we would have $t_{i+1} - t_i = h$ for all $i \in \mathbb{N}$, with h the sampling period. However, in ETC the inter-event times $t_{i+1} - t_i$ are varying and determined based on state information. By introducing

$$e(t) = x(t_i) - x(t), \text{ for } t \in [t_i, t_{i+1}), \tag{11}$$

we can write the closed-loop system of both architectures as

$$\dot{x} = f(x, k(x+e), w). \tag{12}$$

To indicate how the ETM will be chosen, we assume that k has been designed such that for the closed-loop system (12) there exists a continuously differentiable function $V : \mathbb{R}^{n_x} \to \mathbb{R}$ satisfying

$$\alpha_1(\|x\|) \leqslant V(x) \leqslant \alpha_2(\|x\|) \tag{13a}$$

$$\frac{\partial V(x)}{\partial x} f(x, k(x+e), w) \leqslant -W(x), \text{ when } \|x\| \geqslant \rho_1(\|e\|) + \rho_2(\|w\|) \tag{13b}$$

for all $(x, e, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$, where α_1, α_2 are $\mathcal{K}_\infty$-functions, ρ_1 and ρ_2 are $\mathcal{K}$-functions, and W is a continuous positive definite function on $\mathbb{R}^{n_x}$. Obviously, according to Lemma 2.1 this implies that (12) is ISS with respect to both e and w.

A common design practice, originating from [2], for finding stabilizing ETMs is enforcing that[1] $\rho_1(\|e\|) \leqslant \sigma\|x\| + \beta$. When $0 \leqslant \sigma < 1$ and $\beta \geqslant 0$, this leads based on (13) to

$$\alpha_1(\|x\|) \leqslant V(x) \leqslant \alpha_2(\|x\|) \tag{14a}$$

$$\frac{\partial V}{\partial x} f(x, k(x+e), w) \leqslant -W(x), \text{ when } \|x\| \geqslant \frac{1}{1-\sigma}\left(\rho_2(\|w\|) + \beta\right). \tag{14b}$$

As a consequence, the triggering mechanism

$$t_{i+1} = \inf\left\{t > t_i \middle| \rho_1(\|e(t)\|) > \sigma\|x(t)\| + \beta\right\} \tag{15}$$

with $0 \leqslant \sigma < 1$ and $\beta \geqslant 0$ renders the system (12) ISpS if $\beta > 0$, and ISS if $\beta = 0$, due to Lemma 2.1 and Lemma 2.2.

We will call (15) a *relative ETM* when $0 < \sigma < 1$ and $\beta = 0$, which is the ETM as proposed in [2] and many follow-ups, including [5]–[7], [10], an *absolute ETM* when $\sigma = 0$ and $\beta > 0$, as used in, e.g., [4], [15], and a *mixed ETM* when $0 < \sigma < 1$ and $\beta > 0$, as proposed in [8].

[1]Note that the ISS characterizations used in this paper are slightly different from the ones used in [2], but are equivalent and result in essentially the same ETMs.

Combining (11), (12) and (15) leads to the closed-loop ETC system

$$\begin{cases} \dot{x}(t) &= f(x(t), k(x(t)+e(t)), w(t)) \\ e(t) &= x(t_i) - x(t), \text{ for } t \in [t_i, t_{i+1}) \\ t_{i+1} &= \inf\left\{t > t_i \mid \rho_1(\|e(t)\|) > \sigma\|x(t)\| + \beta\right\}. \end{cases} \tag{16}$$

The main objective of this paper is to investigate under which conditions the ETC system (16) can be guaranteed to have inter-event times $t_{i+1} - t_i$, $i \in \mathbb{N}$, that are bounded from below by a positive constant, even in the presence of external disturbances w. In order to study this problem, we formally introduce the relevant *event-separation* properties in Section III-B. However, before doing so, we first introduce in Section III-A the special case where the functions f and k are linear.

A. The linear case

When considering instead of the nonlinear plant $\mathcal{P}$ in (6) a linear model of the form $\dot{x} = Ax + Bu + w$, the above derivation would replace (12) by

$$\dot{x} = (A + BK)x + BKe + w, \tag{17}$$

in which we assume that $A + BK$ is Hurwitz, to satisfy the conditions in (13). To design an ETM that results in an IS(p)S closed-loop system, we can use the following procedure, inspired by [2]. Since $A + BK$ is Hurwitz, we can find positive definite matrices Z and Q, satisfying

$$(A + BK)^\top Z + Z(A + BK) = -Q. \tag{18}$$

From this we derive for $V(x) = x^\top Zx$ that

$$\dot{V} \leqslant 2\|ZBK\|\|x\|\|e\| + 2\|Z\|\|x\|\|w\| - \lambda_m(Q)\|x\|^2. \tag{19}$$

By selecting any $0 < \gamma < 1$, it follows that

$$\dot{V} \leqslant (\gamma - 1)\lambda_m(Q)\|x\|^2 \text{ when } \|x\| \geqslant \frac{2\|ZBK\|}{\gamma\lambda_m(Q)}\|e\| + \frac{2\|Z\|}{\gamma\lambda_m(Q)}\|w\|, \tag{20}$$

which is in the form of (13). In this case the ETM (15) is equal to

$$t_{i+1} = \inf\left\{t > t_i \middle| \|e(t)\| > P\|x(t)\| + T\right\} \tag{21}$$

with

$$P = \sigma\gamma\frac{\lambda_m(Q)}{2\|ZBK\|}, \; T = \beta\gamma\frac{\lambda_m(Q)}{2\|ZBK\|}. \tag{22}$$

Combining (11), (17) and (21) leads to the closed-loop ETC system

$$\begin{cases} \dot{x}(t) &= (A + BK)x(t) + BKe(t) + w(t) \\ e(t) &= x(t_i) - x(t), \text{ for } t \in [t_i, t_{i+1}) \\ t_{i+1} &= \inf\left\{t > t_i \mid \|e(t)\| > P\|x(t)\| + T\right\}, \end{cases} \tag{23}$$

which is a special case of (16).

Remark 3.1: Note that also in the nonlinear case we can consider an ETM of the form (21) when ρ_1 is Lipschitz continuous on compacts. By taking $P = \frac{\sigma}{L_1}$ and $T = \frac{\beta}{L_1}$, with $\rho_1(\|e\|) \leqslant L_1\|e\|$, it holds on compact sets that $\|e\| \leqslant P\|x\| + T$ implies $\rho_1(\|e\|) \leqslant \sigma\|x\| + \beta$, thus the IS(p)S properties of ETM (15) are preserved by ETM (21). Because ETM (15) cannot generate an event before ETM (21) does, we can use ETM (21) to find a lower bound on the inter-event times generated by ETM (15) in the closed-loop system (16).

B. Definitions

The resulting event-triggered system can be written as an impulsive system, cf. [8], [18], of the form

$$\dot{\xi} = F(\xi, \omega), \text{ if } \xi \in C, \tag{24a}$$

$$\xi^+ = G(\xi), \text{ if } \xi \in D, \tag{24b}$$

where the sets C and D are determined by the event-triggering mechanism, as we will see. Throughout this section we will assume that C, D, F and G are such that existence and uniqueness of solutions is guaranteed for each initial condition ξ_0 and each disturbance ω of interest. In addition, we assume that all solutions are complete in the sense of [11], i.e., loosely speaking, either the solution is defined for time $t \to \infty$, or the number of jumps $i \to \infty$, or both. See [11] for more details on the definitions of solutions and the hybrid model class (24).

To be precise, the ETC system (16), can be written in the form (24) by taking $\xi = [x, e] \in \mathbb{R}^{n_\xi}$ with $n_\xi = 2n_x$, $\omega = w$, and

$$F(\xi,\omega) = \begin{bmatrix} f(x, k(x+e), w) \\ -f(x, k(x+e), w) \end{bmatrix}, \; G(\xi) = \begin{bmatrix} x \\ 0 \end{bmatrix},$$
$$C = \{\xi \mid \rho_1(\|e\|) \leqslant \sigma\|x\| + \beta\},$$
$$D = \{\xi \mid \rho_1(\|e\|) > \sigma\|x\| + \beta\}.$$

Since we assume that for the ETC system (16) the first event is generated at the time the system is deployed, the corresponding impulsive system (24) must start in the set

$$\Xi_0 := \{\xi_0 \in \mathbb{R}^{n_\xi} \mid \xi_0 = G(\xi) \text{ for some } \xi \in \mathbb{R}^{n_\xi}\}. \tag{25}$$

Given disturbance ω and initial condition $\xi_0 \in \Xi_0$, the system (24) jumps according to (24b) at the jump times included in the set $\{t_i \mid i \in I(\xi_0, \omega)\}$, where $I(\xi_0,\omega)$ is an index set enumerating the jump times. Clearly, $I(\xi_0,\omega) = \mathbb{N}$ or $I(\xi_0,\omega) = \{0,1,2,\ldots,N\}$ for some $N \in \mathbb{N}$. To make the dependence of the jump/event times on the initial condition $\xi_0 \in \Xi_0$ and the disturbance signal ω explicit, we sometimes write $t_i = t_i(\xi_0,\omega)$, $i \in I(\xi_0,\omega)$. In the case that $I(\xi_0,\omega) = \{0,1,2,\ldots,N\}$, we define $t_{N+1} := \infty$ (as we know that solutions are complete).

All definitions below apply to the impulsive system (24).

Definition 3.1: The i-th inter-event time $\tau_i(\xi_0,\omega)$ with $i \in I(\xi_0,\omega)$ corresponding to disturbance signal $\omega : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_\omega}$ and initial condition $\xi_0 \in \Xi_0$ is given as

$$\tau_i(\xi_0,\omega) := t_{i+1}(\xi_0,\omega) - t_i(\xi_0,\omega). \tag{26}$$

Definition 3.2: The minimum inter-event time for disturbance signal $\omega : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_\omega}$ and initial state $\xi_0 \in \Xi_0$ is defined as

$$\tau(\xi_0,\omega) := \inf_{i \in I(\xi_0,\omega)} t_{i+1}(\xi_0,\omega) - t_i(\xi_0,\omega). \tag{27}$$

Based on the above definitions on inter-event times, we introduce the following event-separation properties for system (24).

Definition 3.3: The impulsive system (24) has the global event-separation property if

$$\inf_{\xi \in \Xi_0} \tau(\xi, 0) > 0. \tag{28}$$

Definition 3.4: The impulsive system (24) has the semi-global event-separation property if for all compact subsets $\mathcal{X} \subset \mathbb{R}^{n_\xi}$

$$\inf_{\xi \in \mathcal{X} \cap \Xi_0} \tau(\xi, 0) > 0. \tag{29}$$

Definition 3.5: The impulsive system (24) has the local event-separation property if for all $\xi \in \Xi_0$

$$\tau(\xi, 0) > 0. \tag{30}$$

In addition, we define their robust counterparts as follows.

Definition 3.6: The impulsive system (24) has the robust global event-separation property if there exists $\epsilon > 0$ such that

$$\inf_{\substack{\xi \in \Xi_0 \\ \|\omega\|_{\mathcal{L}_\infty} \leqslant \epsilon}} \tau(\xi, \omega) > 0. \tag{31}$$

Definition 3.7: The impulsive system (24) has the robust semi-global event-separation property if for all compact subsets $\mathcal{X} \subset \mathbb{R}^{n_\xi}$ there exists $\epsilon > 0$ such that

$$\inf_{\substack{\xi \in \mathcal{X} \cap \Xi_0 \\ \|\omega\|_{\mathcal{L}_\infty} \leqslant \epsilon}} \tau(\xi, \omega) > 0. \tag{32}$$

Definition 3.8: The impulsive system (24) has the robust local event-separation property if there exists $\epsilon > 0$ such that for all $\omega \in \mathcal{L}_\infty$ such that $\|\omega\|_{\mathcal{L}_\infty} \leqslant \epsilon$ and all $\xi \in \Xi_0$ it holds that

$$\tau(\xi, \omega) > 0. \tag{33}$$

IV. Main Results for the State-Feedback Case

In this section we study the event-separation properties for ETC systems (16) for the relative, mixed and absolute ETMs (15). Whenever we mention event-separation properties for ETC systems, we naturally mean the event-separation properties of the corresponding impulsive system (24).

A. Relative triggering

It is shown in [2] that if $w = 0$ and the functions f, k, and ρ_1 are Lipschitz continuous on compacts, then on a compact set $\mathcal{X}$, the ETC system (16) with $\sigma > 0$ and $\beta = 0$ has a positive minimum inter-event time

$$\inf_{x \in \mathcal{X}} \tau([x, 0], 0) \geqslant P/(L + LP), \tag{34}$$

where $L > 0$ is such that

$$\|f(x, k(x+e), 0)\| \leqslant L\|x\| + L\|e\|, \tag{35}$$

for all $x \in \mathcal{X}$, and with P according to Remark 3.1. This means that the ETC system (16) using a relative ETM (15) has the semi-global event-separation property if the functions f, k, and ρ_1 are Lipschitz continuous on compacts, and the global event-separation property if the functions f, k, and ρ_1 are globally Lipschitz continuous. This is an interesting and valuable result, but it appears that the guarantee in (34) does not extend to the case where arbitrarily small disturbances are present. Indeed, the system (16) using a relative ETM (15) in general does not have any robust event-separation properties. We prove this statement for the ETC system (23) with $P > 0$ and $T = 0$, i.e., focussing on the linear case.

Theorem 4.1: The closed-loop event-triggered control system (23) with $P > 0$ and $T = 0$ does not have the robust local event-separation property.

Proof: See [19]. ∎

Since the ETC system (23) with $P > 0$ and $T = 0$ does not have the robust local event-separation property, obviously it also does not have the robust (semi-)global event-separation property.

B. Mixed triggering

In the next theorem, which forms an extension to Theorem III.1 of [2] towards mixed ETMs, we state that large class of ETC systems using mixed ETMs have the robust semi-global event-separation property.

Theorem 4.2: Consider the closed-loop event-triggered system (16) with $0 < \sigma < 1$ and $\beta > 0$. If

1) f, k and ρ_1 are Lipschitz continuous on compacts;
2) there exists a continuously differentiable function V for the system satisfying (13),

then the system (16) is ISpS and has the robust semi-global event-separation property.

Proof: See [19]. ∎

Corollary 4.3: Consider the closed-loop event-triggered system (16) with $0 < \sigma < 1$ and $\beta > 0$. If

1) f, k and ρ_1 are globally Lipschitz continuous;
2) there exists a continuously differentiable function V for the system satisfying (13),

then the system (16) is ISpS and has the robust global event-separation property.

The proof of Corollary 4.3 can be directly derived from the proof of Theorem 4.2.

C. Absolute triggering

Next we show that ETC systems (16) with f, k and ρ_1 Lipschitz continuous on compacts, using absolute ETMs have the robust semi-global event-separation property.

Theorem 4.4: Consider the closed-loop event-triggered control system (16), with $\sigma = 0$ and $\beta > 0$. If

1) f, k and ρ_1 are Lipschitz continuous on compacts;
2) there exists a continuously differentiable function V for the system satisfying (13),

then the system (16) is ISpS and has the robust semi-global event-separation property.

Proof: See [19]. ∎

The MIET τ^* is a function of x_{max}, indicating that the semi-global result cannot be readily extended to a global result along the lines of the proof above. In fact, in the next theorem we state that ETC systems (23) using absolute ETM do not have the global event-separation property.

Theorem 4.5: The closed-loop event-triggered control system (23) with $A + BK \neq 0$, $P = 0$ and $T > 0$ does not have the global event-separation property.

Proof: See [19]. ∎

D. Overview of the linear case

Summarizing the above results for the linear case, the event-separation properties of closed-loop ETC systems (23) are shown in Table I for relative, mixed and absolute ETMs.

ETM	rob. glb.	glb.	rob. s-glb.	s-glb.	rob. loc	loc
rel	×	✓	×	✓	×	✓
mix	✓	✓	✓	✓	✓	✓
abs	×	×	✓	✓	✓	✓

TABLE I
EVENT-SEPARATION PROPERTIES FOR CLOSED-LOOP ETC SYSTEMS (23).

V. ILLUSTRATIVE EXAMPLE

As is shown above, closed-loop ETC systems (23), have no robust event-separation property when relative ETMs are used, but have the robust global event-separation property when a mixed ETM is used. We illustrate these findings by studying the example of [2] and adding disturbances w. This leads to the ETC system

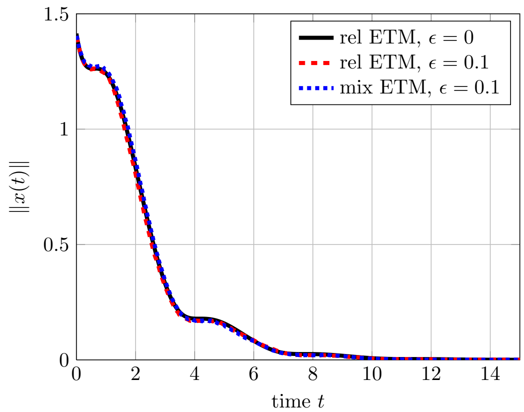
$$\begin{cases} \dot{x} &= Ax + BK(x+e) + w \\ e &= x(t_i) - x, \text{ for } t \in [t_i, t_{i+1}) \\ t_{i+1} &= \inf\{t > t_i \mid \|e(t)\| > P\|x(t)\| + T\}, \end{cases} \tag{36}$$

where

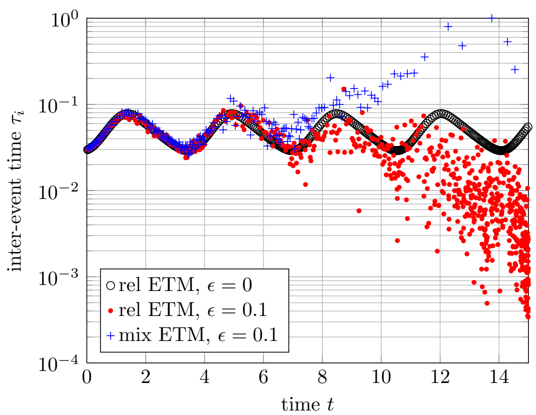
$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ K = \begin{bmatrix} 1 & -4 \end{bmatrix}, \tag{37}$$

and w is zero-mean white noise satisfying $\|w\|_{\mathcal{L}_\infty} \leqslant \epsilon$. We compare two ETMs, the relative ETM with $P = 0.05$ and $T = 0$, and the mixed ETM with $P = 0.05$ and $T = 0.001$. The relative ETM renders the closed-loop ISS with respect to w, and the mixed ETM renders the closed-loop ISpS with respect to w.

The ETC system (36) is simulated using the relative ETM for the cases $\epsilon = 0$ and $\epsilon = 0.1$, and using the mixed ETM for the case $\epsilon = 0.1$. Figure 2 shows the evolution of $\|x(t)\|$ and the inter-event times τ_i. For all three cases, the control performance of the system is comparable. For the relative ETM we find, using the results of [2], that $\inf_{x \in \mathbb{R}^{n_x}} \tau([x, 0], 0) = 0.028$. It can be seen in Figure 2(b) that this lower bound is indeed satisfied when $\epsilon = 0$. However, the inter-event times drop well below this bound for $\epsilon = 0.1$, and many events are generated when $\|x\|$ becomes small. Clearly, this is unsatisfactorily in view of the ETC philosophy that few or no events should be generated when the system is performing satisfactorily. The mixed ETM however is robust to disturbances and generates less and less events as the system approaches the origin, despite the presence of disturbances.



(a)



(b)

Fig. 2. The evolution of $\|x(t)\|$ (a) and the inter-event times τ_i (b) for the example (36).

VI. Conclusions

We have introduced the (robust) global, semi-global and local event-separation properties for impulsive systems, and have studied these event-separation properties for ETC systems based on both Architecture I and II, for relative, absolute and mixed event-triggering mechanisms.

It was found that relative ETMs are not robust to disturbances and generate many events in the presence of arbitrarily small disturbances when the system is operating close to the origin. Absolute ETMs, while robust to disturbances, generate many events when the system is operating far away from the origin (the desired equilibrium). Mixed ETMs yield the most desirable event-separation properties by combining the advantages of both relative and absolute ETMs, i.e., robustness to disturbances and global MIETs.

We conclude that, since every physical system is subject to external disturbances, the effect of these disturbances should not be disregarded in the stability and robustness analysis of both the control properties of the system and its computation/communication properties.

In future research we intend to extend the results of this paper to output-based control and to also consider the presence of measurement noise.

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