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Tradeoffs between quality-of-control and quality-of-service in large-scale nonlinear networked control systems[☆]

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ABSTRACT

In this paper we study input-to-state stability (ISS) of large-scale networked control systems (NCSs) in which sensors, controllers and actuators are connected via multiple (local) communication networks which operate asynchronously and independently of each other. We model the large-scale NCS as an interconnection of hybrid subsystems, and establish rather natural conditions which guarantee that all subsystems are ISS, and have an associated ISS Lyapunov function. An ISS Lyapunov function for the overall system is constructed based on the ISS Lyapunov functions of the subsystems and the interconnection gains. The control performance, or "*quality-of-control*", of the overall system is then viewed in terms of the convergence rate and ISS gain of the associated ISS Lyapunov function. Additionally, the "*quality-of-service*" of the communication networks is viewed in terms of the maximum allowable transmission interval (MATI) and the maximum allowable delay (MAD) of the network, and we show that the allowable quality-of-service of the communication networks is constrained by the required ISS gains and convergence rate of the hybrid subsystem corresponding to that network. Our results show that the quality-of-control of the overall system can be improved (or degraded) by improving (or relaxing) the quality-of-service of the communication networks. Alternatively, when relaxing the quality-of-service of one communication network, we can retain the quality-of-control of the overall system by improving the quality-of-service of one or more of the other communication networks. Our general framework will formally show these intuitive and insightful tradeoffs.

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1. Introduction

In networked control systems (NCSs), sensor and actuator data is transmitted via shared (wired or wireless) communication networks. This offers several advantages over conventional control systems, in which sensor and actuation data is transmitted using dedicated point-to-point wired links, including reduced installation costs, better maintainability and greater flexibility. On the other hand, shared communication networks also introduce communication errors as a result of network imperfections such as varying transmission intervals and delays, and quantization errors. Additionally, since a network is usually shared by multiple sensor, controller and actuator nodes, there is a need for a medium access control

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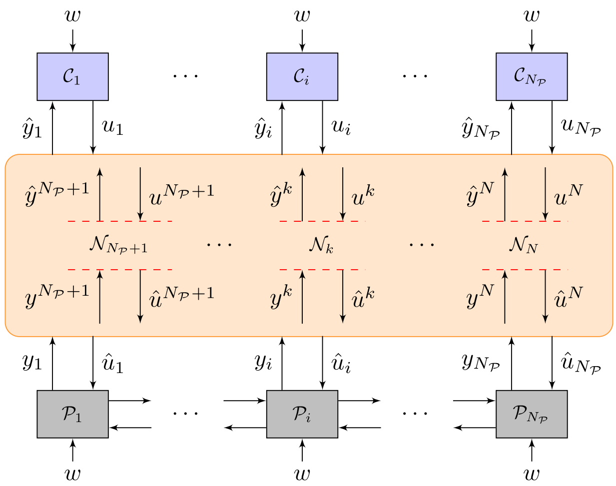


Fig. 1. The networked control setup considered in this paper, in which $N_{\mathscr{P}} \in \mathbb{N}$ (coupled) plants $\mathscr{P}_i$ and their controllers $\mathscr{C}_i$, $i \in \{1, 2, \ldots, N_{\mathscr{P}}\}$, are connected via $N_{\mathscr{N}} \in \mathbb{N}$ communication networks $\mathscr{N}_k$, $k \in \{N_{\mathscr{P}} + 1, N_{\mathscr{P}} + 2, \ldots, N_{\mathscr{P}} + N_{\mathscr{N}}\}$. The variable w denotes the disturbance acting on the plants.

(MAC) protocol that governs the access of the nodes to the network in order to prevent packet losses as much as possible. As a result, one needs to design the communication networks and controllers in such a way that the NCS displays desirable behavior in terms of stability and performance that is robust to these network-induced phenomena.

In most of the available literature on NCSs it is assumed that all sensor and actuation data is transmitted over *one* single communication network, see, e.g., [1–6]. These works provide insightful tradeoffs of the network performance ("*quality-of-service*", expressed in terms of, e.g., maximum allowable transmission interval (MATI), maximum allowable delay (MAD), network reliability, etc.) versus a certain control performance. However, it is not always reasonable to assume that there is one global communication network. For example, in the control of large-scale systems it is often more natural and cost-efficient to use a local controller for each subsystem than one global controller for the whole system. In such a system, it is much more reasonable to close the local control loops over several local communication networks, instead of (assuming the presence of only) one global communication network. This leads to large-scale NCSs with multiple local communication networks operating independently and asynchronously. Clearly, the required network parameters are to be formulated locally for each individual network, preferably based on local conditions involving only local dynamics of the subsystems and possibly a condition on the interconnection. The reason is that global conditions based on the dynamics of the complete large-scale system quickly become intractable when the overall system contains a large number of subsystems.

In this paper the objective is to provide a general framework for the stability analysis of large-scale networked control systems with multiple local communication networks, and to provide, based on local conditions, network parameters for each local communication network such that stability of the overall system is guaranteed. In particular, we focus on the general NCS setup shown in Fig. 1, which generalizes the setups considered in [1,7,8]. However, our results are also applicable to other NCS setups that are not captured by Fig. 1.

We are interested in input-to-state stability (ISS) [9] of the complete system of Fig. 1, which is a very useful concept of stability for nonlinear systems with inputs and interconnected systems, and has been studied in the context of NCSs in, e.g., [3]. The theory on interconnections of (hybrid) ISS systems is already well-developed (see, e.g., [10–18]) and seems well-suited for the stability analysis of large-scale networked control systems, as long as they can be modeled as an interconnection of ISS subsystems. This was already demonstrated in [19], which considers a (networked) interconnection of subsystems, and derives upper bounds on the gains for the communication links guaranteeing a small-gain condition of the overall network. Besides, in [19] extensions of the small-gain theorems of [11,13] are given that include subsystems that are only pre-globally stable (pre-GS) and not ISS. However, the paper does not connect local quality-of-service parameters such as MATI and MAD to the ISS gains (or pre-GS gains) of the communication links.

In this paper, we show that communication networks using a uniformly globally exponentially stable (UGES) [2] MAC protocol give rise to (hybrid) network-induced error systems that are ISS, and that the convergence rate and ISS gains of the network-induced error systems are related to the MATI and MAD of the corresponding communication network. By considering the MATI and MAD of the communication networks as design parameters, the convergence rate and ISS gains of the related hybrid network-induced error system can be scaled. Showing that the network-induced error systems are ISS enables us to model the large-scale NCS as an interconnection of hybrid ISS subsystems, and to extend the work in [7] to a more general networked control setup using the (hybrid) small-gain theory of [13,17]. Moreover, based on the ISS Lyapunov functions of the subsystems, we provide an ISS Lyapunov function for the overall system, an upper bound on its ISS gain, and a lower bound on its convergence rate. The control performance, or "*quality-of-control*" of the overall system is then viewed in terms of the convergence rate and ISS gain of the corresponding ISS Lyapunov function. As a result, the quality-of-control of the overall system can be tuned by varying the ISS gains and convergence rates of the network-induced error systems (which are related to the quality-of-service of the communication networks). More specifically, the quality-of-control of the overall system can be improved (or degraded) by improving (or relaxing) the quality-of-service of the communication networks. Alternatively, when relaxing the quality-of-service of one communication network, we can retain the quality-of-control of the overall system by improving the quality-of-service of one or more of the other communication networks. Finally, once the ISS gains and convergence rate of a network-induced error system is fixed, the MATI and MAD of the corresponding communication network can be maximized in a Pareto-optimal sense. As such, we can find a MATI and a MAD for each local network such that the desired quality-of-control of the overall system is guaranteed. This can be accomplished by first making a tradeoff between the quality-of-service of the network-induced error systems and the quality-of-control of the overall system, and secondly making a tradeoff of MATI versus MAD for each communication network individually (based on the Pareto-optimal curves). This approach is demonstrated via a nonlinear example.

The results are formulated in terms of local conditions on the subsystem dynamics, combined with a small-gain condition on the ISS gains, making the proposed framework suitable for large-scale systems. Preliminary results of this work have appeared in [20], in which we only guaranteed ISS of the overall system, *without* providing an ISS Lyapunov function of the overall system. Moreover, no estimates of the ISS gain and convergence rate were presented, and only a linear example was provided.

The paper is organized as follows. In Section 2, we present preliminaries on input-to-state stability and ISS Lyapunov functions. In Section 3, we present the large-scale networked control setup with multiple communication networks considered in this paper, give the problem formulation, and sketch how we will approach the problem. In Section 4, we model the controlled subsystems and network-induced error dynamics as hybrid systems, and in Section 5, we develop a variation of the small-gain theory given in [17] for constructing ISS Lyapunov functions for interconnections of hybrid systems. This step is necessary, since NCSs with multiple local networks cannot be accommodated in the framework of [17]. Finally, in Section 6 we show how the results of Section 5 can be used to tradeoff

(i) the quality-of-control of the overall system versus the quality-of-service of the communication networks,
(ii) the quality-of-service between the different communication networks, and
(iii) the MATI versus MAD within each communication network.

We demonstrate our approach via a nonlinear example in Section 7, and draw our conclusions in Section 8.

1.1. Notation

For a vector $x \in \mathbb{R}^{n_x}$, we denote by $|x| := \sqrt{x^\top x}$ its Euclidean norm, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^n$. For a matrix $A \in \mathbb{R}^{n\times m}$, we denote by $|A|$ its induced Euclidean norm. Given a closed set $\mathcal{A} \subset \mathbb{R}^{n_x}$ we denote $|x|_{\mathcal{A}} = \min_{y\in\mathcal{A}} |x-y|$. By $\mathbb{N}$ we denote the set of natural numbers including zero, i.e., $\mathbb{N} := \{0, 1, 2, \ldots\}$. For vectors $x, y \in \mathbb{R}^n_{\geqslant 0}$ we write $x \not\geqslant y$ if $x_i < y_i$ for at least one $i \in \{1, 2, \ldots, n\}$, and we write $x < y$ (resp. $x \leqslant y$) if $x_i < y_i$ (resp. $x_i \leqslant y_i$) for all $i \in \{1, 2, \ldots, n\}$. A function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$, and a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and, in addition, $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{KL}$-function if for each fixed $t \in \mathbb{R}_{\geqslant 0}$ the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function and for each fixed $s \in \mathbb{R}_{\geqslant 0}$, $\beta(s, t)$ is decreasing in t and $\beta(s, t) \to 0$ as $t \to \infty$. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{KLL}$-function if for each fixed t the functions $\beta(\cdot, t, \cdot)$ and $\beta(\cdot, \cdot, t)$ are $\mathcal{KL}$-functions. For vectors $x_i \in \mathbb{R}^{n_i}$, $i \in \{1, 2, \ldots, N\}$, we denote by $(x_1, x_2, \ldots, x_N)$ the vector $[x_1^\top x_2^\top \cdots x_N^\top]^\top \in \mathbb{R}^n$ with $n = \sum_{i=1}^N n_i$. For a function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^n$ and $t \in \mathbb{R}_{\geqslant 0}$, we use $f(t^+)$ to denote the limit $f(t^+) = \lim_{s\to t, s>t} f(s)$, provided the limit exists. A function $V : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ is locally Lipschitz continuous if for each $x \in \mathcal{X}$, there exist a neighborhood $\mathcal{U}_x$ and a constant $M > 0$ such that $|V(y) - V(z)| \leqslant M|y - z|$ for all $y, z \in \mathcal{U}_x$. Given an open set $\mathcal{X} \subseteq \mathbb{R}^n$ and a locally Lipschitz function $V : \mathcal{X} \to \mathbb{R}_{\geqslant 0}$, $\partial_x^C V(x)$ denotes the Clarke generalized gradient at $x \in \mathcal{X}$, given by $\partial_x^C V(x) = \mathrm{conv}\,\{\zeta \in \mathbb{R}^n \mid \exists \{x_i \in \mathcal{X}\}_{i\in\mathbb{N}}$ for which $\nabla_x V(x_i)$ exists for each $i \in \mathbb{N}$, $\lim_{i\to\infty} x_i = x$, and $\lim_{i\to\infty} \nabla_x V(x_i) = \zeta\}$ [21, Chapter 2, Theorem 8.1]. A set-valued mapping $F : \mathcal{A} \rightrightarrows \mathcal{B}$ is outer semicontinuous if for all $x \in \mathcal{A}$ and all sequences $\{x_i\}_{i\in\mathbb{N}}$, $\{y_i\}_{i\in\mathbb{N}}$ with $\lim_{i\to\infty} x_i = x$, $\lim_{i\to\infty} y_i = y$, and $y_i \in F(x_i)$ for all $i \in \mathbb{N}$, we have $y \in F(x)$. It is locally bounded if for any compact set $K \subset A$ there exists $m > 0$ such that $F(K) \subset m\mathbb{B}$, where $\mathbb{B}$ denotes the closed unit ball. Given sets $\mathcal{A}$ and $\mathcal{X}$ in $\mathbb{R}^n$, $\mathcal{A} \subset \mathcal{X}$ is relatively closed in $\mathcal{X}$ if $\mathcal{A} = \overline{\mathcal{A}} \cap \mathcal{X}$; when $\mathcal{X}$ is open, then $\mathcal{A}$ is relatively closed in $\mathcal{X}$ if and only if $\mathcal{X} \setminus \mathcal{A}$ is open. Given sets $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{Z} \subseteq \mathcal{X} \times \mathcal{Y}$, the projection of $\mathcal{Z}$ onto $\mathcal{X}$ is given by $\{x \in \mathcal{X} \mid$ there exists $y \in \mathcal{Y}$ such that $(x, y) \in \mathcal{Z}\}$.

2. Preliminaries

Consider the hybrid system

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x, w), & (x, w) \in \mathcal{F}, \\ x^+ \in G(x, w), & (x, w) \in \mathcal{J}, \end{cases} \tag{1}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$ is the state, $w \in \mathcal{W} \subset \mathbb{R}^{n_w}$ is a disturbance, $F : \mathcal{X} \times \mathcal{W} \rightrightarrows \mathbb{R}^{n_x}$ is the flow map, $G : \mathcal{X} \times \mathcal{W} \rightrightarrows \mathcal{X}$ is the jump map, $\mathcal{F} \subseteq \mathcal{X} \times \mathcal{W}$ is the flow set, and $\mathcal{J} \subseteq \mathcal{X} \times \mathcal{W}$ is the jump set.

The solutions to (1) are defined on hybrid time domains [22] as follows. A *compact hybrid time domain* is a set $\mathcal{D} = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\} \subset \mathbb{R}_{\geqslant 0} \times \mathbb{N}$ with $J \in \mathbb{N}_{>0}$ and $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_J$. A *hybrid time domain* is a set $\mathcal{D} \subset \mathbb{R}_{\geqslant 0} \times \mathbb{N}$ such that $\mathcal{D} \cap ([0, T] \times \{0, \dots, J\})$ is a compact hybrid time domain for each $(T, J) \in \mathcal{D}$. A *hybrid signal* is a function defined on a hybrid time domain. A hybrid signal $w : \operatorname{dom} w \to \mathbb{R}^{n_w}$ is a *hybrid input* if $w(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each j. A hybrid signal $x : \operatorname{dom} x \to \mathbb{R}^{n_x}$ is a *hybrid arc* if $x(\cdot, j)$ is locally absolutely continuous for each j.

A hybrid arc $x : \operatorname{dom} x \to \mathcal{X}$ and a hybrid input $w : \operatorname{dom} w \to \mathcal{W}$ are a *solution pair* (x, w) to (1) if $\operatorname{dom} x = \operatorname{dom} w$, $(x(0, 0), w(0, 0)) \in \mathcal{F} \cup \mathcal{J}$, and

1. for all $j \in \mathbb{N}$ and almost all t with $(t, j) \in \operatorname{dom} x$
$$\dot{x}(t, j) \in F(x(t, j), w(t, j)), \quad \text{and} \quad (x(t, j), w(t, j)) \in \mathcal{F}$$
2. for all $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$
$$x(t, j + 1) \in G(x(t, j), w(t, j)), \quad \text{and} \quad (x(t, j), w(t, j)) \in \mathcal{J}.$$

We impose the following basic regularity conditions on the data $(G, F, \mathcal{F}, \mathcal{J}, \mathcal{X}, \mathcal{W})$ of the hybrid system (1), cf. [17,22,23].

Assumption 2.1. The hybrid system (1) satisfies the following:

(A0) $\mathcal{X}$ is an open set and $\mathcal{W}$ is closed;
(A1) $\mathcal{F}$ and $\mathcal{J}$ are relatively closed in $\mathcal{X} \times \mathcal{W}$;
(A2) F is outer semicontinuous and locally bounded, and $F(x, w)$ is nonempty and convex for all $(x, w) \in \mathcal{F}$;
(A3) G is outer semicontinuous, and $G(x, w)$ is nonempty for all $(x, w) \in \mathcal{J}$.

The $\mathcal{L}_\infty$-norm of a hybrid signal $w : \operatorname{dom} w \to \mathbb{R}^{n_w}$ is defined by (see, e.g., [14,17])

$$\|w\| := \max \left\{ \operatorname*{ess\,sup}_{(t,j) \in \operatorname{dom} w \setminus \Phi(w)} |w(t, j)|, \sup_{(t,j) \in \Phi(w)} |w(t, j)| \right\}, \tag{2}$$

where $\Phi(w) := \{(t, j) \in \operatorname{dom} w \mid (t, j + 1) \in \operatorname{dom} w\}$. With $\mathcal{L}_\infty^n$ we denote the space of all functions $w : \operatorname{dom} w \to \mathbb{R}^n$ with $\|w\| < \infty$, where $\operatorname{dom} w$ is a hybrid time domain. A solution pair of hybrid systems is called *maximal* if it cannot be extended and *complete* if its hybrid time domain is unbounded.

In this work we will make use of the additional assumption that the system (1) is persistently flowing. We will see in Section 3 that this assumption holds for the networked control systems we consider in this paper.

Definition 2.2 (*Persistence of Flow*)**.** System (1) is *persistently flowing* if all maximal solution pairs (x, w) have unbounded domains in the t direction, i.e., $\sup_t \operatorname{dom} x = \infty$.

Note that so-called *Zeno behavior* [24] cannot occur when system (1) is persistently flowing.

Since many hybrid systems feature states that are purely used for modeling, like timers and counters, that can grow unbounded, we study input-to-state stability with respect to a certain nonempty closed set $\mathcal{A} \subset \mathcal{X}$, cf. [23].

Definition 2.3 (*Input-to-State Stability*)**.** The system (1) is input-to-state stable (ISS) from input w to state x with respect to the nonempty closed set $\mathcal{A} \subset \mathcal{X}$, if it is persistently flowing, and there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that every solution pair (x, w) with $w \in \mathcal{L}_\infty^{n_w}$ satisfies

$$|x(t, j)|_{\mathcal{A}} \leqslant \max\{\beta(|x(0, 0)|_{\mathcal{A}}, t), \gamma(\|w\|)\} \tag{3}$$

for all $(t, j) \in \operatorname{dom} x$.

Remark 2.4. For hybrid systems, ISS is more commonly defined with (3) replaced by

$$|x(t, j)|_{\mathcal{A}} \leqslant \max\{\beta(|x(0, 0)|_{\mathcal{A}}, t, j), \gamma(\|w\|)\}, \tag{4}$$

with $\beta \in \mathcal{KLL}$, see, e.g., [17,23]. However, since we are primarily interested in the evolution of the state x with respect to the continuous time t (irrespective of the jump counter j), we will adopt the ISS definition based on (3). Interestingly, for the NCSs considered in this paper, the two definitions are equivalent, as we will show in Remark 6.11.

In the following definition, $\mathcal{S}$ denotes the projection of the closure of $\mathcal{F} \cup \mathcal{J} \cup (G(\mathcal{J}, \mathcal{W}) \times \mathcal{W})$ onto $\mathcal{X}$, cf. [25].

Definition 2.5 (*ISS Lyapunov Function*)**.** Let $V : \mathcal{X} \to \mathbb{R}_{\geqslant 0}$ be locally Lipschitz and consider system (1). Under the assumption that (1) is persistently flowing, we call V an ISS Lyapunov function for the system (1) with respect to the nonempty closed set $\mathcal{A} \subset \mathcal{X}$ and input w if there exist $\alpha, \underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$ such that

1. for all $x \in \mathcal{S}$,
$$\underline{\alpha}(|x|_{\mathcal{A}}) \leqslant V(x) \leqslant \bar{\alpha}(|x|_{\mathcal{A}}) \tag{5a}$$

2. for all $(x, w) \in \mathcal{F}$ with $V(x) \geqslant \gamma(|w|)$,

$$\langle \zeta, \xi \rangle \leqslant -\alpha(|x|_{\mathcal{A}}) \quad \text{for all } \zeta \in \partial_x^C V(x), \xi \in F(x, w) \tag{5b}$$

3. for all $(x, w) \in \mathcal{J}$,

$$V(\xi) \leqslant \max\{V(x), \gamma(|w|)\} \quad \text{for all } \xi \in G(x, w). \tag{5c}$$

Note that Condition (5c) implies the more common and slightly weaker implication-form inequality

$$V(\xi) - V(x) \leqslant 0 \quad \text{for all } \xi \in G(x, w),\ (x, w) \in \mathcal{J} \text{ with } V(x) \geqslant \gamma(|w|). \tag{6}$$

As shown in [26], the existence of an ISS Lyapunov function satisfying (6) instead of (5c) does not necessarily imply ISS of the system (1) when the jump map G is discontinuous.

As already mentioned, by studying ISS with respect to a set $\mathcal{A}$ we can ignore states that can grow unbounded (like timers and counters), but which we do not care about. Additionally, the notion of practical input-to-state stability can also be captured in Definitions 2.3 and 2.5 by properly defining the set $\mathcal{A}$.

Theorem 2.6. *System (1) is ISS if it is persistently flowing and admits an ISS Lyapunov function.*

The proof of Theorem 2.6 is similar to the proofs of [23, Proposition 2.7] and [17, Proposition 2.9] (which extends [23, Proposition 2.7] to nonsmooth V). Here, the difference with [17,23] is that in (5c), we do not require a strict decrease of the ISS Lyapunov function V along jumps when $V(x) > \gamma(\|w\|)$, similar to, e.g., the works [1,27,28]. Instead, under the assumption of persistent flow, the ISS Lyapunov function has to flow after a finite number of jumps, which ensures a conditional strict decrease by (5b) corresponding to the ISS definition based on (3). Furthermore, as we have $x \in \mathcal{S}$ for all nontrivial solutions, we do not require (5a) to hold for all $x \in \mathcal{X}$, but only for all $x \in \mathcal{S}$, cf. [25].

To prove Theorem 2.6, we make use of the following lemma, which is a variation of [23, Lemma C.1] or [29, Lemma 4.4].

Lemma 2.7. *Let a hybrid arc $z : \operatorname{dom} z \to \mathbb{R}_{\geqslant 0}$ and a function $\breve{\alpha} \in \mathcal{K}_\infty$ satisfy*

1. *for all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom} z$, $\dot{z}(t, j) \leqslant -\breve{\alpha}(z(t, j))$,*
2. *for all $(t, j) \in \operatorname{dom} z$ such that $(t, j+1) \in \operatorname{dom} z$, $z(t, j+1) \leqslant z(t, j)$,*
3. *$\breve{\alpha}(s) \leqslant s$ for all $s \in \mathbb{R}_{\geqslant 0}$,*
4. $\sup_t \operatorname{dom} z = \infty$.

Then, $z(t, j) \leqslant \beta(z(0, 0), t)$ with $\beta \in \mathcal{KL}$ given by

$$\beta(r, t) = \gamma_{\breve{\alpha}}(r, t, 0) \tag{7}$$

where $\gamma_{\breve{\alpha}} \in \mathcal{KLL}$ is given by

$$\gamma_{\breve{\alpha}}(r, t, j) := \begin{cases} 0 & \text{for } r = 0, \\ \eta^{-1}(\eta(r) + t + j) & \text{for } r > 0, \end{cases} \tag{8}$$

and where η is a strictly decreasing differentiable function on $\mathbb{R}_{>0}$, given by

$$\eta(r) := -\int_1^r \frac{1}{\breve{\alpha}(s)} \mathrm{d}s. \tag{9}$$

Proof. See [29, Lemma 4.4]. □

Proof of Theorem 2.6. Consider any maximal solution pair (x, w) to (1). Define the set $\Omega_w := \{\xi \in \mathcal{X} \mid V(\xi) \leqslant \gamma(\|w\|)\}$ and define $\mathcal{T}_{(x,w)} := \sup\{\tau \geqslant 0 \mid x(t, j) \notin \Omega_w \text{ for } (t, j) \in \operatorname{dom} x \text{ with } 0 \leqslant t + j \leqslant \tau\}$. Note that Ω_w contains $\mathcal{A}$.

By definition, for all $(t, j) \in \operatorname{dom} x$ with $t + j < \mathcal{T}_{(x,w)}$ we have that $V(x(t, j)) > \gamma(\|w\|)$. In this case, by defining

$$\breve{\alpha}(s) := \min\{s, \alpha \circ \bar{\alpha}^{-1}(s)\} \tag{10}$$

and using (5a) and (5b), we have that

$$\langle \zeta, \xi \rangle \leqslant -\breve{\alpha}(V(x)) \quad \text{for all } \zeta \in \partial_x^C V(x),\ \xi \in F(x, w),$$

from which it follows that $\dot{V}(x(t, j)) \leqslant -\breve{\alpha}(V(x(t, j)))$ for almost all t such that $(t, j) \in \operatorname{dom} x = \operatorname{dom} V(x)$, see [18], and thus $V(x)$ and $\breve{\alpha}$ satisfy items 1 and 3 of Lemma 2.7. Additionally, item 2 follows from (5c), and item 4 follows by assumption. Application of Lemma 2.7 to $V(x)$ yields that $V(x(t, j)) \leqslant \beta(V(x(0, 0)), t)$ for all $(t, j) \in \operatorname{dom} x$ with $t + j < \mathcal{T}_{(x,w)}$, where β is given by (7)–(10), and thus we have that

$$|x(t, j)|_{\mathcal{A}} \leqslant \underline{\alpha}^{-1} \circ \beta(\bar{\alpha}(|x(0, 0)|_{\mathcal{A}}), t) \quad \text{for all } (t, j) \in \operatorname{dom} x \text{ with } t + j < \mathcal{T}_{(x,w)}. \tag{11}$$

Given any $(t,j) \in \operatorname{dom} x$ such that $x(t,j) \in \Omega_w$,

- if $(t,j+1) \in \operatorname{dom} x$, then (5c) gives $V(x(t,j+1)) \leqslant \gamma(\|w\|)$, and thus $x(t,j+1) \in \Omega_w$,
- if $(t,j+1) \notin \operatorname{dom} x$, then combining the continuity of $x(\cdot,j)$ and the fact that V is nonincreasing outside Ω_w implies that, for all $(t+\varepsilon,j) \in \operatorname{dom} x$ with $\varepsilon > 0$ we have that $x(t+\varepsilon,j) \in \Omega_w$.

By induction, the set Ω_w is strongly forward invariant, and thus it follows that

$$|x(t,j)|_{\mathcal{A}} \leqslant \underline{\alpha}^{-1} \circ \gamma(\|w\|) \quad \text{for all } (t,j) \in \operatorname{dom} x \text{ with } t+j \geqslant \mathcal{T}_{(x,w)}. \tag{12}$$

The combination of (11) and (12) gives

$$|x(t,j)|_{\mathcal{A}} \leqslant \max\left\{\underline{\alpha}^{-1} \circ \beta(\bar{\alpha}(|x(0,0)|_{\mathcal{A}}),t),\, \underline{\alpha}^{-1} \circ \gamma(\|w\|)\right\} \quad \text{for all } (t,j) \in \operatorname{dom} x, \tag{13}$$

which proves (3). Lastly, the system is persistently flowing by assumption, which completes the proof. □

3. Networked control setup and problem formulation

In this section we introduce the large-scale networked control setup with multiple communication networks that we consider in this paper, give the problem formulation, and sketch our approach to solving the problem.

3.1. Networked control setup

We consider the networked control setup as shown in Fig. 1, in which $N_{\mathcal{P}}$ physically coupled nonlinear continuous-time plants are controlled by $N_{\mathcal{P}}$ local controllers over a collection of $N_{\mathcal{N}}$ communication networks.

First, let $N = N_{\mathcal{P}} + N_{\mathcal{N}}$ and define the sets

$$\begin{aligned} \bar{N}_{\mathcal{P}} &:= \{1,2,\ldots,N_{\mathcal{P}}\} \\ \bar{N}_{\mathcal{N}} &:= \{N_{\mathcal{P}}+1, N_{\mathcal{P}}+2,\ldots,N\} \\ \bar{N} &:= \bar{N}_{\mathcal{P}} \cup \bar{N}_{\mathcal{N}} = \{1,2,\ldots,N\}. \end{aligned}$$

The dynamics of the continuous-time plants $\mathcal{P}_i$, $i \in \bar{N}_{\mathcal{P}}$, are given by

$$\mathcal{P}_i : \begin{cases} \dot{x}_i^p &= f_i^p(x^p, \hat{u}_i, w), \\ y_i &= z_i^p(x_i^p), \end{cases} \tag{14}$$

where x_i^p is the state of plant $\mathcal{P}_i$, $x^p = \left(x_1^p, x_2^p, \ldots, x_{N_{\mathcal{P}}}^p\right)$, y_i is the output of plant $\mathcal{P}_i$, $\hat{u}_i$ is the control input received by plant $\mathcal{P}_i$ and w is a disturbance. The dependence of f_i^p on the complete vector x^p describes the physical coupling of the plants. The controllers $\mathcal{C}_i$, $i \in \bar{N}_{\mathcal{P}}$, are given by

$$\mathcal{C}_i : \begin{cases} \dot{x}_i^c &= f_i^c(x_i^c, \hat{y}_i, w), \\ u_i &= z_i^c(x_i^c), \end{cases} \tag{15}$$

where x_i^c is the state of controller $\mathcal{C}_i$, u_i the controller output, and $\hat{y}_i$ is the plant output received by controller $\mathcal{C}_i$, $i \in \bar{N}_{\mathcal{P}}$. It should be noted that when the communication between the plant and controller is perfect (e.g., when the plant and controller are connected via dedicated wired point-to-point links) we have that $\hat{y}_i = y_i$ and $\hat{u}_i = u_i$. However, when the plant and controller are connected via a (shared) packed-based communication network, we have that in general $\hat{y}_i \neq y_i$ and $\hat{u}_i \neq u_i$.

The plant outputs y_i and control inputs u_i, $i \in \bar{N}_{\mathcal{P}}$, are transmitted via $N_{\mathcal{N}}$ communication networks $\mathcal{N}_k$, $k \in \bar{N}_{\mathcal{N}}$, which are operating independently of each other. We allow that multiple control loops are closed over one single network, or that a single control loop is closed over several different networks. This means that the output vector y_i might be transmitted via multiple networks. For notational simplicity, we assume that each *component* in y_i and u_i, $i \in \bar{N}_{\mathcal{P}}$, is transmitted via *exactly one* communication network.

Remark 3.1. Redundant transmissions over multiple communication networks can be easily incorporated into the framework proposed here. However, as the redundant signals will not be exactly equal due to differences in delays and sampling times, it needs to be specified how the controller or actuator should deal with this discrepancy. Regarding the framework proposed in this paper, adding a redundant transmission over a separate communication network will only have an effect on the ISS gain related to that network, and from that point onwards all presented results can be applied without modifications.

We denote by y_i^k the part of y_i that is transmitted over network $\mathcal{N}_k$. Similarly, by u_i^k we denote the part of u_i that is transmitted over network $\mathcal{N}_k$. Thus, network $\mathcal{N}_k$, $k \in \bar{N}_{\mathcal{N}}$, transmits the vector (y^k, u^k), where $y^k = (y_1^k, y_2^k, \ldots, y_{N_{\mathcal{P}}}^k)$ and

$u^k = (u_1^k, u_2^k, \dots, u_{N_\mathcal{P}}^k)$. We introduce the network-induced errors $e_k^y := \hat{y}^k - y^k$ and $e_k^u := \hat{u}^k - u^k$ (where $\hat{y}_k$ and $\hat{u}_k$ evolve according to (21) and (22), as will be explained below), which we group per network in $e_k := (e_k^y, e_k^u)$, $k \in \bar{N}_\mathcal{N}$, and also into one vector $e := \left(e_{N_\mathcal{P}+1}, \dots, e_N\right) \in \mathbb{R}^{n_e}$. As a result, we can write

$$\hat{y}_i = y_i + Y_i e, \quad i \in \bar{N}_\mathcal{P}, \tag{16}$$

$$\hat{u}_i = u_i + U_i e, \quad i \in \bar{N}_\mathcal{P}, \tag{17}$$

where $Y_i \in \{0, 1\}^{n_{y_i} \times n_e}$ and $U_i \in \{0, 1\}^{n_{u_i} \times n_e}$ are appropriately chosen matrices.

Each communication network $\mathcal{N}_k$, $k \in \bar{N}_\mathcal{N}$, has its own sequence of transmission times $\left\{t_{\kappa_k}^k\right\}_{\kappa_k \in \mathbb{N}}$. At each transmission time $t_{\kappa_k}^k$, the *medium access control* (MAC) protocol allows one of the nodes in the network to transmit its corresponding entries in $\left(y^k(t_{\kappa_k}^k), u^k(t_{\kappa_k}^k)\right)$ over the network. This information arrives after a delay of $\tau_{\kappa_k}^k$ time units at its destination, and results in an update of the corresponding values in $\left(\hat{y}^k(t_{\kappa_k}^k + \tau_{\kappa_k}^k), \hat{u}^k(t_{\kappa_k}^k + \tau_{\kappa_k}^k)\right)$. For each network $\mathcal{N}_k$, $k \in \bar{N}_\mathcal{N}$, we assume that the time between two subsequent transmissions is upper bounded by a *maximum allowable transmission interval* (MATI) τ_{mati}^k, and that the delays are upper bounded by a *maximum allowable delay* (MAD) τ_{mad}^k.

Assumption 3.2. The transmission times $\left\{t_{\kappa_k}^k\right\}_{\kappa_k \in \mathbb{N}}$ satisfy

$$0 \leqslant t_0^k \leqslant \tau_{mati}^k, \tag{18}$$

and

$$\delta_k \leqslant t_{\kappa_k+1}^k - t_{\kappa_k}^k \leqslant \tau_{mati}^k, \tag{19}$$

for all $\kappa_k \in \mathbb{N}$, where $\delta_k \in (0, \tau_{mati}^k]$, $k \in \bar{N}_\mathcal{N}$, is arbitrary. Furthermore, the communication delays $\tau_{\kappa_k}^k$ satisfy

$$0 \leqslant \tau_{\kappa_k}^k \leqslant \min\{\tau_{mad}^k, t_{\kappa_k+1}^k - t_{\kappa_k}^k\}, \tag{20}$$

for all $\kappa_k \in \mathbb{N}$.

The role of $\delta_k > 0$ is to prevent Zeno behavior and can be taken arbitrarily small. However, due to hardware limitations in reality a positive lower bound $\delta_k > 0$ on the inter-transmission times always exists. Note that in this paper we only consider the small-delay case, as Assumption 3.2 implies that in each network, all previous packets have arrived at their destination before a new packet is transmitted.

Furthermore, we assume that the communication networks operate in a zero-order hold (ZOH) fashion, meaning that the values of $\hat{y}^k$ and $\hat{u}^k$ are held constant in between the update times $t_{\kappa_k}^k + \tau_{\kappa_k}^k$ and $t_{\kappa_k+1}^k + \tau_{\kappa_k+1}^k$ for all $\kappa_k \in \mathbb{N}$, i.e.,

$$\dot{\hat{y}}^k = 0, \qquad \dot{\hat{u}}^k = 0. \tag{21}$$

The updates of $\hat{y}^k$ and $\hat{u}^k$ can now be described by

$$\hat{y}^k\left(\left(t_{\kappa_k}^k + \tau_{\kappa_k}^k\right)^+\right) = y^k\left(t_{\kappa_k}^k\right) + h_k^y\left(\kappa_k, e_k\left(t_{\kappa_k}^k\right)\right), \tag{22a}$$

$$\hat{u}^k\left(\left(t_{\kappa_k}^k + \tau_{\kappa_k}^k\right)^+\right) = u^k\left(t_{\kappa_k}^k\right) + h_k^u\left(\kappa_k, e_k\left(t_{\kappa_k}^k\right)\right), \tag{22b}$$

where the functions h_k^u and h_k^y are update functions that are related to the MAC protocol (e.g., Try-Once-Discard or Round Robin) of the network that determines on the basis of the transmission counter κ_k and the network-induced error $e_k(t_{\kappa_k}^k)$ which node is granted access to the network, see also [1,2]. We will refer to $h_k = (h_k^y, h_k^u)$ as the network protocol corresponding to network $\mathcal{N}_k$, $k \in \bar{N}_\mathcal{N}$.

Remark 3.3. Note that the presented control setup contains as special cases the class of systems which employ only one global communication network (in which case $N_\mathcal{N} = 1$ and $e_{N_\mathcal{P}+1} = (\hat{y}_1 - y_1, \dots, \hat{y}_{N_\mathcal{P}} - y_{N_\mathcal{P}}, \hat{u}_1 - u_1, \dots, \hat{u}_{N_\mathcal{P}} - u_{N_\mathcal{P}})$), see, e.g., [1–3,27], and the class of systems described in [7] in which each control loop has its own communication network (in which case $N_\mathcal{N} = N_\mathcal{P}$ and $e_{N_\mathcal{P}+i} = (\hat{y}_i - y_i, \hat{u}_i - u_i)$, $i \in \bar{N}_\mathcal{P}$).

3.2. Problem formulation and main ideas

In this paper, we would like to find for each communication network $\mathcal{N}_k$, $k \in \bar{N}_\mathcal{N}$, upper bounds on the MATI τ_{mati}^k and the MAD τ_{mad}^k, such that the overall NCS of Fig. 1 is ISS, and has an ISS Lyapunov function V with a certain ISS-gain γ and convergence rate α.

Remark 3.4. The communication networks $\mathcal{N}_k$, $k \in \bar{N}_\mathcal{N}$, might be of a different nature in the sense that they might employ different network protocols, or even different communication media. For example, some networks might be wired, while

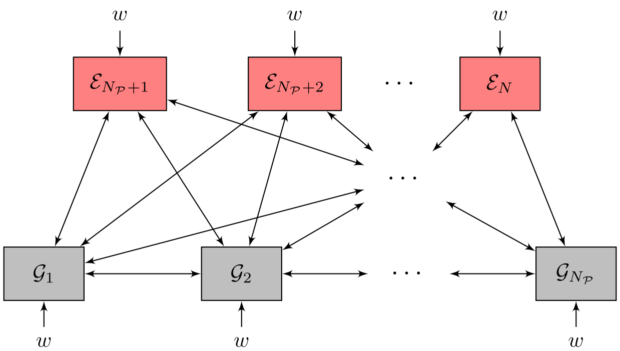


Fig. 2. The networked control setup of Fig. 1, viewed as an interconnection of $N_\mathcal{P}$ controlled subsystems $\mathcal{G}_i$, $i \in \bar{N}_\mathcal{P}$, and $N_\mathcal{N}$ network-induced error systems $\mathcal{E}_k$, $k \in \bar{N}_\mathcal{N}$.

others might be wireless. As such, it is natural to have a different quality-of-service characteristics (e.g., MATI and MAD) for each individual communication network.

In the remainder of the paper, we will derive a MATI and a MAD for each individual communication network in the following way.

As we will show in the next section, the plant-controller pairs $(\mathcal{P}_i, \mathcal{C}_i)$, $i \in \bar{N}_\mathcal{P}$, can be modeled as (continuous-time) controlled subsystems $\mathcal{G}_i$, and the communication networks $\mathcal{N}_k$, $k \in \bar{N}_\mathcal{N}$, induce communication errors which can be described by hybrid subsystems $\mathcal{E}_k$. In this way, the networked control setup of Fig. 1 can be modeled as an interconnection of N hybrid subsystems as shown in Fig. 2. This is a crucial step, as our main results (summarized in Procedure 6.12) hold for any NCS which can be captured by Fig. 2. Interestingly, this also implies that our main results are not limited to the setup of Fig. 1, and apply to *any* NCS setup leading to an interconnection as in Fig. 2. However, in this paper we focus on the setup of Fig. 1 for concreteness. In addition, it already is a generalization of a number of different NCS setups considered in the literature, including those considered in [1,7,8], see Remark 3.3, and it keeps the notation more transparent.

Once the NCS is transformed into the interconnection of Fig. 2, we provide assumptions in Section 6 that guarantee that all closed-loop subsystems $\mathcal{G}_i$, $i \in \bar{N}_\mathcal{P}$, and all network-induced error systems $\mathcal{E}_k$, $k \in \bar{N}_\mathcal{N}$, are ISS with respect to the external disturbance w and all states of the subsystems in the interconnection. Moreover, based on the results in Section 6 we can construct ISS Lyapunov functions for each of these hybrid subsystems. These ISS Lyapunov functions for the hybrid subsystems can be used to construct an ISS Lyapunov function for the overall system as long as a small-gain condition holds, as will be explained in Section 5. Interestingly, it turns out that for each error system $\mathcal{E}_k$, there is a tradeoff between its ISS gains and convergence rate and the quality-of-service in terms of MATI and MAD bounds. We exploit this useful property as follows. As long as the small-gain condition holds, we can improve (or degrade) the quality-of-control of the overall system (in terms of α and γ from Definition 2.5) by improving (or relaxing) the quality-of-service of the communication networks. Alternatively, when relaxing the quality-of-service of one communication network, we can retain the quality-of-control of the overall system by improving the quality-of-service of one or more of the other communication networks. Finally, once the ISS gains and convergence rate of a network-induced error system is fixed, the MATI and MAD of the corresponding communication network can be maximized in a Pareto-optimal sense. Our general framework will formally show these intuitive and insightful tradeoffs.

4. Hybrid modeling of the networked control setup

In this section we derive hybrid models for the controlled subsystems $\mathcal{G}_i$, $i \in \bar{N}_\mathcal{P}$, and the network-induced error dynamics $\mathcal{E}_k$, $k \in \bar{N}_\mathcal{N}$, of Fig. 2.

In this section, we show that the NCS setup described in Section 3, can be described by a collection of N interconnected hybrid subsystems $\mathcal{H}_i$, $i \in \bar{N} := \{1, 2, \ldots, N\}$ given by

$$\mathcal{H}_i : \begin{cases} \dot{x}_i \in F_i(x, w), & (x, w) \in \mathcal{F}_i, \\ x_i^+ \in G_i(x, w), & (x, w) \in \mathcal{J}_i, \end{cases} \tag{23}$$

where $w \in \mathcal{W} \subset \mathbb{R}^{n_w}$ is the disturbance, $x_i \in \mathcal{X}_i \subseteq \mathbb{R}^{n_{x_i}}$ is the state of subsystem $\mathcal{H}_i$, and $x = (x_1, x_2, \ldots, x_N) \in \mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_N \subseteq \mathbb{R}^{n_x}$ is the state of the overall system, with $n_x = \sum_{i=1}^N n_{x_i}$. Moreover, $F_i : \mathcal{X} \times \mathcal{W} \rightrightarrows \mathbb{R}^{n_{x_i}}$ is the flow map, $G_i : \mathcal{X} \times \mathcal{W} \rightrightarrows \mathcal{X}_i$ is the jump map, and $\mathcal{F}_i, \mathcal{J}_i \subseteq \mathcal{X} \times \mathcal{W}$ are the flow set and jump set of subsystem $\mathcal{H}_i$, $i \in \bar{N}$. Note that the dynamics of (23) are dependent on the full state x, indicating the physical coupling of the subsystems, and that the disturbance w is the same for all subsystems, which is not a restriction, see, e.g., [11].

All hybrid subsystems in the interconnection satisfy the following regularity conditions.

Assumption 4.1. All subsystems $\mathcal{H}_i$, $i \in \bar{N}$, given by (23) satisfy the following:

(B0) $\mathcal{X}_i$, $i \in \bar{N}$, are open sets and $\mathcal{W}$ is closed;
(B1) $\mathcal{F}_i$ and $\mathcal{J}_i$ are relatively closed in $\mathcal{X}_i \times \mathcal{W}$ for all $i \in \bar{N}$;
(B2) F_i is outer semicontinuous and locally bounded, and $F_i(x, w)$ is nonempty and convex for all $(x, w) \in \mathcal{F}_i$;
(B3) G_i is outer semicontinuous, and $G_i(x, w)$ is nonempty for all $(x, w) \in \mathcal{J}_i$.

We will start by deriving a model of the form (23) for the controlled subsystems $\mathcal{G}_i$, $i \in \bar{N}_{\mathcal{P}}$. However, in order to do so, we first need to define the state x of the overall system, and thus also the states of all subsystems. For each $i \in \bar{N}_{\mathcal{P}}$ we define $x_i := (x_i^p, x_i^c) \in \mathbb{R}^{n_{x_i}} =: \mathcal{X}_i$ as the state of the controlled subsystem $\mathcal{G}_i$. To model the network-induced error dynamics, we introduce the auxiliary variables $s_k \in \mathbb{R}^{n_{e_k}}$, $\kappa_k \in \mathbb{N}$, $\tau_k \in \mathbb{R}_{\geqslant 0}$, $l_k \in \{0, 1\}$ for each $k \in \bar{N}_{\mathcal{N}}$, similar to [1]. How these auxiliary variables are used will be clarified later. For now, for each $k \in \bar{N}_{\mathcal{N}}$ we define $x_k := (e_k, s_k, \tau_k, \kappa_k, l_k) \in \mathbb{R}^{n_{e_k}} \times \mathbb{R}^{n_{e_k}} \times \mathbb{R}^3 =: \mathcal{X}_k$ as the state of the networked-induced error system $\mathcal{E}_k$. The state of the overall system (1) is then given by $x := (x_1, x_2, \ldots, x_N)$.

The controlled subsystems $\mathcal{G}_i$, $i \in \bar{N}_{\mathcal{P}}$, can now be modeled as

$$\mathcal{G}_i : \dot{x}_i = F_i(x, w), \quad i \in \bar{N}_{\mathcal{P}}, \tag{24}$$

where F_i is given by

$$F_i(x, w) = \begin{bmatrix} f_i^p\left(x^p, z_i^c(x_i^c) + U_i e, w\right) \\ f_i^c\left(x_i^c, z_i^p(x_i^p) + Y_i e, w\right) \end{bmatrix}. \tag{25}$$

As the controlled subsystems $\mathcal{G}_i$ have no discrete dynamics, we have that $\mathcal{F}_i = \mathcal{X}_i \times \mathcal{W}$, $\mathcal{J}_i = \emptyset$, and $G_i(x, w) = \emptyset$.

The modeling of the network-induced error systems $\mathcal{E}_k$, $k \in \bar{N}_{\mathcal{P}}$, builds upon the ideas in [1–3,27]. Each network $\mathcal{N}_k$, $k \in \bar{N}_{\mathcal{N}}$, consists of several (actuator, controller or sensor) nodes, with each node corresponding to one or more elements in $\left(y^k, u^k\right)$. At transmission times $t_{\kappa_k}^k$, $\kappa_k \in \mathbb{N}$, (parts of) the vector (y^k, u^k) are transmitted over network $\mathcal{N}_k$, $k \in \bar{N}_{\mathcal{N}}$. Here, $\kappa_k \in \mathbb{N}$ is a counter keeping track of the transmission number of network $\mathcal{N}_k$, $k \in \bar{N}_{\mathcal{N}}$, and the transmission times satisfy $0 \leqslant t_0^k < t_1^k < \cdots$.

The resets of e_k^y at the update times $t_{\kappa_k}^k + \tau_{\kappa_k}^k$, $\kappa_k \in \mathbb{N}$, can now be described by [1]:

$$\begin{aligned} e_k^y\left((t_{\kappa_k}^k + \tau_{\kappa_k}^k)^+\right) &= \hat{y}^k\left((t_{\kappa_k}^k + \tau_{\kappa_k}^k)^+\right) - y^k\left(t_{\kappa_k}^k + \tau_{\kappa_k}^k\right) \\ &= y^k\left(t_{\kappa_k}^k\right) + h_k^y\left(\kappa_k, e_k\left(t_{\kappa_k}^k\right)\right) - y^k\left(t_{\kappa_k}^k + \tau_{\kappa_k}^k\right) \\ &= h_k^y\left(\kappa_k, e_k\left(t_{\kappa_k}^k\right)\right) + \underbrace{y^k\left(t_{\kappa_k}^k\right) - \hat{y}^k\left(t_{\kappa_k}^k\right)}_{-e_k^y\left(t_{\kappa_k}^k\right)} + \underbrace{\hat{y}^k\left(t_{\kappa_k}^k + \tau_{\kappa_k}^k\right) - y^k\left(t_{\kappa_k}^k + \tau_{\kappa_k}^k\right)}_{e_k^y\left(t_{\kappa_k}^k + \tau_{\kappa_k}^k\right)} \\ &= h_k^y\left(\kappa_k, e_k\left(t_{\kappa_k}^k\right)\right) - e_k^y\left(t_{\kappa_k}^k\right) + e_k^y\left(t_{\kappa_k}^k + \tau_{\kappa_k}^k\right), \end{aligned}$$

where $\hat{y}^k\left(t_{\kappa_k}^k\right) = \hat{y}^k\left(t_{\kappa_k}^k + \tau_{\kappa_k}^k\right)$ in the third equality follows from (21), and where e_k^y is assumed to be left-continuous. A similar derivation holds for e_k^u. This leads to

$$\dot{e}_k(t) = g_k(x, w), \quad \text{for almost all } t \in [t_{\kappa_k}^k, t_{\kappa_k+1}^k] \tag{26a}$$

$$e_k\left((t_{\kappa_k}^k + \tau_{\kappa_k}^k)^+\right) = h_k\left(\kappa_k, e_k\left(t_{\kappa_k}^k\right)\right) - e_k\left(t_{\kappa_k}^k\right) + e_k\left(t_{\kappa_k}^k + \tau_{\kappa_k}^k\right), \tag{26b}$$

where $g_k(x, w) = (-\dot{y}^k, -\dot{u}^k)$.

To model the network-induced error dynamics in a hybrid system formulation [30], we make use of the auxiliary variables s_k, τ_k, κ_k and l_k as already mentioned above. For each network $\mathcal{N}_k$, $k \in \bar{N}_{\mathcal{N}}$, the variable s_k is used to store the value of $h_k\left(\kappa_k, e_k\left(t_{\kappa_k}^k\right)\right) - e_k\left(t_{\kappa_k}^k\right)$ at time $t_{\kappa_k}^k$, κ_k is the counter keeping track of the transmission number as introduced before, τ_k is a timer to constrain both the transmission interval and the delay and essentially model Assumption 3.2, and l_k is a Boolean variable keeping track of whether the next event is a transmission or an update event. To be precise, when $l_k = 0$ the next event will be a transmission event (at times $\left\{t_{\kappa_k}^k\right\}_{\kappa_k \in \mathbb{N}}$) and when $l_k = 1$ the next event will be an update event (at times $\left\{t_{\kappa_k}^k + \tau_{\kappa_k}^k\right\}_{\kappa_k \in \mathbb{N}}$). Further details on this modeling approach can be found in [1].

As we have that $x_k = (e_k, s_k, \tau_k, \kappa_k, l_k)$ for $k \in \bar{N}_{\mathcal{N}}$ and $x = (x_1, \ldots, x_N)$, the network-induced error dynamics for $\mathcal{N}_k$, $k \in \bar{N}_{\mathcal{N}}$, is given by the hybrid model [30]

$$\mathcal{E}_k : \begin{cases} \dot{x}_k = F_k(x, w), & x \in \mathcal{F}_k, \quad \text{(a)} \\ x_k^+ = G_k(x_k), & x \in \mathcal{J}_k \quad \text{(b)}. \end{cases} \tag{27}$$

In (27)(a) the flow map F_k is given by

$$F_k(x, w) = (g_k(x, w), 0, 1, 0, 0) \tag{28}$$

and the flow set $\mathcal{F}_k$ by

$$\mathcal{F}_k := \left\{(x, w) \in \mathcal{X} \times \mathcal{W} \;\middle|\; (l_k = 0 \wedge \kappa_k \in \mathbb{N} \wedge \tau_k \in [0, \tau^k_{mati}]) \vee (l_k = 1 \wedge \kappa_k \in \mathbb{N} \wedge \tau_k \in [0, \tau^k_{mad}])\right\}. \tag{29}$$

The jump equations in (27)(b) are given by the transmission resets (i.e., when $l_k = 0$)

$$G_k(e_k, s_k, \tau_k, \kappa_k, 0) = (e_k, h_k(\kappa_k, e_k) - e_k, 0, \kappa_k + 1, 1) \tag{30}$$

and the update resets (i.e., when $l_k = 1$)

$$G_k(e_k, s_k, \tau_k, \kappa_k, 1) = (s_k + e_k, -s_k - e_k, \tau_k, \kappa_k, 0). \tag{31}$$

Finally, the jump set $\mathcal{J}_k$ in (27)(b) is given by

$$\mathcal{J}_k := \left\{(x, w) \in \mathcal{X} \times \mathcal{W} \;\middle|\; (l_k = 0 \wedge \kappa_k \in \mathbb{N} \wedge \tau_k \in [\delta_k, \infty)) \vee (l_k = 1 \wedge \kappa_k \in \mathbb{N} \wedge \tau_k \in [0, \infty))\right\}. \tag{32}$$

See [1] for more details on this modeling approach, including the definition of the function g_k.

The networked control setup of Fig. 1 can thus be viewed as an interconnection of $N_{\mathcal{P}}$ controlled subsystems $\mathcal{G}_i$, $i \in \bar{N}_{\mathcal{P}}$, given by the continuous model (24)–(25) with states $x_i \in \mathcal{X}_i = \mathbb{R}^{n_{x_i}}$, and $N_{\mathcal{N}}$ network-induced error systems $\mathcal{E}_k$, $k \in \bar{N}_{\mathcal{N}}$, given by the hybrid model (27)–(32) with states $x_k \in \mathcal{X}_k = \mathbb{R}^{n_{e_k}} \times \mathbb{R}^{n_{e_k}} \times \mathbb{R}^3$, leading to the control setup as shown in Fig. 2.

It can be seen that the hybrid subsystems satisfy items B0, B1, and B3 of Assumption 4.1. Additionally, we assume that the flow maps $F_j, j \in \bar{N}$, are such that also item B2 of Assumption 4.1 is satisfied, which is the case when the functions f^p_i, z^p_i, f^c_i, and z^c_i, $i \in \bar{N}_{\mathcal{P}}$, are sufficiently smooth.

We will show in Section 6 that each controlled subsystem $\mathcal{G}_i$, $i \in \bar{N}_{\mathcal{P}}$, and each network-induced error system $\mathcal{E}_k$, $k \in \bar{N}_{\mathcal{N}}$, is ISS with respect to the other subsystems' states and the disturbance w. In the next section, we will show how to connect these hybrid subsystems, and how to guarantee that the overall interconnected system is ISS with respect to the disturbance w.

5. Input-to-state stability of interconnected systems

Now that we have a hybrid model $\mathcal{H}_i$ of the form (23) for each subsystem in the interconnection of Fig. 2, in this section we will connect these models in a suitable way to arrive at a model of the form (1) of the overall system. Additionally, in this section we present a variation of the small-gain theorem of [17], which can be used to guarantee that the complete system of interconnected hybrid subsystems is ISS with respect to the disturbance w.

The interconnection of subsystems $\mathcal{H}_i$, $i \in \bar{N}$, can be described as follows. Define

$$\mathcal{F} = \bigcap_{i \in \bar{N}} \mathcal{F}_i \tag{33}$$

as the overall system's flow set and

$$\mathcal{J} = \bigcup_{i \in \bar{N}} \mathcal{J}_i \tag{34}$$

as the overall system's jump set. This definition is natural as the overall system can flow as long all subsystems can flow, and the overall system can jump if at least one subsystem can jump. The flow map can then be defined by

$$F(x, w) = F_1(x, w) \times F_2(x, w) \times \cdots \times F_N(x, w) \tag{35}$$

and the jump map can be defined by

$$G(x, w) = \bigcup_{\substack{I \in \mathcal{P}(J(x,w)),\\ I \neq \emptyset}} \hat{G}_I(x, w), \tag{36}$$

where $J(x, w) := \{j \in \bar{N} \mid (x, w) \in \mathcal{J}_j\}$ is the index set of all subsystems that can jump, $\mathcal{P}(J(x, w))$ denotes the power set of $J(x, w)$,[1] and

$$\hat{G}_I(x, w) = \hat{G}_{1,I}(x, w) \times \hat{G}_{2,I}(x, w) \times \cdots \times \hat{G}_{N,I}(x, w), \tag{37}$$

$$\hat{G}_{i,I}(x, w) = \begin{cases} G_i(x, w) & \text{if } i \in I, \\ \{x_i\} & \text{else.} \end{cases} \tag{38}$$

This definition of G as in (36) guarantees that a jump in the overall system corresponds to a jump in *at least one* of the subsystems. In fact, the choice of $I \in \mathcal{P}(J(x, w))$ in (36) indicates which of the subsystems will jump. If multiple subsystems are in their jump set, they can jump simultaneously or independently from each other.

[1] I.e., the collection of all subsets of $J(x, w)$.

Remark 5.1. If Assumption 4.1 holds, then the interconnection described by (33)–(38) satisfies Assumption 2.1.

Finally, we assume that the subsystems (23) are such that the overall system (1) given by the interconnection of subsystems (23) is persistently flowing, and require that all subsystems $\mathcal{H}_i$, $i \in \bar{N}$, satisfy the following condition, in which $\mathcal{S}_i$ denotes the projection of the closure of $\mathcal{F}_i \cup \mathcal{J}_i \cup (G_i(\mathcal{J}_i, \mathcal{W}) \times \mathcal{W})$ onto $\mathcal{X}_i$.

Condition 5.2. *Each subsystem $\mathcal{H}_i$, $i \in \bar{N}$, given by (23) has an associated ISS Lyapunov function $V_i : \mathcal{X}_i \to \mathbb{R}_{\geqslant 0}$ with respect to a nonempty closed set $\mathcal{A}_i$ and inputs w and x_j, $j \in \bar{N} \setminus \{i\}$, in the sense that there exist $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $j \in \bar{N}$, with $\gamma_{ii} \equiv 0$, $\gamma_{iw} \in \mathcal{K} \cup \{0\}$ and $\alpha_i, \underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ such that*

1. *for all $x_i \in \mathcal{S}_i$ we have*
$$\underline{\alpha}_i(|x_i|_{\mathcal{A}_i}) \leqslant V_i(x_i) \leqslant \bar{\alpha}_i(|x_i|_{\mathcal{A}_i}), \tag{39a}$$
2. *for all $(x, w) \in \mathcal{F}_i$ satisfying $V_i(x_i) \geqslant \max\{\max_{j \in \bar{N}\setminus\{i\}}\{\gamma_{ij}(V_j(x_j))\}, \gamma_{iw}(|w|)\}$ we have*
$$\langle \zeta_i, \xi_i \rangle \leqslant -\alpha_i(|x_i|_{\mathcal{A}_i}), \quad \text{for all } \zeta_i \in \partial^C_{x_i} V_i(x_i), \xi_i \in F_i(x, w) \tag{39b}$$
3. *for all $(x, w) \in \mathcal{J}_i$ we have*
$$V_i(\xi_i) \leqslant \max\left\{V_i(x_i), \max_{j \in \bar{N}\setminus\{i\}}\{\gamma_{ij}(V_j(x_j))\}, \gamma_{iw}(|w|)\right\}, \quad \text{for all } \xi_i \in G_i(x, w). \tag{39c}$$

Remark 5.3. The functions $\gamma_{ij} \in \mathcal{K}_\infty$, $\gamma_{iw} \in \mathcal{K}$ in (39b) and (39c) are assumed to be the same. Allowing different functions $\tilde{\gamma}_{ij}$ in (39b) and $\bar{\gamma}_{ij}$ in (39c) would make sense in terms of conservatism, but will not be considered in this paper for reasons of exposition. Clearly, if we would take different functions $\tilde{\gamma}_{ij} \neq \bar{\gamma}_{ij}$ in (39b) and (39c), respectively, the function $\gamma_{ij} := \max\{\tilde{\gamma}_{ij}, \bar{\gamma}_{ij}\}$, obviously satisfies (39b) and (39c).

The functions $\gamma_{ij} \in \mathcal{K}_\infty$, $\gamma_{iw} \in \mathcal{K}$ are called (nonlinear) gains. Following [13,17], we define the map $\Gamma : \mathbb{R}^N_{\geqslant 0} \to \mathbb{R}^N_{\geqslant 0}$ by

$$\Gamma(s) := \begin{bmatrix} \max_{j \in \bar{N}} \gamma_{1j}(s_j) \\ \vdots \\ \max_{j \in \bar{N}} \gamma_{Nj}(s_j) \end{bmatrix}. \tag{40}$$

Definition 5.4 (*Small-Gain Condition [11]*)**.** The map Γ defined in (40) satisfies the *small-gain condition* if

$$\Gamma(s) \not\geqslant s \quad \text{for all } s \in \mathbb{R}^N_{\geqslant 0}, s \neq 0. \tag{41}$$

The small-gain condition (41) can be checked using the following result.

Lemma 5.5 (*[31, Theorem 6.4]*)**.** *For Γ defined in (40), the small-gain condition (41) is equivalent to the cycle condition*

$$\gamma_{i_0 i_1} \circ \gamma_{i_1 i_2} \circ \cdots \circ \gamma_{i_{k-1} i_k}(s) < s \tag{42}$$

for all $s \in \mathbb{R}_{>0}$, $i_0, i_1, \ldots, i_k \in \bar{N}$ with $i_0 = i_k$ and $k \in \bar{N}$.

Remark 5.6. In Condition 5.2, we use the 'max' formulation of ISS, i.e., the gains γ_{ij} in (39b) and (39c) are aggregated via maximization. Alternatively, one could generalize the inequalities (39b) and (39c) using monotone aggregation functions instead of maximization, see [13]. For instance, one could use summation of gains. In this case also the map Γ in (40) has to be changed, and an equivalent characterization of the small-gain condition (41) via the cycle condition (42) (which in some cases is easier to check) is no longer possible.

The next result is a small-gain result that constructs an ISS Lyapunov function for the interconnected system based on the ISS Lyapunov functions of the subsystems. We note that this result is close in nature to the one proposed in [17, Theorem 3.6], which extends the small-gain theorem proposed in [13] to hybrid systems. However, a distinctive difference with [17] is that the authors in [17] consider a different class of hybrid systems as they require that all jump sets coincide, i.e., $\mathcal{J}_i = \mathcal{J}_j$ for all i, j, and that all subsystems in (23) can jump simultaneously only, in order to rule out the case that solutions "freeze", see [17, Lemma 2.1]. These requirements are not satisfied here and for the NCS applications we have in mind. By assuming persistence of flow (which is a reasonable assumption for the NCS applications we have in mind, and many other applications), we avoid the freeze-problem of [17] and do not need that all subsystems jump simultaneously. As such, we consider a different class of interconnected systems than in [17].

Lemma 5.7. *Consider system (1) as an interconnection of subsystems (23) following (33), (34), (35), and (36). Suppose that system (1) is persistently flowing and that Condition 5.2 holds. If Γ given by (40) satisfies the small-gain condition (41), then the system (1) is ISS from w to x with respect to the set $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_N$. Furthermore, there exists $\sigma \in \mathcal{K}_\infty^N$ such that the function $V : \mathcal{X} \to \mathbb{R}_{\geqslant 0}$ defined by*

$$V(x) = \max_{i\in\bar{N}} \sigma_i^{-1}(V_i(x_i)) \tag{43}$$

is an ISS Lyapunov function for the overall system (1).

Proof. The proof follows the lines of the proof of [17, Theorem 3.6]. Therefore, we only give a sketch of it here.

First note that the small-gain condition (41) implies the existence of a so-called Ω-path $\sigma \in \mathcal{K}_\infty^N$, see [13, Theorem 5.2-(iii)]. In particular, we have

1. for every $i \in \bar{N}$, the function $\sigma_i^{-1} \in \mathcal{K}_\infty$ is locally Lipschitz on $(0, \infty)$;
2. for every $i \in \bar{N}$ and every compact set $K \subset (0, \infty)$ there exist $0 < k_1 < k_2 < \infty$ such that for all $r \in K$ we have
$$0 < k_1 \leq (\sigma_i^{-1})'(r) \leq k_2; \tag{44}$$
3. for all $r > 0$ we have
$$\Gamma(\sigma(r)) < \sigma(r). \tag{45}$$

We will now show that V defined in (43) satisfies (5) for suitable $\alpha, \underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$. From the proof of [17, Theorem 3.6] we immediately get (5a) and (5b). So here, we only have to prove (5c).

In contrast to [17, Theorem 3.6] we do not assume that all subsystems' jump sets coincide, since (5c) does not require a strict decrease of V at jumps.

Define $\rho(s) := \max_{i,j\in\bar{N}} \sigma_i^{-1} \circ \gamma_{ij} \circ \sigma_j(s)$ and $\gamma(s) := \max_{i\in\bar{N}} \sigma_i^{-1}(\gamma_{iw}(s))$ for all $s \geqslant 0$. Note that (45) implies $\rho(s) < s$ for all $s > 0$, which follows from the definition of Γ in (40) and by considering (45) componentwise.

Eventually, we have that for all $(x, w) \in \mathcal{J}$, all $I \in \mathcal{P}(J(x, w))$, and all $\xi \in \hat{G}_I(x, w)$

$$\begin{aligned}
V(\xi) &= \max_{i\in\bar{N}} \sigma_i^{-1}(V_i(\xi_i)) \\
&\leqslant \max\left\{\max_{i\in\bar{N}\setminus I} \sigma_i^{-1}(V_i(x_i)), \max_{i\in I} \sigma_i^{-1} \max\left\{V_i(x_i), \max_{j\in\bar{N}\setminus\{i\}} \gamma_{ij}(V_j(x_j)), \gamma_{iw}(|w|)\right\}\right\} \\
&\leqslant \max\left\{\max_{i\in\bar{N}} \sigma_i^{-1}(V_i(x_i)), \max_{i,j\in\bar{N}} \sigma_i^{-1} \circ \gamma_{ij}(V_j(x_j)), \gamma(|w|)\right\} \\
&= \max\left\{\max_{i\in\bar{N}} \sigma_i^{-1}(V_i(x_i)), \max_{i,j\in\bar{N}} \sigma_i^{-1} \circ \gamma_{ij} \circ \sigma_j \circ \sigma_j^{-1}(V_j(x_j)), \gamma(|w|)\right\} \\
&\leqslant \max\{V(x), \rho(V(x)), \gamma(|w|)\} = \max\{V(x), \gamma(|w|)\},
\end{aligned}$$

which proves (5c).

Furthermore, we have (from [17]) that

$$\bar{\alpha}(s) = \max_{i\in\bar{N}} \sigma_i^{-1} \circ \bar{\alpha}_i(s), \tag{46}$$

$$\underline{\alpha}(s) = \min_{i\in\bar{N}} \sigma_i^{-1} \circ \underline{\alpha}_i\left(s/\sqrt{N}\right), \tag{47}$$

$$\alpha(s) = \hat{\alpha} \circ \bar{\alpha}^{-1}(s), \tag{48}$$

where

$$\hat{\alpha}(s) = \min\left\{\tilde{\alpha}_i(|x_i|_{\mathcal{A}_i}) \mid |x|_{\mathcal{A}} = s, V(x) = \sigma_i^{-1}(V_i(x_i))\right\} \tag{49}$$

and $\tilde{\alpha}_i(\rho)$, $\rho > 0$, is defined as

$$\tilde{\alpha}_i(\rho) = c_{\rho,i}\alpha_i(\rho) \tag{50}$$

and $c_{\rho,i} := k_1$, with k_1 corresponding to the set $K := [\rho/2, 2\rho]$ given by (44). □

Remark 5.8. The function $\sigma \in \mathcal{K}_\infty^N$ can be numerically constructed as outlined in [32]. Note that the analytic construction of $\sigma \in \mathcal{K}_\infty^N$ given in [17, Theorem 3.6] in general only gives an estimate of the form $\Gamma(\sigma(r)) \leq \sigma(r)$ (with nonstrict inequality) for all $r \geq 0$, while the proof requires the strict inequality (45). For instance, consider $\Gamma(s) = \begin{pmatrix} s_2/4 \\ \max\{2s_1, 2s_3\} \\ s_2/4 \end{pmatrix}$ and $a = (1, 1, 1)$. Then the construction proposed in [17, Theorem 3.6] yields

$$\sigma(r) = \begin{pmatrix} r \\ 2r \\ r \end{pmatrix} \not> \begin{pmatrix} 0.5r \\ 2r \\ 0.5r \end{pmatrix} = \Gamma(\sigma(r)) \quad \text{for all } r > 0.$$

6. Stability analysis

When all controlled subsystems (24) and all network-induced error systems (27) are ISS (and have an associated ISS Lyapunov function satisfying Condition 5.2), we can check the small-gain condition (41) to determine ISS of the complete system, and use Lemma 5.7 to find an ISS Lyapunov function for the overall system (1). In this section, we establish rather natural conditions such that all controlled subsystems and all network-induced error systems are indeed ISS.

6.1. ISS of the controlled subsystems

For all of the controlled subsystems $\mathcal{G}_i$, $i \in \bar{N}_{\mathcal{P}}$, given by (24), we assume that the controller $\mathcal{C}_i$ has been designed such that $\mathcal{G}_i$ satisfies Property 6.1, and hence is ISS from inputs w and x_j, $j \in \bar{N} \setminus \{i\}$, to state x_i with respect to the origin (i.e., the set $\mathcal{A}_i := \{0\}$).

Property 6.1. *For each controlled subsystem $\mathcal{G}_i$, $i \in \bar{N}_{\mathcal{P}}$, given by (24), there exist a locally Lipschitz function $V_i : \mathcal{X}_i \to \mathbb{R}_{\geqslant 0}$, functions $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $j \in \bar{N}$, with $\gamma_{ii} \equiv 0$, $\gamma_{iw} \in \mathcal{K} \cup \{0\}$ and $\alpha_i, \underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ such that*

1. *for all $x_i \in \mathcal{S}_i$ we have*
$$\underline{\alpha}_i(|x_i|) \leqslant V_i(x_i) \leqslant \bar{\alpha}_i(|x_i|), \tag{51a}$$
2. *for all $(x, w) \in \mathcal{F}_i$ satisfying $V_i(x_i) \geqslant \max\{\max_{j \in \bar{N}\setminus\{i\}}\{\hat{\gamma}_{ij}(|x_j|_{\mathcal{A}_j})\}, \gamma_{iw}(|w|)\}$ we have*
$$\langle \zeta_i, F_i(x, w) \rangle \leqslant -\alpha_i(|x_i|) \quad \text{for all } \zeta_i \in \partial^C_{x_i} V_i(x_i), \tag{51b}$$

where $\mathcal{A}_i := \{x_i \in \mathcal{X}_i \mid x_i = 0\}$, $i \in \bar{N}_{\mathcal{P}}$, and $\mathcal{A}_k := \{x_k \in \mathcal{X}_k \mid (e_k, s_k) = 0\}$, $k \in \bar{N}_{\mathcal{N}}$.

Note that under the assumption that all subsystems in the interconnection of Fig. 2 have an associated ISS Lyapunov function satisfying (39a), we have that (51b) implies that (39b) holds with

$$\gamma_{ij}(s) = \hat{\gamma}_{ij} \circ \bar{\alpha}_j^{-1}(s) \tag{52}$$

for all $j \in \bar{N}$. In this case Property 6.1 is essentially a special case of Condition 5.2 with $\mathcal{J}_i = \varnothing$, i.e., the case where the system does not jump. However, while Property 6.1 is a *local* condition, depending only on the dynamics of subsystem $\mathcal{G}_i$, Condition 5.2 is a *global* condition since it applies to all subsystems in the interconnection simultaneously.

The problem of finding a controller $\mathcal{C}_i$ such that Property 6.1 is satisfied will not be discussed in this paper. Readers interested in this problem are referred to [33] and the references therein.

6.2. ISS of the network-induced error systems

To be able to claim input-to-state stability of the network-induced error systems $\mathcal{E}_k$, $k \in \bar{N}_{\mathcal{N}}$, given by (27), we assume that the communication networks employ a *uniformly globally exponentially stable* (UGES) MAC protocol [2]. This class of protocols includes the well-known Round Robin (RR) protocol, which gives the nodes access to the network in a cyclic manner, and the Try-Once-Discard (TOD) protocol, which gives the node with the largest error access to the network, see, e.g., [2]. In [34,35] it is described how the TOD protocol can be implemented over wireless networks.

Other examples of UGES protocols are the sampled-data protocol and the TOD-tracking protocol [5].

Assumption 6.2 (*Uniformly Globally Exponentially Stable (UGES) Protocols*)**.** For each network-induced error system (27), the protocol given by h_k is UGES, meaning that there exists a function $W_k : \mathbb{N} \times \mathbb{R}^{n_{e_k}} \to \mathbb{R}_{\geqslant 0}$ that is locally Lipschitz in its second argument such that for all $\kappa_k \in \mathbb{N}$ and all $e_k \in \mathbb{R}^{n_{e_k}}$

$$\underline{a}_k |e_k| \leqslant W_k(\kappa_k, e_k) \leqslant \bar{a}_k |e_k| \tag{53}$$
$$W_k(\kappa_k + 1, h_k(\kappa_k, e_k)) \leqslant \lambda_k W_k(\kappa_k, e_k) \tag{54}$$

for constants $0 \leqslant \underline{a}_k \leqslant \bar{a}_k$ and $0 < \lambda_k < 1$.

Furthermore, we assume that the growth of W_k is bounded during flows and along jumps in κ_k.

Assumption 6.3. For each network-induced error system (27), the function W_k satisfies for all $\kappa_k \in \mathbb{N}$ and all $e_k \in \mathbb{R}^{n_{e_k}}$

$$W_k(\kappa_k + 1, e_k) \leqslant \lambda_k^W W_k(\kappa_k, e_k) \tag{55}$$

for some constant $\lambda_k^W \geqslant 1$, and satisfies for all $e_k \in \mathbb{R}^{n_{e_k}}$ and all $\kappa_k \in \mathbb{N}$

$$|\zeta_k| \leqslant M_k^W \quad \text{for all } \zeta_k \in \partial_{e_k}^C W_k(\kappa_k, e_k) \tag{56}$$

for some $M_k^W > 0$. Additionally, for all $x \in \mathcal{X}$, the growth of e_k is bounded during flow in the sense that for all $x \in \mathcal{X}$ and all $w \in \mathbb{R}^{n_w}$ there exist $M_k^e \geqslant 0$ and $\hat{\gamma}_{kj} \in \mathcal{K}_\infty \cup \{0\}$, $\hat{\gamma}_{kw} \in \mathcal{K} \cup \{0\}$ such that

$$|g_k(x, w)| \leqslant M_k^e |e_k| + \max_{j \in \bar{N} \setminus \{k\}} \left\{ \hat{\gamma}_{kj}(|x_j|_{\mathcal{A}_j}), \hat{\gamma}_{kw}(|w|) \right\}. \tag{57}$$

As mentioned in [1], all protocols discussed in [2,3,27,36] (including the TOD and RR protocol) satisfy (55) and (56). From Assumptions 6.2 and 6.3 it follows that for almost all e_k and all κ_k

$$\langle \zeta_k, g_k(x, w) \rangle \leqslant L_k W_k(\kappa_k, e_k) + M_k^W \max_{j \in \bar{N} \setminus \{k\}} \left\{ \hat{\gamma}_{kj}(|x_j|_{\mathcal{A}_j}), \hat{\gamma}_{kw}(|w|) \right\}, \quad \text{for all } \zeta_k \in \partial_{e_k}^C W_k(\kappa_k, e_k), \tag{58}$$

where

$$L_k := \frac{M_k^W M_k^e}{\underline{a}_k}. \tag{59}$$

In the remainder of this section it will be shown that when Assumptions 3.2, 6.2 and 6.3 hold, and τ_{mati}^k and τ_{mad}^k are small enough, then the network-induced error systems satisfy the following property, which states that each network-induced error system $\mathcal{E}_k$, $k \in \bar{N}_\mathcal{N}$, given by (27), is ISS from inputs w and x_j, $j \in \bar{N} \setminus \{k\}$, to state x_k with respect to the set $\mathcal{A}_k := \{x_k \in \mathcal{X}_k \mid (e_k, s_k) = 0\}$. We do not restrict the states τ_k, κ_k and l_k, since these are only used for modeling purposes and do not show up in the network output $(\hat{y}^k, \hat{u}^k)$ (and thus do not affect the other subsystems).

Property 6.4. *For each network-induced error system $\mathcal{E}_k$, $k \in \bar{N}_\mathcal{N}$, given by (27), and each $\chi_k, \nu_k \in \mathbb{R}_{>0}$, there exist a locally Lipschitz function $V_k : \mathcal{X}_k \to \mathbb{R}_{\geqslant 0}$, functions $\underline{\alpha}_k, \bar{\alpha}_k, \alpha_k \in \mathcal{K}_\infty$, and functions $\gamma_{kj} \in \mathcal{K}_\infty \cup \{0\}$, $\gamma_{kw} \in \mathcal{K} \cup \{0\}$ with $\gamma_{kk} \equiv 0$ such that*

1. *for all $x_k \in \mathcal{S}_k$*

$$\underline{\alpha}_k(|x_k|_{\mathcal{A}_k}) \leqslant V_k(x_k) \leqslant \bar{\alpha}_k(|x_k|_{\mathcal{A}_k}) \tag{60a}$$

2. *for all $(x, w) \in \mathcal{F}_k$ satisfying $V_k(x_k) \geqslant \max\{\max_{j \in \bar{N} \setminus \{k\}}\{\chi_k \gamma_{kj}(V_j(x_j))\}, \chi_k \gamma_{kw}(|w|)\}$ we have*

$$\langle \zeta_k, F_k(x, w) \rangle \leqslant -\nu_k \alpha_k(|x_k|_{\mathcal{A}_k}) \quad \text{for all } \zeta_k \in \partial_{x_k}^C V_k(x_k) \tag{60b}$$

3. *for all $(x, w) \in \mathcal{J}_k$*

$$V_k(G_k(x, w)) \leqslant \max\{V_k(x_k), \max_{j \in \bar{N} \setminus \{k\}} \{\chi_k \gamma_{kj}(V_j(x_j))\}, \chi_k \gamma_{kw}(|w|)\}, \tag{60c}$$

where $\mathcal{A}_k := \{x_k \in \mathcal{X}_k \mid (e_k, s_k) = 0\}$.

The scalars $\chi_k, \nu_k \in \mathbb{R}_{>0}$ are tuning parameters that will be used to tradeoff quality-of-control and quality-of-service as explained in Procedure 6.12 and as demonstrated in Section 7. Note that in (60b) we separate ν_k from α_k, and χ_k from γ_{kj} and γ_{kw}, as we will see in Theorems 6.5 and 6.8 that the functions α_k, γ_{kj}, and γ_{kw} follow from the function g_k and the MAC protocol (or more precisely, from the function W_k in Assumption 6.2 in the delay-free case and the function $\tilde{W}_k$ in Assumption 6.6 when delays are present), while the tuning parameters $\chi_k, \nu_k \in \mathbb{R}_{>0}$ in (60b) are directly related to the network parameters τ_{mati}^k, τ_{mad}^k, as we will see. When τ_{mati}^k and τ_{mad}^k have not yet been fixed, χ_k and ν_k can be chosen *freely*, where ν_k can be used to tune the decay rate of V_k (see (60b)), and χ_k scales the ISS gain of V_k (see items 2 and 3 in Property 6.4).

As will be shown in Theorems 6.5 and 6.8, choosing $\chi_k \to 0$ and/or $\nu_k \to \infty$ leads to $\tau_{mati}^k \to 0$ and $\tau_{mad}^k \to 0$, i.e., communication errors can only be avoided by having perfect communication. On the other hand, increasing χ_k or decreasing ν_k will allow for larger τ_{mati}^k and τ_{mad}^k (up to certain asymptotic values).

We are now ready to derive the ISS Lyapunov functions for the network-induced error systems, which we will first do for the case without transmission delays in Section 6.2.1, and then for the case with delays in Section 6.2.2.

6.2.1. Varying transmission intervals

First we state our results for communication networks without delays, i.e., $\tau_{\kappa_k}^k = 0$ for all $\kappa_k \in \mathbb{N}$. Since there are no delays, we do not need the variables s_k and l_k to fully describe the network-induced error dynamics, and (27) reduces to

$$\mathcal{E}_k : \begin{cases} \left.\begin{aligned} \dot{e}_k &= g_k(x, e, w) \\ \dot{\tau}_k &= 1 \\ \dot{\kappa}_k &= 0 \end{aligned}\right\}, & \tau_k \in [0, \tau^k_{mati}], \\ \left.\begin{aligned} e_k^+ &= h_k(\kappa_k, e_k) \\ \tau_k^+ &= 0 \\ \kappa_k^+ &= \kappa_k + 1 \end{aligned}\right\}, & \tau_k \in [\delta_k, \infty). \end{cases} \tag{61}$$

Theorem 6.5. *Consider all network-induced error systems (61), $k \in \bar{N}_{\mathcal{N}}$, and let $\chi_k, \nu_k \in \mathbb{R}_{>0}$ be given. If all controlled subsystems $\mathcal{G}_i$, $i \in \bar{N}_{\mathcal{P}}$ satisfy Property 6.1, Assumptions 3.2, 6.2 and 6.3 hold, and*

$$\tau^k_{mati} \leqslant c_k^{-1} \ln\left(\frac{\lambda_k^{-1} c_k + \nu_k}{\lambda_k c_k + \nu_k}\right), \tag{62}$$

where $c_k = 2\left(L_k + M_k^W \sqrt{\frac{1}{\lambda_k \chi_k}}\right)$ with L_k as in (59), then Property 6.4 is satisfied by the ISS Lyapunov functions

$$V_k(x_k) = \phi_k(\tau_k) W_k^2(\kappa_k, e_k), \tag{63}$$

where the functions $\phi_k : \mathbb{R} \to \mathbb{R}$ are given by

$$\phi_k(\tau) = \left(\lambda_k^{-1} + c_k^{-1}\nu_k\right) e^{-c_k \tau} - c_k^{-1}\nu_k. \tag{64}$$

Furthermore, (60) holds with[2]

$$\alpha_k(s) = \underline{a}_k^2 s^2 \tag{65}$$
$$\underline{\alpha}_k(s) = \lambda_k \underline{a}_k^2 s^2 \tag{66}$$
$$\bar{\alpha}_k(s) = \lambda_k^{-1} \bar{a}_k^2 s^2 \tag{67}$$
$$\gamma_{kj}(s) = \hat{\gamma}_{kj}^4 \circ \underline{\alpha}_j^{-1}(s), \quad \text{for all } j \in \bar{N} \tag{68}$$
$$\gamma_{kw}(s) = \hat{\gamma}_{kw}^2(s). \tag{69}$$

The proof of Theorem 6.5 can be obtained by drawing inspiration from [1,2,37] and is given below.

Proof. The function ϕ_k given by (64) is the unique solution to the differential equation

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}\phi_k(\tau) = -c_k \phi_k(\tau) - \nu_k \tag{70}$$

with $\phi_k(0) = \lambda_k^{-1}$, which shows that ϕ_k is continuous and decreasing. Furthermore, from (64) it follows that $\phi_k(\tau^k_{mati}) \geqslant \lambda_k$ as long as τ^k_{mati} satisfies (62), and thus $\phi_k(\tau) \in [\lambda_k, \lambda_k^{-1}]$ for all $\tau \in [0, \tau^k_{mati}]$. Combining this with (53) and the fact that $|e_k| = |x_k|_{\mathcal{A}_k}$ leads to

$$\lambda_k \underline{a}_k^2 |x_k|^2_{\mathcal{A}_k} \leqslant V_k(x_k) \leqslant \lambda_k^{-1} \bar{a}_k^2 |x_k|^2_{\mathcal{A}_k} \tag{71}$$

which proves that (60a) holds for all $x_k \in \mathcal{S}_k$ with $\underline{\alpha}_k(s) = \lambda_k \underline{a}_k^2 s^2$ and $\bar{\alpha}_k(s) = \lambda_k^{-1} \bar{a}_k^2 s^2$.

Using that [38, Proposition 2.3.13]

$$\partial^C_{x_k} V_k(x_k) \subseteq \nabla_{x_k} \phi_k(\tau_k) W_k^2(\kappa_k, e_k) + 2\phi_k W_k(\kappa_k, e_k) \partial^C_{x_k} W_k(\kappa_k, e_k),$$

we obtain based on (58), (59), and (70), that for all $x_k \in \mathcal{F}_k$

$$\begin{aligned} \langle \zeta, F_k(x, w)\rangle \leqslant{}& \left(-2\left(L_k + M_k^W \sqrt{\frac{1}{\lambda_k \chi_k}}\right)\phi_k - \nu_k\right) W_k^2 \\ &+ 2\phi_k W_k \left(L_k W_k + M_k^W \max_{j \in \bar{N}\setminus\{k\}} \left\{\hat{\gamma}_{kj}(|x_j|_{\mathcal{A}_j}), \hat{\gamma}_{kw}(|w|)\right\}\right) \\ \leqslant{}& -\nu_k W_k^2, \quad \text{if } \sqrt{\frac{1}{\lambda_k \chi_k}} W_k \geqslant \max_{j \in \bar{N}\setminus\{k\}} \left\{\hat{\gamma}_{kj}(|x_j|_{\mathcal{A}_j}), \hat{\gamma}_{kw}(|w|)\right\} \end{aligned}$$

[2] Here we make some slight abuse of notation, since V_k is not differentiable with respect to κ_k. This is justified since the components in $F_k(x, w)$ corresponding to κ_k are zero.

for all $\zeta \in \partial^C_{x_k} V_k(x_k)$, where we omitted the arguments of ϕ_k and W_k for notational convenience. From this it follows that

$$V_k(x_k) \geqslant \chi_k \max_{j\in\bar{N}\setminus\{k\}} \left\{\hat{\gamma}^2_{kj}(|x_j|_{\mathcal{A}_j}), \hat{\gamma}^2_{kw}(|w|)\right\} \Rightarrow \langle \zeta, F_k(x,w)\rangle \leqslant -\nu_k \left(\underline{a}_k |x_k|_{\mathcal{A}_k}\right)^2 \quad \text{for all } \zeta \in \partial^C_{x_k} V_k(x_k), \tag{72}$$

which proves that (60b) holds with $\gamma_{kj}(s) = \hat{\gamma}^4_{kj} \circ \underline{\alpha}_j^{-1}(s)$ for all $j \in \bar{N}$, $\gamma_{kw}(s) = \hat{\gamma}^2_{kw}(s)$, and $\alpha_k(s) = \underline{a}_k^2 s^2$, where we used that (51a) holds for all V_i, $i \in \bar{N}_{\mathcal{P}}$ and that (60a) holds for all V_k, $k \in \bar{N}_{\mathcal{N}}$.

If a jump occurs, it holds that $\tau_k \in [\delta_k, \tau^k_{mati}]$, and thus

$$\begin{aligned} V_k(G_k(x,w)) &= \phi_k(0) W_k^2(\kappa_k+1, h_k(\kappa_k, e_k)) \\ &\leqslant \lambda_k^{-1}\lambda_k^2 W_k^2(\kappa_k, e_k) = \lambda_k W_k^2(\kappa_k, e_k) \\ &\leqslant \phi_k(\tau^k_{mati}) W_k^2(\kappa_k, e_k) \\ &\leqslant \phi_k(\tau_k) W_k^2(\kappa_k, e_k) = V_k(x_k). \end{aligned} \tag{73}$$

In particular, we see that (60c) is implied. □

6.2.2. Varying transmission intervals and delays

Next we also allow nonzero communication delays upper bounded by τ^k_{mad} and $t^k_{\kappa_k+1} - t^k_{\kappa_k}$ as in Assumption 3.2, next to time-varying transmission intervals. The obtained network-induced error is now described by (27), and in order to guarantee ISS of the network-induced errors, we need the following assumption, which is based on [1, Condition IV.1].

Assumption 6.6. For each network-induced error system (27) there exist a function $\tilde{W}_k : \mathbb{N} \times \{0,1\} \times \mathbb{R}^{n_{e_k}} \times \mathbb{R}^{n_{e_k}} \to \mathbb{R}_{\geqslant 0}$ with $\tilde{W}_k(\kappa_k, l_k, \cdot, \cdot)$ locally Lipschitz for all fixed $\kappa_k \in \mathbb{N}$ and $l_k \in \{0,1\}$, functions $\underline{b}_k, \bar{b}_k \in \mathcal{K}_\infty$, $\hat{\gamma}_{kj} \in \mathcal{K}_\infty \cup \{0\}$, $\hat{\gamma}_{kw} \in \mathcal{K} \cup \{0\}$ and constants $\lambda_k \in (0,1)$, $H^k_0, H^k_1, L^k_0, L^k_1 \in \mathbb{R}_{\geqslant 0}$, such that

- for all $\kappa_k \in \mathbb{N}$ and all $s_k, e_k \in \mathbb{R}^{n_{e_k}}$ it holds that

$$\tilde{W}_k(\kappa_k+1, 1, e_k, h_k(\kappa_k, e_k) - e_k) \leqslant \lambda_k \tilde{W}_k(\kappa_k, 0, e_k, s_k) \tag{74a}$$

$$\tilde{W}_k(\kappa_k, 0, s_k + e_k, -s_k - e_k) \leqslant \tilde{W}_k(\kappa_k, 1, e_k, s_k); \tag{74b}$$

- for all $\kappa_k \in \mathbb{N}$, $l_k \in \{0,1\}$ and all $s_k, e_k \in \mathbb{R}^{n_{e_k}}$ it holds that

$$\underline{b}_k(|(e_k, s_k)|) \leqslant \tilde{W}_k(\kappa_k, l_k, e_k, s_k) \leqslant \bar{b}_k(|(e_k, s_k)|); \tag{75}$$

- for all $\kappa_k \in \mathbb{N}$, $l_k \in \{0,1\}$, $s_k \in \mathbb{R}^{n_{e_k}}$, $x \in \mathbb{R}^{n_x}$ and almost all $e \in \mathbb{R}^{n_e}$ it holds that

$$\left\langle \frac{\partial \tilde{W}_k(\kappa_k, l_k, e_k, s_k)}{\partial e_k}, g_k(x,w) \right\rangle \leqslant L^k_{l_k} \tilde{W}_k(\kappa_k, l_k, e_k, s_k) + H^k_{l_k} \max_{j\in\bar{N}\setminus\{k\}} \left\{\hat{\gamma}_{kj}(|x_j|_{\mathcal{A}_j}), \hat{\gamma}_{kw}(|w|)\right\}. \tag{76}$$

Lemma 6.7. *If Assumptions 3.2, 6.2 and 6.3 hold, then the function $\tilde{W}_k : \mathbb{N} \times \{0,1\} \times \mathbb{R}^{n_{e_k}} \times \mathbb{R}^{n_{e_k}} \to \mathbb{R}_{\geqslant 0}$ given by*

$$\tilde{W}_k(\kappa_k, l_k, e_k, s_k) := \begin{cases} \max\{W_k(\kappa_k, e_k), W_k(\kappa_k, e_k + s_k)\}, & l_k = 0, \\ \max\left\{\frac{\lambda_k}{\lambda^W_k} W_k(\kappa_k, e_k), W_k(\kappa_k, e_k + s_k)\right\}, & l_k = 1, \end{cases} \tag{77}$$

satisfies Assumption 6.6 with

$$H^k_0 = H^k_1 = M^W_k, \tag{78}$$

$$L^k_0 = \frac{M^W_k M^e_k}{\underline{a}_k}, \tag{79}$$

$$L^k_1 = \frac{\lambda^W_k M^W_k M^e_k}{\lambda_k \underline{a}_k}, \tag{80}$$

and $\underline{b}_k(r) = \underline{b}_k r$ and $\bar{b}_k(r) = \bar{b}_k r$, where $\underline{b}_k$ and $\bar{b}_k$ are positive constants, given by

$$\underline{b}_k = \frac{1}{2}\frac{\lambda_k}{\lambda^W_k} \underline{a}_k^2 \bar{a}_k^{-1} \left\| \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \right\|^{-1}, \tag{81}$$

$$\bar{b}_k = \sqrt{2}\bar{a}_k. \tag{82}$$

Proof. ([1, Theorem V.3]). □

The function $\tilde{W}$ defined in (77) can now be used to define an ISS Lyapunov function for the network-induced error system (27).

Theorem 6.8. *Consider all network-induced error systems (27), $k \in \bar{N}_{\mathcal{N}}$, and let $\chi_k, \nu_k \in \mathbb{R}_{>0}$ be given. Suppose the following conditions hold:*

- *All controlled subsystems $\mathcal{G}_i$, $i \in \bar{N}_{\mathcal{P}}$ satisfy Property 6.1,*
- *Assumptions 3.2 and 6.6 (or alternatively, Assumptions 3.2, 6.2 and 6.3),*
- *τ^k_{mati} and τ^k_{mad} satisfy*

$$\tau^k_{mati} \leqslant (c^k_0)^{-1} \ln\left(\frac{\Phi^k_0 c^k_0 + \nu_k}{\lambda^2_k \Phi^k_1 c^k_0 + \nu_k}\right), \tag{83}$$

$$\left(\Phi^k_1 + (c^k_1)^{-1}\nu_k\right) e^{-c^k_1 \tau^k_{mad}} - (c^k_1)^{-1}\nu_k \geqslant \left(\Phi^k_0 + (c^k_0)^{-1}\nu_k\right) e^{-c^k_0 \tau^k_{mad}} - (c^k_0)^{-1}\nu_k \tag{84}$$

for $c^k_l := 2\left(L^k_l + H^k_l\sqrt{\frac{1}{\lambda_k \chi_k}}\right)$, $l \in \{0, 1\}$, some $\Phi^k_0 \in \mathbb{R}_{>0}$ and some $\Phi^k_1 \in [\Phi^k_0, \lambda^{-2}_k \Phi^k_0]$,

then Property 6.4 is satisfied for the ISS Lyapunov functions

$$V_k(x_k) = \phi^k_{l_k}(\tau_k)\tilde{W}^2_k(\kappa_k, l_k, e_k, s_k), \tag{85}$$

where $\tilde{W}_k$ is given by (77), and where the functions $\phi^k_l : \mathbb{R} \to \mathbb{R}$ are given by

$$\phi^k_l(\tau) = \left(\Phi^k_l + (c^k_l)^{-1}\nu_k\right) e^{-c^k_l \tau} - (c^k_l)^{-1}\nu_k, \qquad l \in \{0, 1\}. \tag{86}$$

Furthermore, (60) holds with

$$\alpha_k(s) = \underline{b}^2_k s^2 \tag{87}$$

$$\underline{\alpha}_k(s) = \lambda^2_k \Phi_1 \underline{b}^2_k s^2 \tag{88}$$

$$\bar{\alpha}_k(s) = \Phi_1 \bar{b}^2_k s^2 \tag{89}$$

$$\gamma_{kj}(s) = \hat{\gamma}^4_{kj} \circ \underline{\alpha}^{-1}_j(s), \quad \text{for all } j \in \bar{N} \tag{90}$$

$$\gamma_{kw}(s) = \hat{\gamma}^2_{kw}(s). \tag{91}$$

Contrary to τ^k_{mati}, there is no explicit formula for τ^k_{mad}, thus (84) needs to be solved numerically. Note that both τ^k_{mati} and τ^k_{mad} depend on Φ^k_0 and Φ^k_1, and different choices for these parameters lead to different values for τ^k_{mati} and τ^k_{mad}. In this way, one can trade τ^k_{mati} for τ^k_{mad}, or vice versa, while keeping the quality-of-control constant.

Remark 6.9. Note that by taking $\Phi^k_0 = \lambda^{-1}_k$ and $\Phi^k_1 = \Phi^k_0$, we find $\tau^k_{mad} = 0$ and recover (62).

Proof. The proof of Theorem 6.8 is similar to the proof of Theorem 6.5 and also uses elements from [1, Theorem IV.2].

The functions ϕ^k_l, $l \in \{0, 1\}$, given by (86), are the unique solutions to the differential equation

$$\dot{\phi}^k_l(\tau_k) = -c^k_l \phi^k_l(\tau_k) - \nu_k \tag{92}$$

with $\phi^k_l(0) = \Phi^k_l$, which shows that the functions ϕ^k_l, $l \in \{0, 1\}$ are continuous and decreasing. Furthermore, from (86) it follows that $\phi^k_0(\tau^k_{mati}) \geqslant \lambda^2_k \Phi^k_1 = \lambda^2_k \phi^k_1(0)$ as long as τ^k_{mati} satisfies (83), and that $\phi^k_1(\tau^k_{mad}) \geqslant \phi^k_0(\tau^k_{mad})$ as long as τ^k_{mad} satisfies (84). Together with (74) this proves that V_k given by (85) satisfies (60c). Additionally, we have that

$$\Phi^k_0 \geqslant \phi^k_0(\tau_k) \geqslant \lambda^2_k \Phi^k_1, \qquad \tau_k \in [0, \tau^k_{mati}], \tag{93a}$$

$$\Phi^k_1 \geqslant \phi^k_1(\tau_k) \geqslant \lambda^2_k \Phi^k_1, \qquad \tau_k \in [0, \tau^k_{mad}], \tag{93b}$$

which together with $\Phi^k_1 \geqslant \Phi^k_0$ and (75), proves that V_k given by (85) satisfies (60a) with $\underline{\alpha}_k(s) = \lambda^2_k \Phi_1 \underline{b}^2_k s^2$ and $\bar{\alpha}_k(s) = \Phi_1 \bar{b}^2_k s^2$.

Lastly, (60b) follows directly from (76), (85), (92) and Lemma 6.7. □

6.3. ISS of the overall system

Now, all subsystems are modeled, and the overall system (1) is defined as the interconnection of $N_{\mathcal{P}}$ controlled subsystems $\mathcal{G}_i, i \in \bar{N}_{\mathcal{P}}$, and $N_{\mathcal{N}}$ network-induced error systems $\mathcal{E}_k, k \in \bar{N}_{\mathcal{N}}$. Furthermore, we are able to verify if Properties 6.1 and 6.4 are satisfied, and thus we can verify for given $\chi_k, k \in \bar{N}_{\mathcal{N}}$, if the overall system (1) is ISS, using the following theorem.

Theorem 6.10. *Let $\chi_k, \nu_k > 0, k \in \bar{N}_{\mathcal{N}}$, be given. If*

- *Assumption 4.1 holds for all controlled subsystems $\mathcal{G}_i, i \in \bar{N}_{\mathcal{P}}$, and all network-induced error systems $\mathcal{E}_k, k \in \bar{N}_{\mathcal{N}}$,*
- *Property 6.1 holds for all controlled subsystems $\mathcal{G}_i, i \in \bar{N}_{\mathcal{P}}$,*
- *Assumption 3.2 holds for all communication networks $\mathcal{N}_k, k \in \bar{N}_{\mathcal{N}}$,*
- *Assumptions 6.2 and 6.3 hold for all communication networks $\mathcal{N}_k, k \in \bar{N}_{\mathcal{N}}$ with $\tau^k_{mad} = 0$,*
- *Assumption 6.6 holds for all communication networks $\mathcal{N}_k, k \in \bar{N}_{\mathcal{N}}$ with $\tau^k_{mad} > 0$,*
- *and the gain operator Γ, given by*

$$\Gamma(s) := \begin{bmatrix} \max\limits_{j\in\bar{N}} \gamma_{1j}(s_j) \\ \vdots \\ \max\limits_{j\in\bar{N}} \gamma_{N_{\mathcal{P}}j}(s_j) \\ \chi_{N_{\mathcal{P}}+1} \max\limits_{j\in\bar{N}} \gamma_{N_{\mathcal{P}}+1,j}(s_j) \\ \vdots \\ \chi_N \max\limits_{j\in\bar{N}} \gamma_{Nj}(s_j) \end{bmatrix} \tag{94}$$

satisfies the small-gain condition (41),

then the overall system (1) is persistently flowing and ISS from w to x with respect to the set $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_N$.

Furthermore, there exists an Ω-path $\sigma \in \mathcal{K}^N_\infty$ such that the function $V : \mathcal{X} \to \mathbb{R}_{\geqslant 0}$ defined by (43) is an ISS Lyapunov function for the overall system according to Definition 2.5 with ISS gain

$$\gamma(s) = \max\{\max_{i\in\bar{N}_{\mathcal{P}}} \sigma_i^{-1}(\gamma_{iw}(s)), \max_{k\in\bar{N}_{\mathcal{N}}} \sigma_k^{-1}(\chi_k\gamma_{kw}(s))\} \tag{95}$$

and where $\bar{\alpha}$, $\underline{\alpha}$ and α are defined by (46)–(49) with

$$\tilde{\alpha}_i(\rho) = \begin{cases} c_{\rho,i}\alpha_i(\rho), & \text{when } i \in \bar{N}_{\mathcal{P}} \\ c_{\rho,i}\nu_i\alpha_i(\rho), & \text{when } i \in \bar{N}_{\mathcal{N}}. \end{cases} \tag{96}$$

Proof. As the controlled subsystems $\mathcal{G}_i$, $i \in \bar{N}_{\mathcal{P}}$, do not jump, Condition 5.2 is trivially satisfied when Property 6.1 holds. Moreover, Property 6.4 is satisfied for all network-induced error systems $\mathcal{E}_k$, $k \in \bar{N}_{\mathcal{N}}$, by Theorem 6.5 if $\tau^k_{mad} = 0$, and by Theorem 6.8 if $\tau^k_{mad} > 0$. Hence we can apply Lemma 5.7 to construct an ISS Lyapunov function for the overall system, and thus the overall system (1) is ISS from w to x w.r.t. the set $\mathcal{A}$. It remains to show that the overall system (1) is persistently flowing.

Since Assumption 4.1 holds for all subsystems, also Assumption 2.1 holds for the overall system, see Remark 5.1. Additionally, we have that $(G(x, w), w^+) \subset \mathcal{F} \cup \mathcal{J}$ for all $(x, w) \in \mathcal{J}$, and that flow is always possible for all $(x, w) \in \mathcal{F} \setminus \mathcal{J}$. As finite escape times are ruled out by the existence of the ISS Lyapunov function, it follows from [22, Proposition 2.4] that all maximal solutions of the overall system (1) are complete. Finally, Zeno behavior cannot occur since the $\delta_k > 0$ pose a lower bound on the transmission intervals in the network-induced error systems $\mathcal{E}_k$, and thus all maximal solutions are unbounded in the t direction, which means that the overall system (1) is persistently flowing. □

Concluding, once we have all the functions $\gamma_{ij}, i, j \in \bar{N}$, and scalars $\chi_k, k \in \bar{N}_{\mathcal{N}}$, we can check the input-to-state stability of the complete NCS and generate the corresponding ISS Lyapunov function. Recalling that increasing χ_k allows for larger τ^k_{mati} and τ^k_{mad}, it makes sense to maximize the values of χ_k, $k \in \bar{N}_{\mathcal{N}}$, in a Pareto optimal sense, such that the condition $\Gamma(s) \not\geqslant s$ still holds, see also [19]. On the other hand, while increasing χ_k relaxes the quality-of-service requirements of network $\mathcal{N}_k$ (in terms of MATI and MAD), it will also increase the ISS gain γ of the ISS Lyapunov function (43) related to the overall system (and thus it will decrease the quality-of-control of the overall system).

However, it should be noted that changing χ_k might lead to a change in the Ω-path $\sigma \in \mathcal{K}^N_\infty$, and thus in the Lyapunov function V, in which case different gains from w to $V(x)$ cannot be compared. In this case we can make use of the bound $|x| \leqslant \underline{\alpha}^{-1}(V(x))$ to calculate the gain from w to $|x|$ in order to assess the change in the quality-of-service.

The scaling ν_k does not affect the stability of the overall system, and can thus be chosen *freely*. Small ν_k allow for large τ^k_{mati} and τ^k_{mad}, however, at the cost of a slow decay α of the overall system. Increasing ν_k leads to smaller τ^k_{mati} and/or τ^k_{mad}, but, up to a certain point, will also increase the decay rate α. A further increase in ν_k will not affect α, as α will be limited by α_i, $i \in \bar{N}_{\mathcal{P}}$.

This shows that there is a tradeoff between the quality-of-service of the communication networks (in terms of τ^k_{mati} and τ^k_{mad}) and the quality-of-control of the overall system (in terms of α and γ). Once the values of χ_k and ν_k have been fixed, the corresponding values for τ^k_{mati} and τ^k_{mad} can be chosen using the results of Section 6.2.

Remark 6.11. Assuming that $\tau_k(0, 0) = 0$ for all $k \in \bar{N}_{\mathcal{N}}$, we have that for all solution pairs (x, w) to (1) it holds that $j \leqslant (2N_{\mathcal{N}}/\delta)t$ for all $(t, j) \in \operatorname{dom} x$, where $\delta = \min_{k\in\bar{N}_{\mathcal{N}}} \delta_k$ is the global lower bound between two successive transmissions in a communication network. Following the proof of Theorem 2.6, we find

$$|x(t, j)|_{\mathcal{A}} \leqslant \max\left\{\underline{\alpha}^{-1} \circ \beta(\bar{\alpha}(|x(0, 0)|_{\mathcal{A}}), t, j), \underline{\alpha}^{-1}\gamma(\|w\|)\right\} \quad \text{for all } (t, j) \in \operatorname{dom} x, \tag{97}$$

with $\beta \in \mathcal{KLL}$ given by

$$\beta(r, t, j) = \gamma_{\breve{\alpha}}\left(r, \frac{1}{2}t, \frac{\delta}{4N_{\mathcal{N}}}j\right), \tag{98}$$

which shows that the conditions (3) and (4) are equivalent for the NCSs considered in this paper.

In short, one can find a MATI and a MAD for each local network that guarantee ISS (with the desired convergence rate and ISS gain) of the complete NCS by using the following procedure.

Procedure 6.12. *Step* 1: *For each controlled subsystem $\mathcal{G}_i$, $i \in \bar{N}_{\mathcal{P}}$, given by* (24)*, determine an ISS Lyapunov function V_i satisfying* Property 6.1.

Step 2: *For each communication network $\mathcal{N}_k$, $k \in \bar{N}_{\mathcal{N}}$, choose a MAC protocol h_k, and for each resulting network-induced error system $\mathcal{E}_k$, $k \in \bar{N}_{\mathcal{N}}$, given by* (27)*, find a function $\tilde{W}_k$ satisfying* Assumption 6.6 *(or a function W_k satisfying* Assumptions 6.2 *and* 6.3 *for communication networks without delays).*

Step 3: *Using* Theorem 6.8 *(or* Theorem 6.5 *for communication networks without delays), find an ISS Lyapunov function V_k satisfying* Property 6.4 *for each network-induced error system $\mathcal{E}_k$, $k \in \bar{N}_{\mathcal{N}}$.*

Step 4: *Collect the functions γ_{ij}, γ_{iw}, and α_i, for all $i, j \in \bar{N}$. Tune χ_k and ν_k, $k \in \bar{N}_{\mathcal{N}}$, such that the small-gain condition holds and the overall Lyapunov function given by* Theorem 6.10 *guarantees the desired quality-of-control of the overall system.*

Step 5: *Based on χ_k and ν_k, choose appropriate network parameters τ^k_{mati} and τ^k_{mad} for each network $\mathcal{N}_k$, $k \in \bar{N}_{\mathcal{N}}$, using* (62) *in the case without delays, or* (83) *and* (84) *in the case with delays.*

In the next section, Procedure 6.12 will be demonstrated using an illustrative example.

7. Illustrative example

In this section we illustrate our results and Procedure 6.12 via a *nonlinear* example. For ease of exposition we consider the case without delays. Note however that delays can be included in a straightforward manner, see, e.g., the linear example in [20].

Consider the two coupled plants $\mathcal{P}_1$, $\mathcal{P}_2$ with states $x_1, x_2 \in \mathbb{R}$ and disturbance $w = (w_1, w_2) \in \mathbb{R}^2$ given by

$$\mathcal{P}_1 : \dot{x}_1 = x_1 + ax_2^2 + u_1 + w_1, \tag{99}$$

$$\mathcal{P}_2 : \dot{x}_2 = x_2 + a\sqrt{|x_1|} + u_2 + w_2, \tag{100}$$

where $a = 0.2$, and the inputs $u_1, u_2 \in \mathbb{R}$ are given by the controllers

$$\mathcal{C}_1 : u_1 = -2\hat{x}_1, \tag{101}$$

$$\mathcal{C}_2 : u_2 = -2\hat{x}_2, \tag{102}$$

where $\hat{x}_1 = x_1 + e_3$ and $\hat{x}_2 = x_2 + e_4$, and e_3, e_4 are the network-induced errors. This yields the controlled subsystems

$$\mathcal{G}_1 : \dot{x}_1 = -x_1 + ax_2^2 - 2e_3 + w_1, \tag{103}$$

$$\mathcal{G}_2 : \dot{x}_2 = -x_2 + a\sqrt{|x_1|} - 2e_4 + w_2, \tag{104}$$

and network-induced error systems

$$\mathcal{E}_3 : \begin{cases} \left.\begin{aligned} \dot{e}_3 &= x_1 - ax_2^2 + 2e_3 - w_1 \\ \dot{\tau}_3 &= 1 \\ \dot{\kappa}_3 &= 0 \end{aligned}\right\}, & \tau_3 \in [0, \tau^3_{mati}], \\ \left.\begin{aligned} e_3^+ &= 0 \\ \tau_3^+ &= 0 \\ \kappa_3^+ &= \kappa_3 + 1 \end{aligned}\right\}, & \tau_3 \in [\delta_3, \infty), \end{cases} \tag{105}$$

$$\mathcal{E}_4 : \begin{cases} \left.\begin{aligned} \dot{e}_4 &= x_2 - a\sqrt{|x_1|} + 2e_4 - w_2 \\ \dot{\tau}_4 &= 1 \\ \dot{\kappa}_4 &= 0 \end{aligned}\right\}, & \tau_4 \in [0, \tau_{mati}^4], \\ \left.\begin{aligned} e_4^+ &= 0 \\ \tau_4{}^+ &= 0 \\ \kappa_4{}^+ &= \kappa_4 + 1 \end{aligned}\right\}, & \tau_4 \in [\delta_4, \infty), \end{cases} \tag{106}$$

with state $x_k = (e_k, \tau_k, \kappa_k)$, $k \in \{3, 4\}$.

Step 1

For the controlled subsystems $\mathcal{G}_1$, $\mathcal{G}_2$, we choose the local Lyapunov functions

$$V_1(x_1) = x_1^2, \qquad V_2(x_2) = x_2^2. \tag{107}$$

The Lyapunov functions V_1 and V_2 satisfy Property 6.1, which can be shown as follows. Obviously, $\underline{\alpha}_1(s) = \bar{\alpha}_1(s) = \underline{\alpha}_2(s) = \bar{\alpha}_2(s) = s^2$. Next, for $\mathcal{G}_1$ we find that

$$\begin{aligned} \dot{V}_1 &= 2x_1\left(-x_1 + ax_2^2 - 2e_3 + w_1\right) \\ &\leqslant -2|x_1|^2 + 2a|x_1|\,|x_2|^2 + 4|x_1|\,|x_3|_{\mathcal{A}_3} + 2|x_1|\,|w_1|, \end{aligned}$$

which yields

$$\dot{V}_1 \leqslant -2(1-\epsilon)|x_1|^2 \quad \text{if } 2\epsilon V_1 > 2a\sqrt{V_1}|x_2|^2 + 4\sqrt{V_1}|x_3|_{\mathcal{A}_3} + 2\sqrt{V_1}|w_1|,$$

and thus

$$\dot{V}_1 \leqslant -2(1-\epsilon)|x_1|^2 \quad \text{if } V_1 > \max\left\{\epsilon^{-2}9a^2|x_2|^4, \epsilon^{-2}36|x_3|^2_{\mathcal{A}_3}, \epsilon^{-2}9|w|^2\right\}, \tag{108}$$

where we used that $|w_1| \leqslant |w|$. In the same way, we find for $\mathcal{G}_2$

$$\dot{V}_2 = 2x_2\left(-x_2 + a\sqrt{|x_1|} - 2e_4 + w_2\right),$$

which yields

$$\dot{V}_1 \leqslant -2(1-\epsilon)|x_2|^2 \quad \text{if } V_2 > \max\left\{\epsilon^{-2}9a^2|x_1|, \epsilon^{-2}36|x_4|^2_{\mathcal{A}_4}, \epsilon^{-2}9|w|^2\right\}, \tag{109}$$

and we choose $\epsilon = 0.8$.

Step 2

For both communication networks we choose the Try-Once-Discard MAC protocol. We could also have chosen the Round-Robin protocol, as in this case the two protocols are identical since in both communication networks there is only one transmitting node.

For the TOD protocol, we can take $W_3(\kappa_3, e_3) = |e_3|$ and $W_4(\kappa_4, e_4) = |e_4|$ as shown in [2], which satisfy Assumption 6.2 with $\underline{a}_k = \bar{a}_k = 1$ and $\lambda_k \in (0, 1)$ arbitrary for $k \in \{3, 4\}$. Here, we choose $\lambda_3 = \lambda_4 = 0.5$.

Furthermore, we find that Assumption 6.3 is satisfied, since (55) holds with $\lambda_3^W = \lambda_3^W = 1$, (56) holds with $M_3^W = M_4^W = 1$, and (57) holds with

$$|g_3(x, w)| \leqslant 2|e_3| + \max\left\{3|x_1|, 3a|x_2|^2, 3|w|\right\} \tag{110}$$

$$|g_4(x, w)| \leqslant 2|e_4| + \max\left\{3a\sqrt{|x_1|}, 3|x_2|, 3|w|\right\}, \tag{111}$$

and thus $M_3^e = M_4^e = 2$ and $L_3 = L_4 = 2$.

Step 3

By applying Theorem 6.5 we find the ISS Lyapunov functions

$$V_3(x_3) = \phi_3(\tau_3)|e_3|^2, \qquad V_4(x_4) = \phi_4(\tau_4)|e_4|^2, \tag{112}$$

where ϕ_3 and ϕ_4 satisfy (64) with $\lambda_3, \lambda_4 = 0.5$, $L_3 = L_4 = 2$, and $M_3^W = M_4^W = 1$.

Step 4

From Theorem 6.5 and (52), we find the gain operator

$$\Gamma(s) = \begin{bmatrix} \max\{\gamma_{12}(s_2), \gamma_{13}(s_3)\} \\ \max\{\gamma_{21}(s_1), \gamma_{24}(s_4)\} \\ \max\{\chi_3\gamma_{31}(s_1), \chi_3\gamma_{32}(s_2)\} \\ \max\{\chi_4\gamma_{41}(s_1), \chi_4\gamma_{42}(s_2)\} \end{bmatrix} \tag{113}$$

with

$$\gamma_{12}(s_2) = 9\epsilon^{-2}a^2s_2^2 \qquad \gamma_{21}(s_1) = 9\epsilon^{-2}a^2\sqrt{s_1} \qquad \gamma_{31}(s_1) = 9s_1 \qquad \gamma_{41}(s_1) = 9a^2\sqrt{s_1}$$
$$\gamma_{13}(s_3) = 36\epsilon^{-2}\lambda_3^{-1}s_3 \qquad \gamma_{24}(s_4) = 36\epsilon^{-2}\lambda_4^{-1}s_4 \qquad \gamma_{32}(s_2) = 9a^2s_2^2 \qquad \gamma_{42}(s_2) = 9s_2.$$

Furthermore, we find that

$$\gamma_{1w}(s) = 9\epsilon^{-2}s^2, \qquad \gamma_{2w}(s) = 9\epsilon^{-2}s^2, \qquad \gamma_{3w}(s) = 9s^2, \qquad \gamma_{4w}(s) = 9s^2,$$

and

$$\alpha_1(s) = 2(1-\epsilon)s^2, \qquad \alpha_2(s) = 2(1-\epsilon)s^2, \qquad \alpha_3(s) = s^2, \qquad \alpha_4(s) = s^2.$$

In the case of perfect communication we have that $\chi_3 = \chi_4 = 0$, and by making use of Lemma 5.5, the small-gain condition (41) reduces to

$$\gamma_{12} \circ \gamma_{21}(s) = 9\epsilon^{-2}a^2 \left(9\epsilon^{-2}a^2\sqrt{s}\right)^2 < s, \quad \text{for all } s \in \mathbb{R}_{>0}, \tag{114}$$

which is equivalent to

$$9^3\epsilon^{-6}a^6 < 1. \tag{115}$$

Note that the condition $\gamma_{12} \circ \gamma_{21}(s) < s$ for all $s \in \mathbb{R}_{>0}$ is equivalent to the condition $\gamma_{21} \circ \gamma_{12}(s) < s$ for all $s \in \mathbb{R}_{>0}$, for all functions $\gamma_{12}, \gamma_{21} \in \mathcal{K}_\infty \cup \{0\}$. Clearly, (115) holds for the values $a = 0.2$, $\epsilon = 0.8$. In other words, the overall system is ISS when the communication is perfect, and thus ISS of the overall system can be retained as long as τ_{mati}^3 and τ_{mati}^4 are small enough.

For this example we can construct the Ω-path $\sigma \in \mathcal{K}_\infty^N$ analytically, as the condition $\Gamma(\sigma(r)) < \sigma(r)$ can be rewritten as

$$\gamma_{12}^{-1}(\sigma_1(r)) > \sigma_2(r) > \gamma_{21}(\sigma_1(r)) \tag{116}$$
$$\gamma_{13}^{-1}(\sigma_1(r)) > \sigma_3(r) > \chi_3 \max\{\gamma_{31}(\sigma_1(r)), \gamma_{32}(\sigma_2(r))\} \tag{117}$$
$$\gamma_{24}^{-1}(\sigma_2(r)) > \sigma_4(r) > \chi_4 \max\{\gamma_{41}(\sigma_1(r)), \gamma_{42}(\sigma_2(r))\}. \tag{118}$$

For example, we can choose σ as

$$\sigma_1(r) = r \tag{119}$$
$$\sigma_2(r) = \mu\gamma_{12}^{-1}(\sigma_1(r)) = \mu\left(9\epsilon^{-2}a^2\right)^{-1/2}\sqrt{r} =: \varsigma_2\sqrt{r} \tag{120}$$
$$\sigma_3(r) = \mu\gamma_{13}^{-1}(\sigma_1(r)) = \mu\left(36\epsilon^{-2}\lambda_3^{-1}\right)^{-1} r =: \varsigma_3 r \tag{121}$$
$$\sigma_4(r) = \mu\gamma_{24}^{-1}(\sigma_2(r)) = \mu\left(36\epsilon^{-2}\lambda_4^{-1}\right)^{-1} \varsigma_2\sqrt{r} =: \varsigma_4\sqrt{r}, \tag{122}$$

where $\varsigma_2, \varsigma_3, \varsigma_4 \in \mathbb{R}_{>0}$, and $\mu \in (0, 1)$. Note that this choice of σ satisfies items 1–3 in the proof of Lemma 5.7 and thus can be used to construct the ISS Lyapunov function $V(x) = \max_{i\in\bar{N}} \sigma_i^{-1}(V_i(x_i))$ for the overall system. Furthermore, this choice of σ works for every χ_3, χ_4 we choose in the allowed range

$$0 < \chi_3 < \min\left\{\varsigma_3/9, \varsigma_3/(9a^2\varsigma_2^2)\right\} = \chi_{3,\max} \tag{123}$$
$$0 < \chi_4 < \min\left\{\varsigma_4/(9a^2), \varsigma_4/(9\varsigma_2)\right\} = \chi_{4,\max}, \tag{124}$$

which means that we can use the same overall Lyapunov function $V(x)$ for all χ_3, χ_4 in this range, allowing for easy comparison of the quality-of-control of the overall system for varying χ_3, χ_4. We choose μ close to 1, which results in $\chi_{3,\max} = \chi_{4,\max}$. The upper and lower bounds $\bar{\alpha}$ and $\underline{\alpha}$ of the constructed ISS Lyapunov function $V(x)$ are shown in Fig. 3(a).

Based on Theorem 6.10, the ISS gain $\gamma(|w|)$ and decay rate $\alpha(|x|)$ of the overall Lyapunov function $V(x)$ can be determined for varying values of χ_3, χ_4 and ν_3, ν_4. The ISS gain $\gamma(|w|)$ is shown in Fig. 3(b), in which it can be seen that γ is not influenced by χ_3 or χ_4. For this system (and this choice of $V(x)$), the gain γ is dominated by γ_{1w} and γ_{2w} corresponding to the controlled subsystems $\mathcal{G}_1, \mathcal{G}_2$. Thus, the ISS gains of the network-induced error systems do not effect γ, which means that we can choose $\chi_3 = \chi_{3,\max}$ and $\chi_4 = \chi_{4,\max}$ in order to maximize τ_{mati}^3 and τ_{mati}^4, *without* any penalty on the guaranteed ISS gain γ of the overall system.

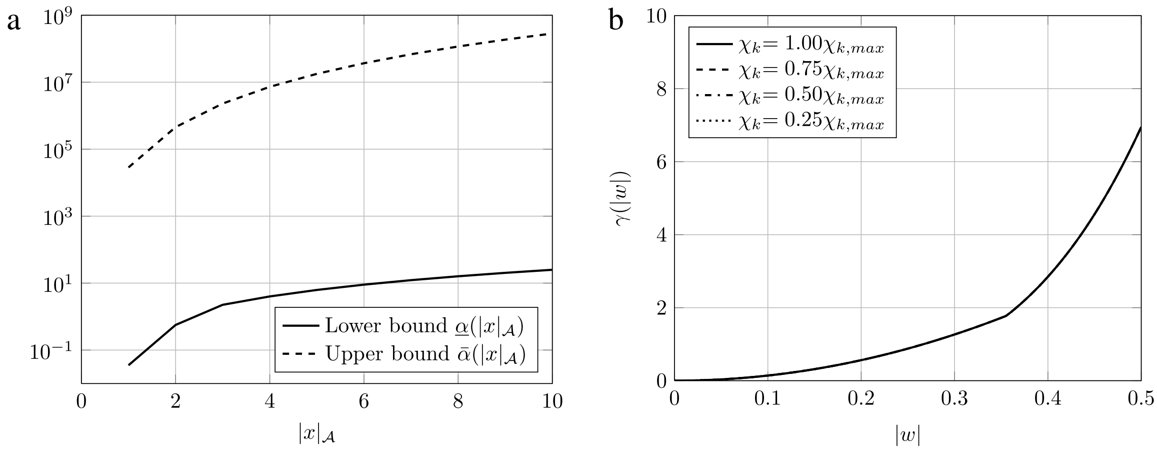


Fig. 3. Upper and lower bounds of the Lyapunov function $V(x)$ for the overall system (a), and the ISS gain $\gamma(|w|)$ for various values of χ_3, χ_4 (b).

On the other hand, the decay rate $\alpha(|x|)$ (which is shown in Fig. 4) is quite sensitive to ν_3 and ν_4. As there are no constraints on ν_3 and ν_4, they can be chosen such that the desired convergence rate α is achieved. Fig. 4 also shows that α cannot be increased indefinitely, but is limited by α_1 and α_2, e.g., Fig. 4(a) and (b) are identical, so increasing ν_3 and ν_4 above $\nu_3 = 0.05$ and $\nu_4 = 1.4 \cdot 10^{-4}$ does not further improve α.

Step 5

Finally, τ^k_{mati} follows from (62) and the choice of χ_k and ν_k, $k \in \{3, 4\}$. Fig. 5 shows the level sets of τ^k_{mati}, $k \in \{3, 4\}$, from which we can see that, as expected, an increase in χ_k or a decrease in ν_k leads to an increase in τ^k_{mati}. Since $L_3 = L_4$, $M^W_3 = M^W_4$, $\lambda_3 = \lambda_4$, and $\chi_{3,\max} = \chi_{4,\max}$, the level curves of τ^3_{mati} and τ^4_{mati} are identical, meaning that $\tau^3_{mati} = \tau^4_{mati}$ whenever $\chi_3 = \chi_4$ and $\nu_3 = \nu_4$. However, based on Fig. 4 it makes sense to choose $\nu_3 \neq \nu_4$, leading to $\tau^3_{mati} \neq \tau^4_{mati}$.

8. Conclusions and extensions

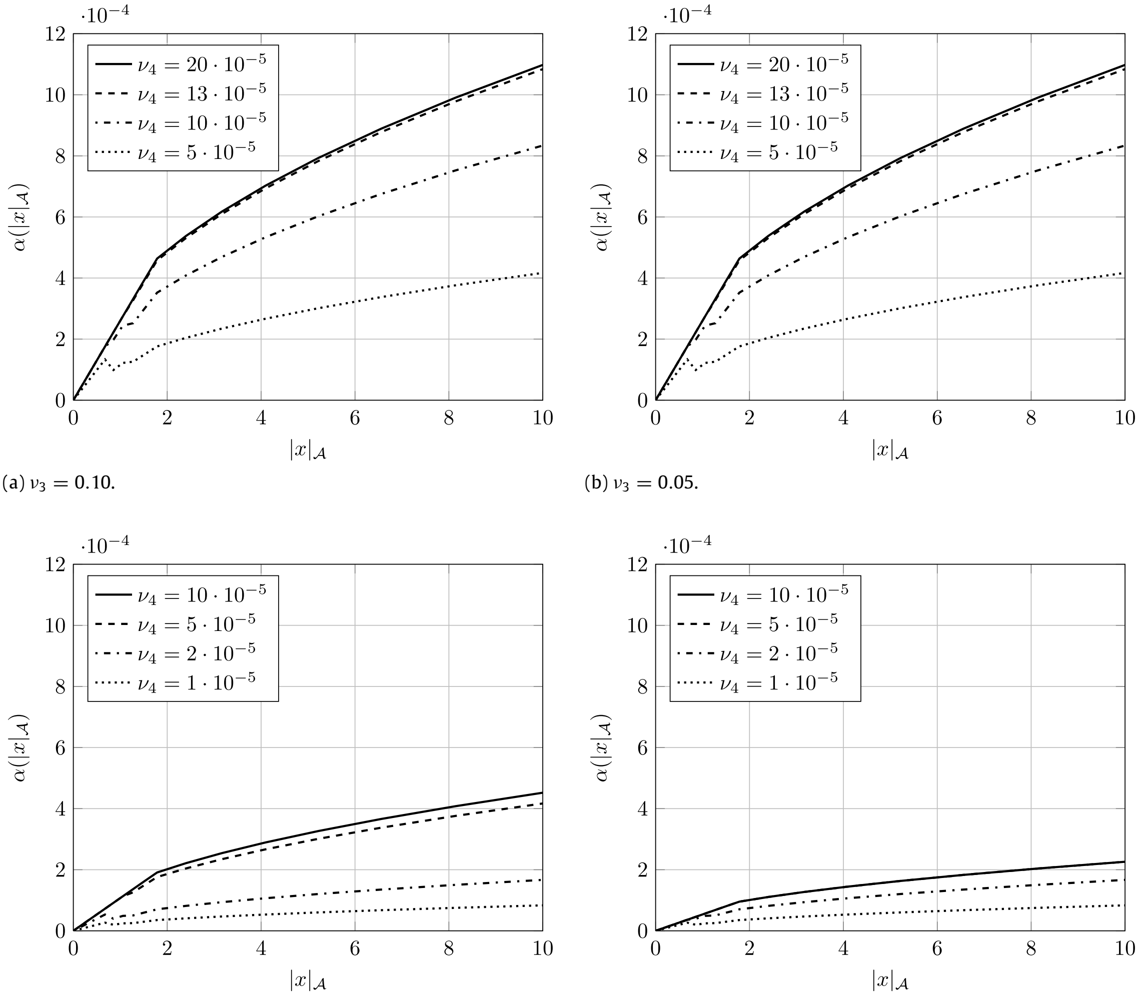
In this paper we studied ISS of large-scale NCSs with multiple local communication networks. We modeled the overall system as an interconnection of controlled subsystems and network-induced error systems, which are described as hybrid systems. We assumed that each controlled subsystem has an associated ISS Lyapunov function, and, under some assumptions on the medium access control protocol and the dynamics of the signals transmitted over the network, we construct ISS Lyapunov functions for the network-induced error systems. Moreover, for each communication network the maximum allowable transmission interval (MATI) and the maximum allowable delay (MAD) are related to the ISS gains and the convergence rate of the constructed ISS Lyapunov functions corresponding to that network. An ISS Lyapunov function for the overall system was then constructed based on the ISS Lyapunov functions of the subsystems and the interconnection gains. As a result, the quality-of-control of the overall system can be tuned by varying the ISS gains and convergence rates of the network-induced error systems (which are related to the quality-of-service of the local communication networks). More specifically, the quality-of-control of the overall system can be improved (or degraded) by improving (or relaxing) the quality-of-service of the communication networks. In addition, when relaxing the quality-of-service of one communication network, we can retain the quality-of-control of the overall system by improving the quality-of-service of one or more of the other communication networks. Finally, once the ISS gains and the convergence rate of a network-induced error system are fixed, the MATI and MAD of the corresponding communication network can be maximized in a Pareto-optimal sense. As such, we can find a MATI and a MAD for each local network such that the desired quality-of-control of the overall system is guaranteed by trading off

(i) the quality-of-control of the overall system versus the quality-of-service of the communication networks,
(ii) the quality-of-service between the different communication networks, and
(iii) the MATI versus MAD within each communication network, based on the Pareto-optimal curves.

The proposed framework is based on local conditions combined with a small-gain condition on the ISS gains, making the framework suitable for large-scale systems. The results were demonstrated via a nonlinear example.

The presented stability analysis is not limited to the communication networks described in this paper. For instance, as shown in [19], we can derive similar results if the networked-induced error systems are only globally stable (GS). Certainly, non-UGES MAC protocols can be allowed, as long as the resulting network-induced error systems are GS, see, e.g., [2].

Furthermore, instead of only considering time-triggered communication networks (characterized by MATIs), also event-triggered networks can be allowed. In an event-triggered network, the transmission times are not determined (or restricted)



(a) $\nu_3 = 0.10$.

(b) $\nu_3 = 0.05$.

(c) $\nu_3 = 0.02$.

(d) $\nu_3 = 0.01$.

Fig. 4. Guaranteed convergence rates for $\alpha(|x|_{\mathcal{A}})$ for various values of ν_3, ν_4.

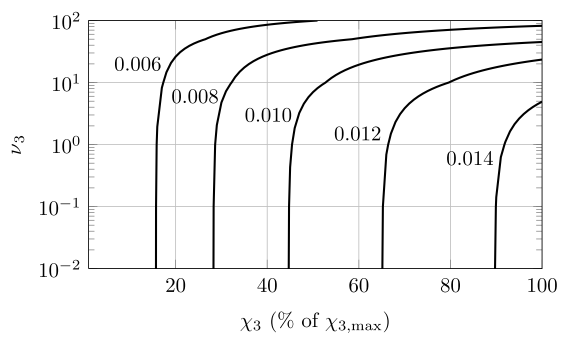


Fig. 5. Level sets of τ^k_{mati}, $k \in \{3, 4\}$.

based on time, but based on the actual information content of the involved signals, see, e.g., [39] for an overview. We believe that the dynamic event-triggering techniques of [8,40] can be easily included in the general framework proposed in this paper, which allows to extend the transmission intervals in the communication networks (thereby reducing the load on the communication network) without degrading the guaranteed quality-of-control of the overall system.

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