Dynamic Periodic Event-Triggered Control for Linear Systems

ABSTRACT

In event-triggered control systems, events are typically generated when a *static* function of the output (or state) of the system exceeds a given threshold. Recently, event-generators have been proposed that generate events based on an additional dynamic variable, with dynamics that depend on the output of the system. It is shown that these *dynamic* event-generators are able to guarantee the same performance as their static counterparts, while typically generating significantly fewer events. However, all dynamic event-generators available in literature require continuous measuring of the output of the plant, which is difficult to realize on digital platforms. In this paper, we propose new dynamic event-generators for linear systems, which require only periodic sampling of the output, and are therefore easy to implement on digital platforms. Based on hybrid modelling techniques combined with constructive designs of Lyapunov/storage functions for the resulting hybrid models, it is shown that these (dynamic periodic) event-generators lead to closed-loop systems which are globally exponentially stable (GES) with a guaranteed decay rate and $\mathcal{L}_2$-stable with a guaranteed $\mathcal{L}_2$-gain. The benefits of these new event-generators are also demonstrated via a numerical example.

CCS Concepts

•Theory of computation → Timed and hybrid models; •Mathematics of computing → *Numerical analysis; Differential equations;*

Keywords

Event-triggered control; hybrid systems; impulsive systems; Riccati differential equations; $\mathcal{L}_2$ stability; global exponential stability

1. INTRODUCTION

In most digital control systems, the measured output of the plant is periodically transmitted to the controller. Hence, the transmission times $\{t_k\}_{k\in\mathbb{N}}$ are determined in open-loop, as they are independent of the state of the system. This possibly leads to a waste of (e.g., computation, communication, and energy) resources, as many of the transmissions are actually not necessary to achieve the desired performance guarantees. In recent years, many *event-triggered control* (ETC) strategies have been proposed which generate the transmission times based on a triggering condition involving the current state or output measurement of the plant and the most recently transmitted measurement data, see, e.g., [4, 14, 17, 22] and the references therein. This brings a *feedback* mechanism into the sampling and communication process, such that measurement data is only transmitted to the controller when needed in order to guarantee the required stability and performance properties of the system.

The ETC strategies in the works mentioned above can be categorized as *continuous* event-triggered control (CETC) strategies, as the triggering condition (and thus the state or output of the system) has to be monitored continuously. As this can be difficult to realize on digital platforms, many CETC controllers are implemented using a discretized version in practice. A better solution to this problem is to use *periodic* event-triggered control (PETC), in which the triggering condition is only checked periodically at fixed equidistant time instances. This enables (easier) implementation on digital platforms, such that the event-generator that is implemented in practice is identical to its original design, instead of a discrete approximation as in the CETC case. Note that PETC differs from standard periodic sampled-data control, as in PETC the event/transmission times are only a subset of the sampling times and can be aperiodic. Of course, event-triggered control schemes for discrete-time systems (e.g., [6, 8, 15, 18, 25]) can also be interpreted as PETC schemes, but these do not take into account the inter-sample behavior of the underlying continuous process.

In the past few years, various PETC strategies have been proposed for linear systems [5, 11–14] and for nonlinear systems, see, e.g., [3, 19, 23]. In all these works, events are triggered whenever a certain *static* function of the state or output exceeds a given threshold. Hence, the event-generators do not have any dynamics of their own, and can be categorized as *static* periodic event-generators.

More recently, *continuous* event-generators have been proposed which generate events based on an additional dynamic variable (with dynamics that depend on the state or output of the system), leading to *dynamic* continuous event-generators. Dynamic continuous event-generators have been proposed for nonlinear systems in [7,9,20], and for linear sys-

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tems in [2]. In these works, it is shown that dynamic event-generators are able to guarantee the same performance as their static counterparts, while typically generating significantly fewer events. However, the proposed event-generators in [2,7,9,20] are all CETC solutions, and therefore difficult to implement in practice. To the best of the authors' knowledge, *dynamic* PETC solutions have not yet been proposed in literature.

As dynamic event-generators show great potential in CETC systems, we would like to exploit their benefits also in the context of PETC systems. Therefore, in this work, we propose two designs of dynamic periodic event-generators for linear systems. Our designs are based on ideas from [2] and [11], making use of hybrid modeling techniques and matrix Ricatti differential equations. The first variant requires that the event-generator has access to the complete state of the plant, while the second is purely based on output measurements. Both dynamic event-generators we propose lead to closed-loop systems which are globally exponentially stable (GES) with a guaranteed decay rate and $\mathcal{L}_2$-stable with a guaranteed $\mathcal{L}_2$-gain. We show via a numerical example that both dynamic event-generators we propose outperform the static event-generator of [11], in the sense that identical decay rate and $\mathcal{L}_2$-gain guarantees are achieved with significantly fewer events.

1.1 Notation

For a vector $x \in \mathbb{R}^{n_x}$, we denote by $|x| := \sqrt{x^\top x}$ its Euclidean norm. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we denote by $\lambda_{max}(A)$ and $\lambda_{min}(A)$ its maximum and minimum eigenvalue, respectively. For a matrix $P \in \mathbb{R}^{n\times n}$, we write $P \succ 0$ ($P \succeq 0$) if P is symmetric and positive (semi-)definite, and $P \prec 0$ ($P \preceq 0$) if P is symmetric and negative (semi-)definite. By I and O we denote the identity and zero matrix of appropriate dimensions, respectively. For a measurable signal $w : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_w}$, we write $w \in \mathcal{L}_2$ if $\|w\|_{\mathcal{L}_2} < \infty$, where $\|w\|_{\mathcal{L}_2} := \left(\int_0^\infty |w(t)|^2 \mathrm{d}t\right)^{1/2}$ denotes its $\mathcal{L}_2$-norm. By $\mathbb{N}$ we denote the set of natural numbers including zero, i.e., $\mathbb{N} := \{0, 1, 2, \dots\}$. A function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$, and a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and, in addition, $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{KL}$-function if for each fixed $t \in \mathbb{R}_{\geqslant 0}$ the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function and for each fixed $s \in \mathbb{R}_{\geqslant 0}$, $\beta(s, t)$ is decreasing in t and $\beta(s, t) \to 0$ as $t \to \infty$. For vectors $x_i \in \mathbb{R}^{n_i}$, $i \in \{1, 2, \dots, N\}$, we denote by $(x_1, x_2, \cdots, x_N)$ the vector $[x_1^\top x_2^\top \cdots x_N^\top]^\top \in \mathbb{R}^n$ with $n = \sum_{i=1}^N n_i$. For brevity, we sometimes write symmetric matrices of the form $\left[\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right]$ as $\left[\begin{smallmatrix} A & B \\ \star & C \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} A & \star \\ B^\top & C \end{smallmatrix}\right]$. For a left-continuous signal $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^n$ and $t \in \mathbb{R}_{\geqslant 0}$, we use $f(t^+)$ to denote the limit $f(t^+) = \lim_{s\to t, s>t} f(s)$.

2. CONTROL SETUP

In this paper, we consider the event-triggered control setup as shown in Figure 1, in which the plant $\mathcal{P}$ is given by

$$\mathcal{P} : \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_p(t) = A_p x_p(t) + B_p u(t) + B_{pw} w(t) \\ y(t) = C_y x_p(t) + D_y u(t) \\ z(t) = C_z x_p(t) + D_z u(t) + D_{zw} w(t) \end{cases} \tag{1}$$

and the controller $\mathcal{C}$ is given by

$$\mathcal{C} : \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_c(t) = A_c x_c(t) + B_c \hat{y}(t) \\ u(t) = C_u x_c(t) + D_u \hat{y}(t). \end{cases} \tag{2}$$

Here, $x_p(t) \in \mathbb{R}^{n_{x_p}}$ denotes the state of the plant $\mathcal{P}$, $y(t) \in \mathbb{R}^{n_y}$ its measured output, $z(t) \in \mathbb{R}^{n_z}$ the performance output, and $w(t) \in \mathbb{R}^{n_w}$ a disturbance at time $t \in \mathbb{R}_{\geqslant 0}$. Furthermore, $x_c(t) \in \mathbb{R}^{n_{x_c}}$ denotes the state of the controller $\mathcal{C}$, $u(t) \in \mathbb{R}^{n_u}$ is the control input at time $t \in \mathbb{R}_{\geqslant 0}$, and $\hat{y}(t) \in \mathbb{R}^{n_y}$ denotes the output that is available at the controller, given by

$$\hat{y}(t) = y(t_k), \quad t \in (t_k, t_{k+1}], \tag{3}$$

where the sequence $\{t_k\}_{k\in\mathbb{N}}$ denotes the event (or transmission) times which are generated by the event-generator specified below.

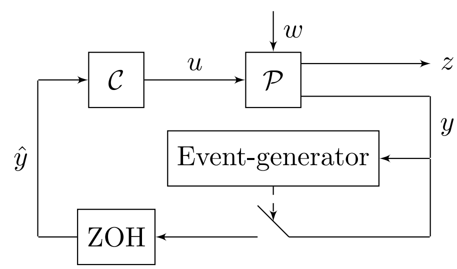


Figure 1: Event-triggered control setup.

In this work, the plant's output y is sampled periodically at fixed sample times $s_n = nh$, $n \in \mathbb{N}$, where $h \in \mathbb{R}_{>0}$ is the sample period. At each sample time s_n, $n \in \mathbb{N}$, the event-generator decides whether or not the measured output $y(s_n)$ should be transmitted to the controller. Hence, the sequence of event times $\{t_k\}_{k\in\mathbb{N}}$ is a subsequence of the sequence of sample times $\{s_n\}_{n\in\mathbb{N}}$.

Define the state $\xi := (x_p, x_c, \hat{y}) \in \mathbb{R}^{n_\xi}$, with $n_\xi = n_{x_p} + n_{x_c} + n_y$, and introduce a timer variable $\tau \in [0, h]$, which keeps track of the time that has elapsed since the latest sample time, and a dynamic variable $\eta \in \mathbb{R}$, which will be included in the event-generator. Lastly, define the matrix $Y \in \mathbb{R}^{2n_y \times n_\xi}$ as

$$Y := \begin{bmatrix} C_y & D_y C_u & D_y D_u \\ O & O & I \end{bmatrix} \tag{4}$$

such that $\zeta := (y, \hat{y}) = Y\xi$, and the signal $\hat{o} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{2n_y} \times [0, h] \times \mathbb{R}$ as

$$\hat{o}(t) := (\zeta(s_n), \tau(t), \eta(t)), \quad t \in (s_n, s_{n+1}], \ n \in \mathbb{N}, \tag{5}$$

which is the information that is available to the event-generator at time $t \in \mathbb{R}_{\geqslant 0}$.

The dynamic variable η will evolve according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(t) = \Psi(\hat{o}(t)), \qquad t \in (s_n, s_{n+1}), \ n \in \mathbb{N}, \tag{6a}$$

$$\eta(t^+) = \eta_T(\hat{o}(t)), \qquad t \in \{t_k\}_{k\in\mathbb{N}}, \tag{6b}$$

$$\eta(t^+) = \eta_N(\hat{o}(t)), \qquad t \in \{s_n\}_{n\in\mathbb{N}} \setminus \{t_k\}_{k\in\mathbb{N}}, \tag{6c}$$

where the functions $\Psi : \mathbb{R}^{2n_y} \times [0, h] \times \mathbb{R} \to \mathbb{R}$, $\eta_T : \mathbb{R}^{2n_y} \times [0, h] \times \mathbb{R} \to \mathbb{R}$ and $\eta_N : \mathbb{R}^{2n_y} \times [0, h] \times \mathbb{R} \to \mathbb{R}$ are to be designed. Note that at transmission times t_k, $k \in \mathbb{N}$, the variable η is updated differently than at the other sample times $s_n \neq t_k$, $n, k \in \mathbb{N}$, at which no transmission occurs.

Now, we can write the hybrid closed-loop system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \xi(t) \\ \tau(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} A\xi(t) + Bw(t) \\ 1 \\ \Psi(\hat{o}(t)) \end{bmatrix}, \quad \begin{matrix} t \in (s_n, s_{n+1}), \\ n \in \mathbb{N}, \end{matrix} \tag{7a}$$

$$\begin{bmatrix} \xi(t^+) \\ \tau(t^+) \\ \eta(t^+) \end{bmatrix} = \begin{bmatrix} J\xi(t) \\ 0 \\ \eta_T(\hat{o}(t)) \end{bmatrix}, \quad t \in \{t_k\}_{k\in\mathbb{N}} \tag{7b}$$

$$\begin{bmatrix} \xi(t^+) \\ \tau(t^+) \\ \eta(t^+) \end{bmatrix} = \begin{bmatrix} \xi(t) \\ 0 \\ \eta_N(\hat{o}(t)) \end{bmatrix}, \quad \begin{matrix} t \in \\ \{s_n\}_{n\in\mathbb{N}} \setminus \{t_k\}_{k\in\mathbb{N}} \end{matrix} \tag{7c}$$

$$z(t) = C\xi(t) + Dw(t), \tag{7d}$$

where

$$A = \begin{bmatrix} A_p & B_pC_u & B_pD_u \\ O & A_c & B_c \\ O & O & O \end{bmatrix}, \; B = \begin{bmatrix} B_{pw} \\ O \\ O \end{bmatrix},$$

$$C = \begin{bmatrix} C_z & D_zC_u & D_zD_u \end{bmatrix}, \; D = D_{zw}, \text{ and}$$

$$J = \begin{bmatrix} I & O & O \\ O & I & O \\ C_y & D_yC_u & D_yD_u \end{bmatrix}.$$

At sample times $s_n = nh$, $n \in \mathbb{N}$, the reset (7b) occurs when an event is triggered by the event-generator, otherwise the state (ξ, τ, η) jumps according to (7c). In between the sample times $s_n = nh$, $n \in \mathbb{N}$, the system evolves according to the differential equation (7a).

In this work, the sequence of event/transmission times $\{t_k\}_{k\in\mathbb{N}}$ is generated by *dynamic* periodic event-generators of the form

$$t_0 = 0,\; t_{k+1} = \min\{t > t_k \mid \eta_N(\hat{o}(t)) \leqslant 0 \wedge \zeta^\top(t) Q \zeta(t) \geqslant 0,\; t = nh,\; n \in \mathbb{N}\}, \tag{8}$$

where the scalar $h \in \mathbb{R}_{>0}$ and the matrix $Q \in \mathbb{R}^{2n_y \times 2n_y}$ are design parameters, in addition to the functions Ψ, η_T, and η_N. Note that the function η_N appears both in the update dynamics (6c), as well as in the triggering condition in (8).

With the model (6), (8), we can also capture *static* periodic event-generators by choosing $\eta(0) = 0$ and

$$\Psi(\hat{o}) = 0 \tag{9a}$$

$$\eta_T(\hat{o}) = 0 \tag{9b}$$

$$\eta_N(\hat{o}) = 0 \tag{9c}$$

for all $\hat{o} \in \mathbb{R}^{2n_y} \times \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}$, as then we have that $\eta(t) = 0$ for all $t \in \mathbb{R}_{\geqslant 0}$, and the dynamic periodic event-generator (8) reduces to static periodic event-generators as in [11], given by

$$t_0 = 0,\; t_{k+1} = \min\{t > t_k \mid \zeta^\top(t) Q \zeta(t) \geqslant 0,\; t = nh,\; n \in \mathbb{N}\}, \tag{10}$$

which only has h and Q as design parameters. A possible choice for Q is given by

$$Q = \begin{bmatrix} (1-\sigma^2)I & -I \\ -I & I \end{bmatrix} \tag{11}$$

with $\sigma \in (0, 1)$, such that (10) reduces to

$$t_0 = 0,\; t_{k+1} = \min\{t > t_k \mid |\hat{y}(t) - y(t)|^2 \geqslant \sigma^2 |y(t)|^2,\; t = nh,\; n \in \mathbb{N}\},$$

which can be seen as the digital version of static continuous event-generators [22] of the type

$$t_0 = 0,\; t_{k+1} = \inf\{t \geqslant t_k \mid |\hat{y}(t) - y(t)|^2 \geqslant \sigma^2 |y(t)|^2,\; t \in \mathbb{R}_{\geqslant 0}\}.$$

Other control setups and other choices of Q are also possible, see e.g., [11].

We will consider the following two notions of stability.

Definition 2.1. *The PETC system* (7)-(8) *is said to be globally exponentially stable (GES), if there exist a function $\beta \in \mathcal{KL}$ and scalars $c > 0$ and $\rho > 0$ such that for any initial condition $\xi(0) = \xi_0 \in \mathbb{R}^{n_\xi}$, $\tau(0) = 0$, $\eta(0) = 0$, all corresponding solutions to* (7)-(8) *with $w = 0$ satisfy $|\xi(t)| \leqslant ce^{-\rho t}|\xi_0|$ and $|\eta(t)| \leqslant \beta(|\xi_0|, t)$ for all $t \in \mathbb{R}_{\geqslant 0}$. In this case, we call ρ a (lower bound on the) decay rate.*

Note that we only require exponential decay of the state variable ξ, as we are mainly interested in the control performance regarding the plant and controller states, which are captured in ξ. In addition, we require that η stays bounded by a $\mathcal{KL}$-function for practical implementability. We do not put any constraint on the variable τ as it is only used for modelling purposes.

Definition 2.2. *The PETC system* (7)-(8) *is said to have an $\mathcal{L}_2$-gain from w to z smaller than or equal to θ, if there exists a function $\delta \in \mathcal{K}_\infty$ such that for any initial condition $\xi(0) = \xi_0 \in \mathbb{R}^{n_\xi}$, $\tau(0) = 0$, $\eta(0) = 0$, all corresponding solutions to* (7)-(8) *with $w \in \mathcal{L}_2$ satisfy $\|z\|_{\mathcal{L}_2} \leqslant \delta(|\xi_0|) + \theta\|w\|_{\mathcal{L}_2}$.*

In the next section, we will present the stability analysis of the static PETC system (7) with (9) and (10), which leads to designs for h and Q. Building upon the design and stability analysis of the static PETC system, we then present our design for the functions Ψ, η_T, and η_N in the dynamic event-generator in Section 4.

3. STATIC PETC

To analyze the stability and $\mathcal{L}_2$-gain of the static PETC system (7) with (9) and (10), we will use the Lyapunov/storage function U given by

$$U(\xi, \tau, \eta) = V(\xi, \tau) + \eta, \tag{12}$$

with V given by

$$V(\xi, \tau) = \xi^\top P(\tau) \xi, \quad \tau \in [0, h], \tag{13}$$

where $P : [0, h] \to \mathbb{R}^{n_\xi \times n_\xi}$ is a continuously differentiable function with $P(\tau) \succ 0$ for $\tau \in [0, h]$. The function P will be chosen such that (12) becomes a storage function [21,24] for the PETC system (7), (10) with the supply rate $\theta^{-2} z^\top z - w^\top w$ and decay rate 2ρ.

In order to do so, we select the function $P : [0,h] \to \mathbb{R}^{n_\xi \times n_\xi}$ to satisfy the Riccati differential equation (where we omitted τ for compactness of notation)

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}\tau}P = &-A^\top P - PA - 2\rho P - \theta^{-2}C^\top C \\ &- (PB + \theta^{-2}C^\top D)M(B^\top P + \theta^{-2}D^\top C), \quad (14)\end{aligned}$$

provided the solution exists on $[0,h]$ for the desired values of $\rho > 0$ and θ. Here, $M := (I - \theta^{-2}D^\top D)^{-1}$ is assumed to exist and to be positive definite, which means that $\theta^2 > \lambda_{max}(D^\top D)$.

In order to find the explicit expression for P, we introduce the Hamiltonian matrix

$$H := \begin{bmatrix} A+\rho I+\theta^{-2}BMD^\top C & BMB^\top \\ -C^\top LC & -\left(A+\rho I+\theta^{-2}BMD^\top C\right)^\top \end{bmatrix}$$

in which $L := (\theta^2 I - DD^\top)^{-1}$, and we define the matrix exponential

$$F(\tau) := e^{-H\tau} = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) \\ F_{21}(\tau) & F_{22}(\tau) \end{bmatrix}. \quad (15)$$

Assumption 3.1. *$F_{11}(\tau)$ is invertible for all $\tau \in [0,h]$.*

Assumption 3.1 can always be satisfied by choosing h sufficiently small, as $F_{11}(0) = I$ and F_{11} is a continuous function of τ. The function $P : [0,h] \to \mathbb{R}^{n_\xi \times n_\xi}$ is now explicitly defined for $\tau \in [0,h]$ by

$$\begin{aligned}P(\tau) = &(F_{21}(h-\tau) + F_{22}(h-\tau)P(h)) \\ &(F_{11}(h-\tau) + F_{12}(h-\tau)P(h))^{-1}, \quad (16)\end{aligned}$$

see [1, 11] for further details.

Before stating the next theorem (which is a slight variation of [11, Theorem III.2]), let us introduce the notation $P_0 := P(0)$, $P_h := P(h)$, $\bar{F}_{11} := F_{11}(h)$, $\bar{F}_{12} := F_{12}(h)$, $\bar{F}_{21} := F_{21}(h)$, and $\bar{F}_{22} := F_{22}(h)$, the matrix

$$\bar{G} := \bar{F}_{11}^{-\top} P_h \bar{F}_{11}^{-1} + \bar{F}_{21}\bar{F}_{11}^{-1}, \quad (17)$$

and a matrix $\bar{S}$ that satisfies $\bar{S}\bar{S}^\top := -\bar{F}_{11}^{-1}\bar{F}_{12}$. A matrix $\bar{S}$ exists under Assumption 3.1, because this assumption will guarantee that the matrix $-\bar{F}_{11}^{-1}\bar{F}_{12}$ is positive semi-definite.

Theorem 3.2. *If there exist matrices $N_T, N_N \in \mathbb{R}^{2n_y \times 2n_y}$ with $N_T, N_N \succeq 0$ and $P_h \in \mathbb{R}^{n_\xi \times n_\xi}$ with $P_h \succ 0$, and scalars $\beta, \mu, \theta, \rho \in \mathbb{R}_{\geqslant 0}$, such that*

$$\begin{bmatrix} P_h - Y^\top(N_T + \mu Q)Y - J^\top \bar{G} J & J^\top \bar{F}_{11}^{-\top} P_h \bar{S} \\ \star & I - \bar{S}^\top P_h \bar{S} \end{bmatrix} \succ 0, \quad (18)$$

$$\begin{bmatrix} P_h - Y^\top(N_N - \beta Q)Y - \bar{G} & \bar{F}_{11}^{-\top} P_h \bar{S} \\ \star & I - \bar{S}^\top P_h \bar{S} \end{bmatrix} \succ 0, \quad (19)$$

and Assumption 3.1 hold, then the static PETC system (7) with (9) and (10) is GES with decay rate ρ, and has an $\mathcal{L}_2$-gain from w to z smaller than or equal to θ.

The proof of Theorem 3.2 is given in [11] for the system (7) with (9) and (10) with η absent. However, as here we have added the variable η to the closed loop, and our results in Section 4 build upon the proof of Theorem 3.2, we provide a sketch of the proof below.

Proof. The proof is based on the storage function U given by (12) with V as defined in (13). However, we only need to consider the function V, as it holds that $\eta(t) = 0$ for all $t \in \mathbb{R}_{\geqslant 0}$ and thus in this case $U = V$.

The proof consists of showing that the function V is a proper function and satisfies for all $\xi \in \mathbb{R}^{n_\xi}$ and all $\tau \in [0,h]$,

$$c_1|\xi|^2 \leqslant V(\xi,\tau) \leqslant c_2|\xi|^2 \quad (20)$$

with $c_2 \geqslant c_1 > 0$, has a supply rate $\theta^{-2}z^\top z - w^\top w$ [21, 24] and decay rate 2ρ during flow (7a), and is nonincreasing along jumps (7b) and (7c).

The first property follows from Assumption 3.1, as this assumption guarantees that $P(\tau) \succ 0$ for all $\tau \in [0,h]$, see [1, 11]. Hence, (20) holds with

$$c_1 = \min_{\tau\in[0,h]} \lambda_{min}(P(\tau)), \text{ and} \quad (21a)$$

$$c_2 = \max_{\tau\in[0,h]} \lambda_{max}(P(\tau)), \quad (21b)$$

where $c_2 \geqslant c_1 > 0$.

For brevity, we will use the notation $V(t) = V(\xi(t), \tau(t))$ in the remainder.

The second property follows directly from (14) and (7a) as these differential equations guarantee that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V(t) \leqslant -2\rho V(t) - \theta^{-2}z(t)^\top z(t) + w(t)^\top w(t) \quad (22)$$

during flow (7a) when $\tau \in [0,h]$, (see [11]).

Finally, we show that V does not increase along jumps. In [11], it is shown that

$$\begin{aligned}P_0 = &\bar{F}_{21}\bar{F}_{11}^{-1} + \\ &\bar{F}_{11}^{-\top}\left(P_h + P_h\bar{S}\left(I - \bar{S}^\top P_h \bar{S}\right)^{-1}\bar{S}^\top P_h\right)\bar{F}_{11}^{-1}. \quad (23)\end{aligned}$$

By applying a Schur complement it follows from (18) that along jumps (7b) (when $\tau = h$ and $\zeta^\top Q\zeta \geqslant 0$) we have

$$\begin{aligned}V\left(t^+\right) &= \xi(t)^\top J^\top P_0 J\xi(t) \\ &\leqslant \xi(t)^\top P_h \xi(t) - \zeta(t)^\top(N_T + \mu Q)\zeta(t) \quad (24a)\\ &\leqslant \xi(t)^\top P_h \xi(t) = V(t), \quad (24b)\end{aligned}$$

and it follows from (19) that along jumps (7c) (when $\tau = h$ and $\zeta^\top Q\zeta \leqslant 0$) we have

$$\begin{aligned}V\left(t^+\right) &= \xi(t)^\top P_0 \xi(t) \\ &\leqslant \xi(t)^\top P_h \xi(t) - \zeta(t)^\top(N_N - \beta Q)\zeta(t) \quad (25a)\\ &\leqslant \xi(t)^\top P_h \xi(t) = V(t). \quad (25b)\end{aligned}$$

Combining (20), (22), (25b), and (24b) indeed establishes the upper bound θ on the $\mathcal{L}_2$-gain of the PETC system (7) with (9) and (10) [11]. Furthermore, when $w = 0$, it follows that for all $t \in \mathbb{R}_{\geqslant 0}$

$$|\xi(t)| \leqslant ce^{-\rho t}|\xi(0)| \quad (26)$$

with $c = \sqrt{c_1/c_2}$, which proves that the system is GES with decay rate ρ. □

Note that the conditions of Theorem 3.2 depend nonlinearly on the design variables h and Q, and the control performance measures ρ and θ. However, by fixing the variables h, Q, ρ, and θ, inequalities (18) and (19) become LMIs, in which case the parameters P_h, P_0, N_T, N_N, μ, and β can be synthesized numerically via semi-definite programming (e.g., using Yalmip/SeDuMi[16] in MATLAB).

The $\mathcal{L}_2$-gain estimate θ can be optimized via bisection when h, Q, and ρ are fixed. Although the optimization is non-convex and we should expect to find local optima, good results can be found with proper initial estimates. The same holds for the decay rate ρ (when h, Q, and θ are fixed) and the sample period h (when Q, ρ and θ are fixed). Finally, when Q is given by (11), the design freedom in Q is reduced to the scalar σ, which can also be optimized via bisection when h, ρ, and θ are fixed.

4. DYNAMIC PETC

In this section we present our design for the dynamics (6) of the variable η, which is included in the dynamic event-generator (8). The idea is as follows. In Section 3, the function V (and, hence, also the Lyapunov/storage function U) is often strictly decreasing along jumps (7b) and (7c). However, to guarantee GES and $\mathcal{L}_2$-stability, it is sufficient if U is nonincreasing along jumps [10]. To get less conservative results, we will store this 'unnecessary' decrease of V as much as possible in the dynamic variable η, which acts as a buffer. When a transmission is necessary according to the static event-generator, we might choose not to transmit at this sample time. As the state then jumps according to (7c), we can no longer guarantee that V does not increase along this jump. However, an increase of V can be compensated by reducing η, and hence we can defer the transmission until the buffer η is no longer large enough. The transmission only needs to occur if the buffer η would become negative otherwise. As a result, the conservatism in the stability analysis is reduced, and the same $\mathcal{L}_2$-gain and decay rate can be guaranteed with typically fewer transmissions. In this way, our design leads to a dynamic PETC system (7), (8) with the same $\mathcal{L}_2$-gain θ and decay rate ρ as the static PETC system (7), (10), but with a significant reduction in the number of transmissions, as we will see in Section 5.

First, we choose the flow dynamics (6a) of η as

$$\Psi(\hat{o}) = -2\rho\eta, \text{ for } \tau \in (0, h]. \tag{27}$$

Remark 4.1. *As Ψ is given by (27), it follows that $\eta(s_{n+1}) = e^{-2\rho h}\eta(s_n^+)$. Thus, since the event-generator only needs to know the value of η at sample times s_n, $n \in \mathbb{N}$, the variable η does not need to continuously evolve according to (27) in the event-generator. Instead we can use the discrete-time dynamics just described.*

For the functions η_T and η_N, we provide the following two designs.

1) State-based dynamic PETC:

$$\eta_T(\hat{o}) = \eta + \xi^\top (P_h - J^\top P_0 J)\xi, \tag{28a}$$
$$\eta_N(\hat{o}) = \eta + \xi^\top (P_h - P_0)\xi. \tag{28b}$$

2) Output-based dynamic PETC:

$$\eta_T(\hat{o}) = \eta + \zeta^\top (N_T + \mu Q)\zeta, \tag{29a}$$
$$\eta_N(\hat{o}) = \eta + \zeta^\top (N_N - \beta Q)\zeta. \tag{29b}$$

Here, the scalars ρ, μ, and β, and the matrices N_T, N_N, P_0, and P_h follow from the stability analysis of the static PETC system in Theorem 3.2.

The first design requires that the full state $\xi(s_n)$ is known to the event-generator at sample time s_n, $n \in \mathbb{N}$. This is the case when $y = (x_p, x_c)$ (e.g., when $\mathcal{C}$ is a static state-feedback controller in which case $y = x_p$ and $n_{x_c} = 0$), as then $\zeta = \xi$. When $y = x_p$ and $n_{x_c} \neq 0$, a copy of the controller could be included in the event-generator in order to track the controller state x_c.

The second design is more conservative, but can also be used in case the event-generator does not have access to the complete vector (x_p, x_c), in which case $\zeta \neq \xi$. Hence, this choice can be used for output-based dynamic PETC.

Theorem 4.2. *If the conditions of Theorem 3.2 hold, then the dynamic PETC system (7) with (8), (27), and (28) or (29) is GES with decay rate ρ, and has an $\mathcal{L}_2$-gain from w to z smaller than or equal to θ.*

Proof. Consider again the Lyapunov/storage function U given by (12) with V as defined in (13).

First, we show that U is a proper storage function, by showing that $\eta(t) \geqslant 0$ for all $t \in \mathbb{R}_{\geqslant 0}$, and that U satisfies for all $\xi \in \mathbb{R}^{n_\xi}$, $\tau \in [0, h]$, and all $\eta \in \mathbb{R}_{\geqslant 0}$,

$$c_1|\xi|^2 + |\eta| \leqslant U(\xi, \tau, \eta) \leqslant c_2|\xi|^2 + |\eta|, \tag{30}$$

where c_1 and c_2 are given by (21). As $\eta(0) = 0$, it follows from (27) that $\eta(t) \geqslant 0$ for all $t \in [0, h]$, and hence, that $\eta(s_1) \geqslant 0$. Next, given event-generator (8), a transmission (7b) occurs in case $\zeta(s_1)^\top Q\zeta(s_1) \geqslant 0$ and $\eta_N(\hat{o}(s_1)) \leqslant 0$. In this case, $\eta_T(\hat{o}(s_1)) \geqslant 0$ follows from (24b) when η_T is given by (28a), or from $N_T \succeq 0$ and $\mu \geqslant 0$ when η_T is given by (29a). Otherwise, if $\zeta(s_1)^\top Q\zeta(s_1) \leqslant 0$ or $\eta_N(\hat{o}(s_1)) \geqslant 0$, no transmission occurs, and the state jumps according to (7c). Observe however that when $\zeta(s_1)^\top Q\zeta(s_1) \leqslant 0$ it holds that $\eta_N(\hat{o}(s_1)) \geqslant 0$, which follows from (25b) when η_N is given by (28b), or from $N_N \succeq 0$ and $\beta \geqslant 0$ when η_N is given by (29b). Hence, in all cases it holds that $\eta(s_1^+) \geqslant 0$. It follows by induction that $\eta(t) \geqslant 0$ for all $t \in \mathbb{R}_{\geqslant 0}$. Property (30) then follows by combining (20) and (12).

It remains to show that U has a supply rate $\theta^{-2}z^\top z - w^\top w$ and decay rate 2ρ during flow (7a), and is nonincreasing along jumps (7b) and (7c). For brevity, we will use the notation $V(t) = V(\xi(t), \tau(t))$ and $U(t) = U(\xi(t), \tau(t), \eta(t))$ in the remainder.

From (14) and (27) it follows (using (22)) that

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}U(t) &\leqslant -2\rho V(t) - 2\rho\eta(t) - \theta^{-2}z(t)^\top z(t) + w(t)^\top w(t) \\ &= -2\rho U(t) - \theta^{-2}z(t)^\top z(t) + w(t)^\top w(t) \end{aligned} \tag{31}$$

during flow (7a).

Finally, we show that

$$U(t^+) \leqslant U(t) \tag{32}$$

holds along jumps. When using (28), we find along transmissions (7b) that

$$\begin{aligned} U(t^+) &= \xi^\top J^\top P_0 J\xi + \eta + \xi^\top (P_h - J^\top P_0 J)\xi \\ &= \xi^\top P_h \xi + \eta = U(t) \end{aligned}$$

and along non-transmission jumps (7c) that

$$\begin{aligned} U(t^+) &= \xi^\top P_0 \xi + \eta + \xi^\top (P_h - P_0)\xi \\ &= \xi^\top P_h \xi + \eta = U(t). \end{aligned}$$

Hence, (32) holds with equality. Alternatively, when using (29), we find (using (24a)) that along transmissions (7b) it holds that

$$\begin{aligned} U(t^+) &= \xi^\top J^\top P_0 J\xi + \eta + \zeta^\top (N_T + \mu Q)\zeta \\ &\leqslant \xi^\top P_h \xi + \zeta^\top (N_T + \mu Q - N_T - \mu Q)\zeta + \eta \\ &= U(t) \end{aligned}$$

and along non-transmission jumps (7c) (using (25a)) that

$$\begin{aligned} U(t^+) &= \xi^\top P_0 \xi + \eta + \zeta^\top (N_N - \beta Q)\zeta \\ &\leqslant \xi^\top P_h \xi + \zeta^\top (N_N - \beta Q - N_N + \beta Q)\zeta + \eta \\ &= U(t). \end{aligned}$$

Equations (30), (31), and (32) together prove that the system has an $\mathcal{L}_2$-gain from w to z smaller than or equal to θ [21, 24]. Furthermore, in case $w = 0$, we obtain

$$U(t) \leqslant e^{-2\rho t} U(0) \tag{33}$$

$$c_1 |\xi(t)|^2 + \eta(t) \leqslant e^{-2\rho t}(c_2 |\xi(0)|^2 + \eta(0)), \tag{34}$$

and, since $\eta(0) = 0$ and $\eta(t) \geqslant 0$ for all $t \in \mathbb{R}_{\geqslant 0}$, we find

$$|\xi(t)| \leqslant c e^{-\rho t} |\xi(0)|, \text{ and} \tag{35}$$

$$|\eta(t)| \leqslant c_2 e^{-2\rho t} |\xi(0)|^2 \tag{36}$$

with $c = \sqrt{c_1/c_2}$, which proves that the system is GES with decay rate ρ. □

While the static periodic event-generator only has design parameters h and Q, the state-based dynamic event-generator has design parameters h, Q, ρ, P_0, and P_h, and the output-based dynamic event-generator has design parameters h, Q, ρ, N_T, N_N, μ, and β. However, as already mentioned in Section 3, for fixed h, Q, ρ, and θ, inequalities (18) and (19) are LMIs, in which case the parameters P_h, P_0, N_T, N_N, μ, and β can be synthesized (and optimized) numerically via semi-definite programming (e.g., using Yalmip/SeDuMi in MATLAB). Hence, the design of these extra parameters follows directly and naturally from the design and stability analysis of the static event-generator. Of course, manual tuning of one or more of these parameters is also possible, but can be difficult given the large design space.

5. NUMERICAL EXAMPLE

In this section, we consider the example from [22], with open-loop unstable plant $\mathcal{P}$ given by (1) with $n_{x_p} = n_y = 2$ and

$$A_p = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_{pw} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

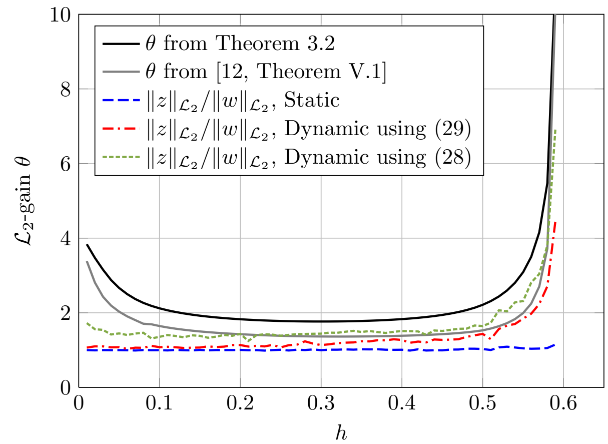
$$C_y = C_z = I, \quad D_y = D_z = D_{zw} = O,$$

and controller $\mathcal{C}$ given by (2) with $n_{x_c} = 0$ and

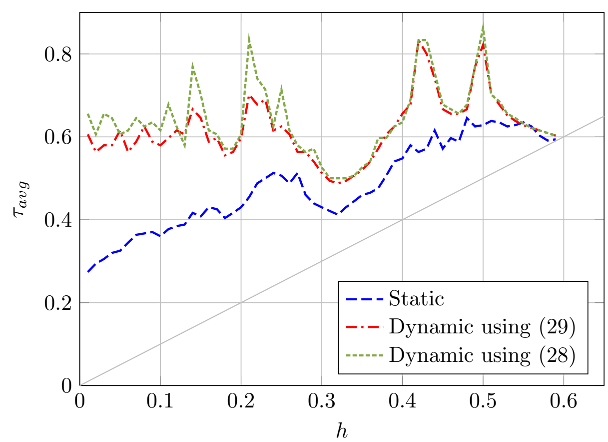
$$D_u = \begin{bmatrix} 1 & -4 \end{bmatrix}.$$

Note that this is a static state-feedback controller, and thus both (28) and (29) can be used.

Figure 2(a) shows the guaranteed $\mathcal{L}_2$-gain θ as a function of the timer threshold h based on Theorem 3.2. This $\mathcal{L}_2$-gain holds for the static event-generator (10) with (9), as well as for the state-based dynamic event-generator (8) with (27) and (28), and the output-based dynamic event-generator (8) with (27) and (29). For each h, the $\mathcal{L}_2$-gain



(a)



(b)

Figure 2: Guaranteed $\mathcal{L}_2$-gains θ (solid lines) and actual $\mathcal{L}_2$-gains $\|z\|_{\mathcal{L}_2}/\|w\|_{\mathcal{L}_2}$ (dashed lines) for varying h (a) and average inter-event times τ_{avg} for varying h (b).

θ is minimized via a bisection algorithm, and the matrices N_T, N_N, P_h and P_0 and scalars μ and β are found numerically by solving the LMIs (18) and (19). Additionally, for the three different event-generators, Figure 2(a) shows the actual $\mathcal{L}_2$-gains $\|z\|_{\mathcal{L}_2}/\|w\|_{\mathcal{L}_2}$ for the disturbance w given by

$$w(t) = e^{-0.2t} \sin(t/4), \tag{37}$$

which have been obtained by simulating the PETC systems for 120 time units with $\xi(0) = (0, 0, 0, 0)$ and disturbance w given by (37). Figure 2(b) shows the average inter-event times $\tau_{avg} = $ (total number of events)/(simulation time) for the static and dynamic event-generators, which have been obtained by the same simulations.

As the upper bound θ on the $\mathcal{L}_2$-gain holds for even the worst-case disturbance, the actual $\mathcal{L}_2$-gain $\|z\|_{\mathcal{L}_2}/\|w\|_{\mathcal{L}_2}$ is much lower than θ for most other disturbances when using the static event-generator. The dynamic event-generators store as much as possible of the unnecessary decrease of V in the buffer η, and by generating events only when this buffer would become empty (or negative) otherwise, the actual $\mathcal{L}_2$ gain $\|z\|_{\mathcal{L}_2}/\|w\|_{\mathcal{L}_2}$ is increased and therefore closer to the guaranteed upper bound θ. We clearly see this in Figure 2(a), as $\|z\|_{\mathcal{L}_2}/\|w\|_{\mathcal{L}_2}$ for the dynamic event-generators is closer to the upper bound θ from Theorem 3.2 than when using the static event-generator. As a result, also τ_{avg} of the dynamic event-generators is significantly higher than τ_{avg} of the static event-generator.

When looking at the $\mathcal{L}_2$-gain guarantees of Theorem 3.2, the dynamic event-generators yield larger average inter-event times than the static event-generator, for the same guaranteed control performance and sample time h. However, for the static PETC system (7) with (9) and (10), a tighter upper bound θ on the $\mathcal{L}_2$-gain can be calculated using [12, Theorem V.1], which is also shown in Figure 2(a) (where we partitioned the state-space into two regions using $X_1 = Y^\top QY$ and $X_2 = -Y^\top QY$, see [12] for more details). Our new dynamic PETC systems cannot be captured in the framework of [12] because of the quadratic terms in (28a), (28b), (29a), and (29b).

In Figure 2(a) we can see that when an $\mathcal{L}_2$-gain $\theta = 2$ is acceptable, we can select $h = 0.55$ when using the static event-generator (based on [12, Theorem V.1]), or $h = 0.45$ when using either of the dynamic event-generators (based on Theorem 3.2). In Figure 2(b) we can then see that for the disturbance w given by (37), the average inter-event time τ_{avg} for both dynamic event-generators with $h = 0.45$ is higher than τ_{avg} of the static event-generator with $h = 0.55$. Hence, for identical control performance guarantees, we can find a higher guaranteed minimum inter-event time h when using static PETC, but higher average inter-event times τ_{avg} might be obtained by using dynamic PETC. As a result, there is no clear answer to whether static or dynamic PETC is better, and which PETC variant one should use depends on the constraints and requirements of the control problem at hand.

6. CONCLUSIONS

We proposed two (state-based and output-based) dynamic periodic event-generators for linear systems. Our designs lead to closed-loop hybrid systems which are globally exponentially stable with a guaranteed decay rate and $\mathcal{L}_2$-stable with a guaranteed $\mathcal{L}_2$-gain. Moreover, for identical control performance guarantees, our new dynamic periodic event-generators typically generate fewer events than the corresponding static periodic event-generator. Finally, our new dynamic event-generators are the first in literature that only require periodic sampling of the state or output of the system, making our new designs much easier to implement on digital platforms than other dynamic event-generators that require continuous measuring of the system.

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