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# Dynamic event-triggered control with time regularization for linear systems

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***Abstract*— We present a framework for the analysis and design of dynamic and static event-triggered controllers with time regularization for linear systems. This framework leads to guarantees on global exponential stability, $\mathcal{L}_2$-stability, and a positive minimum inter-event time, in addition to a reduction in the number of events compared to regular time-triggered controllers and other event-triggered controllers in literature. By using new analysis tools tailored to linear systems, we achieve a significant reduction in conservatism, in the sense that the novel framework yields new event-generator designs with much larger inter-event times and much tighter bounds on the $\mathcal{L}_2$-gain and convergence rate of the event-triggered control system compared to previous results for more general nonlinear systems. We demonstrate the benefits of our new results via a numerical example, and show that the conservatism in the estimates of the $\mathcal{L}_2$-gain is indeed small.**

## I. Introduction

In most digital control systems the measured output of the plant is periodically transmitted to the controller. Hence, the transmission times $\{t_k\}_{k\in\mathbb{N}}$ are determined in open-loop, as they are independent of the state of the system. This possibly leads to a waste of (e.g., computation, communication, and energy) resources, as many of the transmissions are actually not necessary to achieve the desired performance guarantees. In recent years, many *event-triggered control* (ETC) strategies have been proposed which generate the transmission times based on the state $x_p(t) \in \mathbb{R}^{n_{x_p}}$ of the plant and the most recently transmitted value $\hat{x}_p(t) \in \mathbb{R}^{n_{x_p}}$ at time $t \in \mathbb{R}_{\geqslant 0}$. This brings a *feedback* mechanism into the sampling and communication process, such that measurement data is only transmitted to the controller when needed in order to guarantee the required stability and performance properties of the system [1]–[5]. Typical event-generators produce transmission times according to $t_0 = 0$, and

$$t_{k+1} = \inf\{t \geqslant t_k \mid \phi(x_p(t), \hat{x}_p(t)) \geqslant 0\},\ k \in \mathbb{N}, \quad (1)$$

for a certain function $\phi : \mathbb{R}^{2n_{x_p}} \to \mathbb{R}$. Hence, the event times are generated based on a *static* condition on the $x_p$ and $\hat{x}_p$. However, static event-generators as in (1) that lead to asymptotic stability of the ETC system typically exhibit Zeno behavior (an accumulation of transmission times, and thus a zero minimum inter-event time (MIET)) in the presence of disturbances, and static event-generators that do not exhibit Zeno behavior often only lead to practical stability and not to asymptotic stability [6]. So far, the only types of event-generators that do not exhibit Zeno behavior in the presence of disturbances and lead to asymptotic stability are *periodic* event-generators [7], and event-generators with *time regularization* (or 'waiting times') that enforce a positive MIET by design, see [8]–[11].

ETC is most effective when the conservatism in the event-generator design is small, as otherwise still too many unnecessary transmission would occur. Therefore, in [12], [13], *dynamic* event-generators have been proposed that generate events based on an additional dynamic variable, which leads to reduced conservatism and enlarged inter-event times. In our work [14], we proposed an event-generator design for a class of *nonlinear* systems, which combines dynamic event-generators and time regularization. In this work the event-generator produces transmission times according to

$$t_{k+1} = \inf\{t \geqslant t_k + h \mid \eta(t) \leqslant 0\},\ k \in \mathbb{N},\ t_0 = 0, \quad (2)$$

where $h \in \mathbb{R}_{>0}$ is the waiting time/guaranteed positive MIET, and $\eta(t) \in \mathbb{R}_{\geqslant 0}$ is an additional dynamical variable, which is included in the event-generator and has dynamics based on the state $x_p$ of the plant and its sampled version $\hat{x}_p$ (or the output $y$ and its sampled version $\hat{y}$). We were able to show that the resulting ETC system is globally asymptotically stable and $\mathcal{L}_p$-stable with a guaranteed $\mathcal{L}_p$-gain, if the dynamics of $\eta$ are properly designed.

The event-generators proposed in [11], [14] are formulated for *nonlinear* systems. In this work, we provide new designs of static and dynamic event-generators with time regularization tailored to *linear* systems, which guarantee global exponential stability (GES) and $\mathcal{L}_2$-stability. By exploiting linearity and making use of matrix Riccati differential equations combined with advanced hybrid system techniques, we achieve a significant reduction in conservatism, and are able to provide new and simplified event-generator designs with much larger inter-event times and much tighter bounds on the $\mathcal{L}_2$-gain and convergence rate of the event-triggered control system compared to the more general nonlinear results of [11], [14].

The results are illustrated with a numerical example that shows that, for identical control performance guarantees, the event-generators designed using the proposed tools for linear systems indeed yield much larger inter-event times than the event-generators proposed in [14]. The example also shows that the conservatism in the $\mathcal{L}_2$-gain estimate is small.

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### A. Notation

For a vector $x \in \mathbb{R}^{n_x}$, we denote by $|x| := \sqrt{x^\top x}$ its Euclidean norm. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we denote by $\lambda_{max}(A)$ and $\lambda_{min}(A)$ its maximum and minimum eigenvalue, respectively. For a matrix $P \in \mathbb{R}^{n\times n}$, we write $P \succ 0$ ($P \succeq 0$) if $P$ is symmetric and positive (semi-)definite, and $P \prec 0$ ($P \preceq 0$) if $P$ is symmetric and negative (semi-)definite. By $I$ and $O$ we denote the identity and zero matrix of appropriate dimensions, respectively. For a signal $w : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_w}$, we denote by $\|w\|_{\mathcal{L}_2} := \left(\int_0^\infty |w(t)|^2 \mathrm{d}t\right)^{1/2}$ its $\mathcal{L}_2$-norm. When $\|w\|_{\mathcal{L}_2} < \infty$, we write $w \in \mathcal{L}_2$. By $\mathbb{N}$ we denote the set of natural numbers including zero, i.e., $\mathbb{N} := \{0, 1, 2, \dots\}$. A function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$, and a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and, in addition, $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathcal{KL}$-function if for each fixed $t \in \mathbb{R}_{\geqslant 0}$ the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function and for each fixed $s \in \mathbb{R}_{\geqslant 0}$, $\beta(s, t)$ is decreasing in $t$ and $\beta(s, t) \to 0$ as $t \to \infty$. For vectors $x_i \in \mathbb{R}^{n_i}$, $i \in \{1, 2, \dots, N\}$, we denote by $(x_1, x_2, \cdots, x_N)$ the vector $[x_1^\top x_2^\top \cdots x_N^\top]^\top \in \mathbb{R}^n$ with $n = \sum_{i=1}^N n_i$. For brevity, we sometimes write symmetric matrices of the form $\left[\begin{smallmatrix} A & B \\ B^\top & C\end{smallmatrix}\right]$ as $\left[\begin{smallmatrix} A & B \\ \star & C\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} A & \star \\ B^\top & C\end{smallmatrix}\right]$. For a left-continuous signal $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^n$ and $t \in \mathbb{R}_{\geqslant 0}$, we use $f(t^+)$ to denote the limit $f(t^+) = \lim_{s\to t, s>t} f(s)$.

## II. Control Setup

In this paper we consider the event-triggered control setup of Figure 1, in which the plant $\mathcal{P}$ is given by

$$\mathcal{P} : \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_p(t) = A_p x_p(t) + B_p u(t) + B_{pw} w(t) \\ y(t) = C_y x_p(t) + D_y u(t) \\ z(t) = C_z x_p(t) + D_z u(t) + D_{zw} w(t) \end{cases} \tag{3}$$

and the controller $\mathcal{C}$ is given by

$$\mathcal{C} : \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_c(t) = A_c x_c(t) + B_c \hat{y}(t) \\ u(t) = C_u x_c(t) + D_u \hat{y}(t). \end{cases} \tag{4}$$

Here, $x_p(t) \in \mathbb{R}^{n_{x_p}}$ denotes the state of the plant $\mathcal{P}$, $y(t) \in \mathbb{R}^{n_y}$ its measured output, $z(t) \in \mathbb{R}^{n_z}$ the performance output, and $w(t) \in \mathbb{R}^{n_w}$ a disturbance at time $t \in \mathbb{R}_{\geqslant 0}$. Furthermore, $x_c(t) \in \mathbb{R}^{n_{x_c}}$ denotes the state of the controller $\mathcal{C}$, $u(t) \in \mathbb{R}^{n_u}$ the control input, and

$$\hat{y}(t) = y(t_k),\ t \in (t_k, t_{k+1}], \tag{5}$$

the sampled output, where the sequence $\{t_k\}_{k\in\mathbb{N}}$ denotes the event (or transmission) times, which are generated by the event-generator as will be specified below.

Define the state $\xi = (x_p, x_c, \hat{y}) \in \mathbb{R}^{n_\xi}$, with $n_\xi = n_{x_p} + n_{x_c} + n_y$, and introduce a timer variable $\tau \in \mathbb{R}_{\geqslant 0}$, which keeps track of the time that has elapsed since the latest event time, and a dynamic variable $\eta \in \mathbb{R}_{\geqslant 0}$, which will be included in the event-generator. Finally, define $\zeta(t) = (y(t), \hat{y}(t))$ and introduce the matrix $Y \in \mathbb{R}^{2n_y \times n_\xi}$

$$Y = \begin{bmatrix} C_y & D_y C_u & D_y D_u \\ O & O & I \end{bmatrix} \tag{6}$$

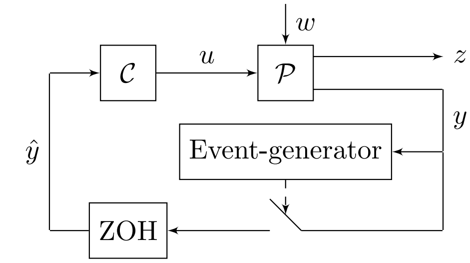



Fig. 1. Event-triggered control setup considered in this paper.

such that $\zeta = Y\xi$.

The variable $\eta$ will evolve according to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\eta(t) = \Psi(o(t)), \qquad t \in (t_k, t_{k+1}), \tag{7a}$$

$$\eta(t^+) = \eta_T(o(t)), \qquad t = t_k, \tag{7b}$$

where $o(t) = (\zeta(t), \tau(t), \eta(t))$ is the information that is available at the event-generator at time $t \in \mathbb{R}_{\geqslant 0}$, and where the functions $\Psi : \mathbb{R}^{2n_y} \times \mathbb{R}^2_{\geqslant 0} \to \mathbb{R}$ and $\eta_T : \mathbb{R}^{2n_y} \times \mathbb{R}^2_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ are to be designed. Now, we can write the closed-loop system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \xi(t) \\ \tau(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} A\xi(t) + Bw(t) \\ 1 \\ \Psi(o(t)) \end{bmatrix}, \quad t \in (t_k, t_{k+1}) \tag{8a}$$

$$\begin{bmatrix} \xi(t^+) \\ \tau(t^+) \\ \eta(t^+) \end{bmatrix} = \begin{bmatrix} J\xi(t) \\ 0 \\ \eta_T(o(t)) \end{bmatrix}, \qquad t = t_k \tag{8b}$$

$$z(t) = C\xi(t) + Dw(t), \tag{8c}$$

where

$$A = \begin{bmatrix} A_p & B_p C_u & B_p D_u \\ O & A_c & B_c \\ O & O & O \end{bmatrix},\ B = \begin{bmatrix} B_{pw} \\ O \\ O \end{bmatrix},$$

$$C = \begin{bmatrix} C_z & D_z C_u & D_z D_u \end{bmatrix},\ D = D_{zw},\ \text{and}$$

$$J = \begin{bmatrix} I & O & O \\ O & I & O \\ C_y & D_y C_u & D_y D_u \end{bmatrix}.$$

In this work, the sequence of jump/event times $\{t_k\}_{k\in\mathbb{N}}$ is generated by *dynamic* event-generators of the form

$$t_{k+1} = \inf\{t \geqslant t_k + h \mid \eta(t) \leqslant 0 \wedge \zeta^\top(t) Q \zeta(t) \geqslant 0\} \tag{9}$$

with $t_0 = 0$, and where $h \in \mathbb{R}_{\geqslant 0}$ is a timer threshold, which enforces a MIET of (at least) $h$ time units, and $Q \in \mathbb{R}^{2n_y \times 2n_y}$.

With the model (7), (9) we can also capture *static* event generators by choosing $\eta(0) = 0$ and

$$\Psi(o) = 0 \tag{10a}$$

$$\eta_T(o) = 0 \tag{10b}$$

for all $o \in \mathbb{R}^{2n_y} \times \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}$, as then we have that $\eta(t) = 0$ for all $t \in \mathbb{R}_{\geqslant 0}$, and the dynamic event-generator (9) reduces to the static event-generator

$$t_{k+1} = \inf\{t \geqslant t_k + h \mid \zeta^\top(t) Q \zeta(t) \geqslant 0\}. \tag{11}$$

In case also $h = 0$, the model (8), (11) reduces to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\xi(t) = A\xi(t) + Bw(t), \qquad \text{when } \zeta^\top Q\zeta \leqslant 0 \qquad (12a)$$

$$\xi(t^+) = J\xi(t), \qquad \text{when } \zeta^\top Q\zeta \geqslant 0, \qquad (12b)$$

as in, e.g., [2], [15], [16]. For instance, if we take $n_{x_c} = 0$, $y = x_p$, and

$$Q = \begin{bmatrix} (1-\sigma^2)I & -I \\ -I & I \end{bmatrix}, \tag{13}$$

we recover the ETC setup from [2] (which results in the event times $t_{k+1} = \inf\{t \geqslant t_k \mid |x_p - \hat{x}_p|^2 \geqslant \sigma^2|x_p|^2\}$). The setup (12) can exhibit Zeno behavior in the presence of disturbances, as shown in [6]. Therefore, we often take $h > 0$, which leads to static event-triggered controllers with time regularization, see, e.g., [8], [9].

We will consider the following two notions of stability.

*Definition 2.1:* The ETC system (8)-(9) is said to be globally exponentially stable (GES), if there exist a function $\beta \in \mathcal{KL}$ and scalars $c > 0$ and $\rho > 0$ such that for any initial condition $\xi(0) = \xi_0 \in \mathbb{R}^{n_\xi}$, $\tau(0) = 0$, $\eta(0) = 0$, all corresponding solutions to (8)-(9) with $w = 0$ satisfy $|\xi(t)| \leqslant ce^{-\rho t}|\xi_0|$ and $|\eta(t)| \leqslant \beta(|\xi_0|, t)$ for all $t \in \mathbb{R}_{\geqslant 0}$. In this case, we call $\rho$ a (lower bound on the) decay rate.

Note that we only require exponential decay of the state variable $\xi$, as we are only interested in the control performance regarding the plant and controller states, which are captured in $\xi$. For closed-loop stability, we then essentially only need that $\eta$ stays bounded in some way. We do not put any constraint on the variable $\tau$ as it is only used for modelling purposes.

*Definition 2.2:* The ETC system (8)-(9) is said to have an $\mathcal{L}_2$-gain from $w$ to $z$ smaller than or equal to $\theta$, if there exists a function $\delta \in \mathcal{K}_\infty$ such that for any initial condition $\xi(0) = \xi_0 \in \mathbb{R}^{n_\xi}$, $\tau(0) = 0$, $\eta(0) = 0$, all corresponding solutions to (8)-(9) with $w \in \mathcal{L}_2$ satisfy $\|z\|_{\mathcal{L}_2} \leqslant \delta(|\xi_0|) + \theta\|w\|_{\mathcal{L}_2}$.

## III. GES and $\mathcal{L}_2$-Gain Analysis

In this section, we first present conditions to analyze stability and performance of the class of static ETC systems given by (8) with (10) and (11) in Section III-A. These conditions give rise to new designs of dynamic event-generators as in (9), as we will see in Section III-B.

### A. Static Event-triggered Control

First we consider *static* event-generators of the form (11), where $Q \in \mathbb{R}^{2n_y \times 2n_y}$ and $h \in \mathbb{R}_{\geqslant 0}$ are design parameters.

To analyze GES and $\mathcal{L}_2$-stability of the system (8) with (10) and (11), we will use a Lyapunov/storage function $U$ of the form

$$U(\xi, \tau, \eta) = V(\xi, \tau) + \eta \tag{14}$$

with $V$ given by

$$V(\xi,\tau) = \begin{cases} \xi^\top P(\tau)\xi, & \text{when } \tau \in [0,h) \\ \xi^\top P(h)\xi, & \text{when } \tau \in [h,\infty), \end{cases} \tag{15}$$

where $P : [0,h] \to \mathbb{R}^{n_\xi \times n_\xi}$ is a continuously differentiable function with $P(\tau) \succ 0$ for $\tau \in [0,h]$.

The function $P : [0,h] \to \mathbb{R}^{n_\xi \times n_\xi}$ will be chosen such that $U$ becomes a storage function [17] for the ETC system (8) with (10) and (11) with the supply rate $\theta^{-2}z^\top z - w^\top w$ and decay rate $2\rho$. In particular, we will select $P : [0,h] \to \mathbb{R}^{n_\xi \times n_\xi}$ to satisfy the Riccati differential equation (where we omitted $\tau$ for compactness of notation)

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}\tau}P = &-A^\top P - PA - Y^\top N_F Y - 2\rho P - \theta^{-2}C^\top C \\ &- (PB + \theta^{-2}C^\top D)M(B^\top P + \theta^{-2}D^\top C) \end{aligned} \tag{16}$$

provided the solution exists on $[0,h]$ for a desired convergence rate $\rho > 0$. Here, $N_F \in \mathbb{R}^{2n_y \times 2n_y}$, $N_F \succeq 0$, is an arbitrary matrix, which we will use as a design variable in Section III-B, and $M := (I - \theta^{-2}D^\top D)^{-1}$ is assumed to exist and to be positive definite, which means that $\theta^2 > \lambda_{max}(D^\top D)$.

In order to find the explicit expression for $P$, we introduce the Hamiltonian matrix

$$H := \begin{bmatrix} A+\rho I+\theta^{-2}BMD^\top C & BMB^\top \\ -C^\top LC - Y^\top N_F Y & -\left(A+\rho I+\theta^{-2}BMD^\top C\right)^\top \end{bmatrix}$$

in which $L := (\theta^2 I - DD^\top)^{-1}$, and we define the matrix exponential

$$F(\tau) := e^{-H\tau} = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) \\ F_{21}(\tau) & F_{22}(\tau) \end{bmatrix}. \tag{17}$$

*Assumption 3.1:* $F_{11}(\tau)$ is invertible for all $\tau \in [0,h]$.

Assumption 3.1 can always be satisfied by choosing $h$ sufficiently small, as $F_{11}(0) = I$ and $F_{11}$ is a continuous function of $\tau$. The function $P : [0,h] \to \mathbb{R}^{n_\xi \times n_\xi}$ is now explicitly defined for $\tau \in [0,h]$ by

$$\begin{aligned} P(\tau) = &\left(F_{21}(h-\tau) + F_{22}(h-\tau)P(h)\right) \\ &\left(F_{11}(h-\tau) + F_{12}(h-\tau)P(h)\right)^{-1}, \end{aligned} \tag{18}$$

see [7], [18] for further details.

Before stating the next theorem, let us introduce the notation $P_0 := P(0)$, $P_h := P(h)$, $\bar{F}_{11} := F_{11}(h)$, $\bar{F}_{12} := F_{12}(h)$, $\bar{F}_{21} := F_{21}(h)$, and $\bar{F}_{22} := F_{22}(h)$, and the matrix $\bar{S}$ that satisfies $\bar{S}\bar{S}^\top := -\bar{F}_{11}^{-1}\bar{F}_{12}$. The matrix $\bar{S}$ exists under Assumption 3.1, because this assumption implies that the matrix $-\bar{F}_{11}^{-1}\bar{F}_{12}$ is positive semi-definite.

*Theorem 3.2:* Consider ETC system (8) with (10), (11), and $Q \in \mathbb{R}^{2n_y \times 2n_y}$. If there exist matrices $N_F, N_T \in \mathbb{R}^{2n_y \times 2n_y}$, $N_F, N_T \succeq 0$, and $P_h \in \mathbb{R}^{n_\xi \times n_\xi}$, $P_h \succ 0$, and scalars $\beta, \mu, \theta, \rho \in \mathbb{R}_{\geqslant 0}$, such that

$$\begin{aligned} &\begin{bmatrix} A^\top P_h + P_h A - \beta Y^\top QY & \star \\ B^\top P_h & O \end{bmatrix} \\ &\preceq \begin{bmatrix} -2\rho P_h - \theta^{-2}C^\top C & \star \\ -\theta^{-2}D^\top C & I - \theta^{-2}D^\top D \end{bmatrix}, \end{aligned} \tag{19}$$

$$\begin{aligned} &\begin{bmatrix} J^\top \bar{F}_{21}\bar{F}_{11}^{-1}J + J^\top \bar{F}_{11}^{-\top} P_h \bar{F}_{11}^{-1}J & \star \\ \bar{S}^\top P_h \bar{F}_{11}^{-1}J & \bar{S}^\top P_h \bar{S} \end{bmatrix} \\ &\prec \begin{bmatrix} P_h - Y^\top(N_T + \mu Q)Y & O \\ O & I \end{bmatrix}, \end{aligned} \tag{20}$$

and Assumption 3.1 hold, then the ETC system is GES with decay rate $\rho$, and the $\mathcal{L}_2$-gain from $w$ to $z$ is smaller than or equal to $\theta$.

*Proof:* The proof is based on the storage function $U$ given by (14) with $V$ as defined in (15). However, we only need to consider the function $V$, as it holds that $\eta(t) = 0$ for all $t \in \mathbb{R}_{\geqslant 0}$ and thus in this case $U = V$.

In the following we will use the notation $V(t) = V(\xi(t), \tau(t))$ for brevity. Following the derivations in the proof of [7, Theorem III.2] it can be shown that (16) implies that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V(t) \leqslant -2\rho V(t) - \theta^{-2}z(t)^\top z(t) + w(t)^\top w(t) - \zeta(t)^\top N_F \zeta(t) \quad (21)$$

during flow (8a) with $\tau \in [0, h)$. Additionally, from (19) it follows that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V(t) \leqslant -2\rho V(t) - \theta^{-2}z(t)^\top z(t) + w(t)^\top w(t) + \beta\zeta(t)^\top Q \zeta(t) \quad (22)$$

during flow (8a) with $\tau \in [h, \infty)$ and $\zeta^\top Q \zeta \leqslant 0$. Together, (21) and (22) show that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V(t) \leqslant -2\rho V(t) - \theta^{-2}z(t)^\top z(t) + w(t)^\top w(t) \quad (23)$$

holds during flow (8a).

Next, we show that $V$ does not increase along jumps. In [7], it is shown that

$$P_0 = \bar{F}_{21}\bar{F}_{11}^{-1} + \bar{F}_{11}^{-\top}\left(P_h + P_h\bar{S}\left(I - \bar{S}^\top P_h \bar{S}\right)^{-1}\bar{S}^\top P_h\right)\bar{F}_{11}^{-1}, \quad (24)$$

and thus by applying a Schur complement it follows from (20) that along jumps (8b) (when $\tau \in [h, \infty)$ and $\zeta^\top Q\zeta \geqslant 0$) we have

$$\begin{aligned} V\left(t_k^+\right) &= \xi(t_k)^\top J^\top P_0 J \xi(t_k) \\ &\leqslant \xi(t_k)^\top P_h \xi(t_k) - \zeta(t_k)^\top (N_T + \mu Q)\zeta(t_k) \\ &\leqslant \xi(t_k)^\top P_h \xi(t_k) = V\left(t_k\right). \end{aligned} \quad (25)$$

Finally, $V$ is a proper function, as

$$c_1|\xi|^2 \leqslant V(\xi, \tau) \leqslant c_2|\xi|^2, \quad (26)$$

where

$$c_1 = \min_{\tau \in [0,h]} \lambda_{min}(P(\tau)), \text{ and} \quad (27a)$$

$$c_2 = \max_{\tau \in [0,h]} \lambda_{max}(P(\tau)) \quad (27b)$$

with $c_2 \geqslant c_1 > 0$ as $P(\tau) \succ 0$ for all $\tau \in [0, h]$ when Assumption 3.1 holds [7], [18]. Combining (23), (25), and (26) indeed establishes the upper bound $\theta$ on the $\mathcal{L}_2$-gain of the ETC system (8), (11) [17]. Furthermore, when $w = 0$, it follows that for all $t \in \mathbb{R}_{\geqslant 0}$

$$|\xi(t)| \leqslant ce^{-\rho t}|\xi(0)| \quad (28)$$

with $c = \sqrt{c_1/c_2}$, which proves that the system is GES with decay rate $\rho$. ■

## B. Dynamic Event-triggered Control

Next, we design the dynamics (7) of the variable $\eta$ with the goal of enlarging the (average) inter-event times compared to the static event-generator of the previous subsection, while maintaining the same stability and performance guarantees.

We now choose the flow dynamics (7a) of $\eta$ as

$$\Psi(o) = \begin{cases} -2\rho\eta + \zeta^\top N_F \zeta, & \text{when } \tau \in [0, h) \quad (29a) \\ -2\rho\eta - \beta\zeta^\top Q\zeta, & \text{when } \tau \in [h, \infty), \quad (29b) \end{cases}$$

and the jump dynamics (7b) of $\eta$ as

$$\eta_T(o) = V(\xi, \tau) - V(J\xi, 0) = \xi^\top\left(P_h - J^\top P_0 J\right)\xi, \quad (30)$$

in case $\zeta = \xi$, (e.g., when $y = x_p$ and $n_{x_c} = 0$, which is the case when $\mathcal{C}$ is a static state-feedback controller). Alternatively, in case $\zeta \neq \xi$, we choose

$$\eta_T(o) = \zeta^\top N_T \zeta. \quad (31)$$

*Theorem 3.3:* Consider ETC system (8) with (9), $\Psi$ given by (29), $\eta_T$ given by (30) or (31), $Q \in \mathbb{R}^{2n_y \times 2n_y}$, $N_F, N_T \in \mathbb{R}^{2n_y \times 2n_y}$, $N_F, N_T \succeq 0$, and $\beta \in \mathbb{R}_{\geqslant 0}$. If there exist a matrix $P_h \in \mathbb{R}^{n_\xi \times n_\xi}$, $P_h \succ 0$, and scalars $\mu, \theta, \rho \in \mathbb{R}_{\geqslant 0}$, such that (19), (20), and Assumption 3.1 hold, then the ETC system is GES with decay rate $\rho$ and the $\mathcal{L}_2$-gain from $w$ to $z$ is smaller than or equal to $\theta$.

*Proof:* Consider the Lyapunov/storage function candidate $U$ given by (14). First, we will show that $U$ is a proper storage function. From (30) or (31) we have that $\eta(t_k^+) \geqslant 0$ for all $k \in \mathbb{N}$, as it follows from (20) that

$$0 \leqslant \zeta^\top N_T \zeta \leqslant \xi^\top\left(P_h - J^\top P_0 J\right)\xi$$

when $\zeta^\top Q \zeta \geqslant 0$. Together with (29a) and $N_F \succeq 0$ it follows from the comparison lemma [19, Lemma 3.4] that $\eta(t) \geqslant 0$ for all $t \in [t_k, t_k + h)$, $k \in \mathbb{N}$, and from (29b) and (9) it follows that $\eta(t) \geqslant 0$ for all $t \in [t_k + h, t_{k+1}]$, $k \in \mathbb{N}$, (as $\eta$ can only become negative when $\eta = 0$ and $\zeta^\top Q\zeta > 0$, in which case a jump (8b) would be triggered). Thus, $\eta(t) \geqslant 0$ for all $t \in \mathbb{R}_{\geqslant 0}$. Combined with (26) and (14) it follows that

$$c_1|\xi|^2 + |\eta| \leqslant U(\xi, \tau, \eta) \leq c_2|\xi|^2 + |\eta|, \quad (32)$$

where $c_1$ and $c_2$ are given by (27).

From (16) and (29a) it follows that

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}U(t) &\leqslant -2\rho V(t) - 2\rho\eta(t) - \theta^{-2}z(t)^\top z(t) + w(t)^\top w(t) \\ &= -2\rho U(t) - \theta^{-2}z(t)^\top z(t) + w(t)^\top w(t) \end{aligned} \quad (33)$$

during flow (8a) with $\tau \in [0, h)$. In addition, (19) and (29b) imply that inequality (33) holds during flow (8a) with $\tau \in [h, \infty)$.

Finally, along jumps (8b) (when $\tau \in [h, \infty)$, $\eta = 0$, and $\zeta^\top Q\zeta \geqslant 0$) we have from (14), (20), and (30) or (31) that

$$U(t_k^+) \leqslant U(t_k). \quad (34)$$

Equations (32), (33), and (34) together prove that the system has an $\mathcal{L}_2$-gain from $w$ to $z$ smaller than or equal to $\theta$ [17]. Furthermore, in case $w = 0$, we obtain

$$U\left(t\right) \leqslant e^{-2\rho t}U\left(0\right), \quad (35)$$

$$c_1|\xi(t)|^2 + \eta(t) \leqslant e^{-2\rho t}(c_2|\xi(0)|^2 + \eta(0)), \quad (36)$$

and, since $\eta(0) = 0$ and $\eta(t) \geqslant 0$ for all $t \in \mathbb{R}_{\geqslant 0}$, we find

$$|\xi(t)| \leqslant ce^{-\rho t}|\xi(0)|, \text{ and} \tag{37}$$

$$|\eta(t)| \leqslant c_2 e^{-2\rho t}|\xi(0)|^2 \tag{38}$$

with $c = \sqrt{c_1/c_2}$, which proves that the system is GES with decay rate $\rho$. ■

While the static event-generator only has design variables $h$ and $Q$, the dynamic event-generator has design variables $h$, $Q$, $N_F$, $N_T$, and $\beta$. However, as we can see, the design of these extra variables follows directly and naturally from the design and stability analysis of the static event-generator.

In principle, the values of $N_T$ and $N_F$ should be chosen as large as possible, as this leads to the largest increase of $\eta$ during flow (29a) and along jumps (31), such that large (average) inter-event times (and thus low resource utilization) might be expected. On the other hand, an increase in $N_F$ or $N_T$ typically demands extra decrease of $V$ during flow or along jumps, respectively, which (for fixed $h$) leads to an increase in the $\mathcal{L}_2$-gain $\theta$ and/or a decrease in the decay rate $\rho$ (and thus a decrease in control performance). This indicates the presence of a tradeoff between resource utilization and control performance.

*Remark 3.1:* For fixed $Q$, the variables $N_T$ and $\beta$ appear linearly in the LMIs (19) and (20), and can be synthesized numerically. Hence, maximizing $N_T$ *without* any penalty on $h$, $\theta$, or $\rho$ can be done in a straightforward manner. For the matrix $N_F$ this is more difficult, as it appears nonlinearly in the LMIs (19) and (20), and thus manual tuning is required. Note that by choosing $N_F = O$ and maximizing $N_T$ as described above, the dynamic event-generator (9) yields the same $\mathcal{L}_2$-gain $\theta$ and exponential decay rate $\rho$ as the corresponding static event-generator (11). Moreover, larger (average) inter-event times might be expected, as will also be demonstrated in the numerical example in Section IV-B.

## IV. Numerical Example

In this section, we consider the example from [2], with open-loop unstable plant $\mathcal{P}$ given by (3) with $n_{x_p} = n_y = 2$ and

$$A_p = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_{pw} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$C_y = C_z = I, \quad D_y = D_z = D_{zw} = O,$$

and controller $\mathcal{C}$ given by (4) with $n_{x_c} = 0$ and $D_u = \begin{bmatrix} 1 & -4 \end{bmatrix}$. Note that this is a static state-feedback controller, and thus both (30) and (31) can be used.

### A. Comparison with [14]

For this example, the event-generator from [14] can be cast in the format of (7), (9) with $\Psi$ given by

$$\Psi(o) = \varrho(x_p) - 2\rho\eta, \tag{39a}$$

when $\tau \in [0, h)$ and

$$\Psi(o) = \varrho(x_p) - 2\rho\eta - \gamma\left(2\lambda L + \gamma(1+\lambda^2)\right)|\hat{y} - y|^2, \tag{39b}$$

when $\tau \in [h, \infty)$, and $\eta_T$ given by

$$\eta_T(o) = \gamma\lambda|\hat{y} - y|^2. \tag{40}$$

Here, $\varrho$ is an arbitrary positive semi-definite function and $\lambda$, $L$, and $\gamma$ are constants, see [14] for more details.

We will now design the functions (29), (31), (39), and (40), such that the function (29) is equal to (39), and (31) is equal to (40). In this way, we ensure that the dynamic event-generator from [14] is identical to the dynamic event-generator (7)-(9) with (29) and (31) proposed here. The same holds for their static counterparts. This allows us to give a fair comparison of the $\mathcal{L}_2$-gain analysis proposed here and in [14].

From [14] it follows that $L = |-B_p D_u| = 4.1231$, we choose $\lambda = 0.01$, $\rho = 0$, and $\varrho(x_p) = 0.2|x_p|^2$, and $\gamma$ follows from optimizing LMI (70) in [14]. In order to make the function $\Psi$ given by (29) identical to (39) we choose

$$N_F = 0.2\begin{bmatrix} I & O \\ O & O \end{bmatrix}, \quad \beta = 1, \text{ and} \tag{41}$$

$$Q = \gamma\left(2\lambda L + \gamma(1+\lambda^2)\right)\begin{bmatrix} I & -I \\ -I & I \end{bmatrix} - N_F. \tag{42}$$

Finally, in order to make the function $\eta_T$ given by (31) identical to (40) we choose

$$N_T = \gamma\lambda\begin{bmatrix} I & O \\ O & O \end{bmatrix}. \tag{43}$$

Figure 2 shows the guaranteed $\mathcal{L}_2$-gain $\theta$ as a function of the timer threshold $h$ for the two different frameworks. Additionally, it shows the corresponding average inter-event times $\tau_{avg}$ for the static and dynamic event-generators, which have been obtained by simulating the system for 40 time units with $\xi(0) = (0, 0, 0, 0)$ and disturbance $w$ given by

$$w(t) = e^{-0.2t}\sin(t/4). \tag{44}$$

For this linear example, we can guarantee the same $\mathcal{L}_2$-gain

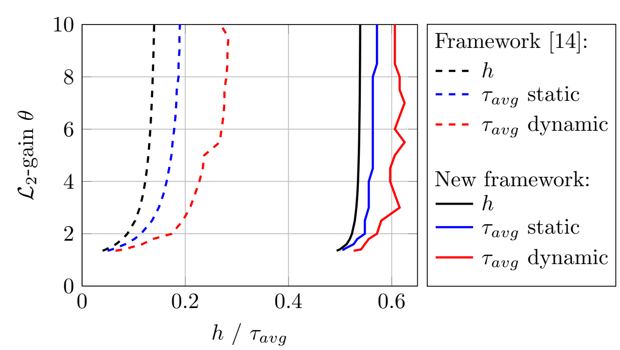


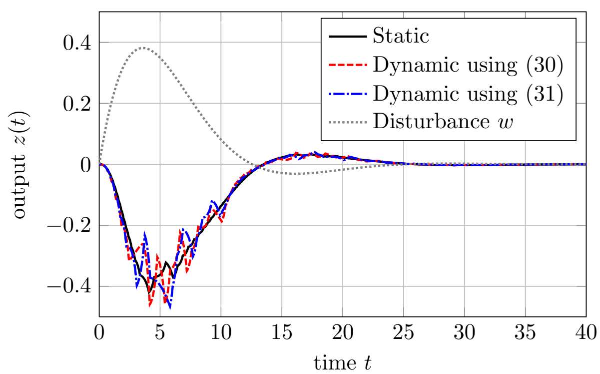

Fig. 2. Tradeoffs of the guaranteed $\mathcal{L}_2$-gain $\theta$ vs. the timer threshold $h$ and the average inter-event times $\tau_{avg}$ using the framework proposed in [14] (dashed lines) and the proposed new framework in this paper (full lines).

$\theta$ for much larger $h$ using the new framework for *linear* systems, compared to the framework of [14] for *nonlinear* systems. In fact, for identical $\theta$, the average inter-event times $\tau_{avg}$ of the static and dynamic event-generators with $h$ determined using [14] are below the MIET $h$ derived with the current framework. Finally, we can see that the dynamic event-generators have higher $\tau_{avg}$ (and thus produce fewer redundant transmissions) than their static counterparts.

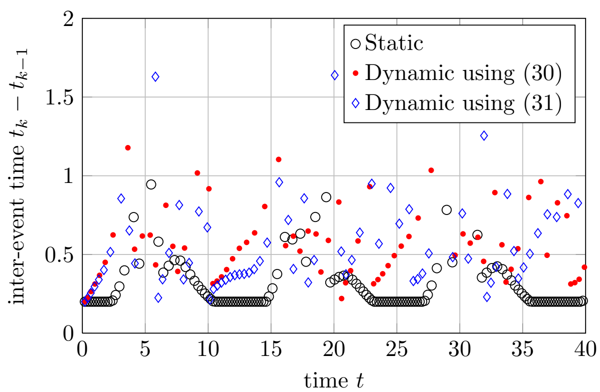
### B. Time response

Finally, we study the time response of the ETC system with the proposed event-generators. We choose $h = 0.2$, $Q$ as in (13) with $\sigma = 0.10$, $\rho = 0.05$, and $N_F = O$. We find the matrix $N_T$ and the guaranteed $\mathcal{L}_2$-gain $\theta$ by means of numerical optimization, resulting in $\theta = 1.2083$.

In Figure 3 we see the time response and the inter-event times of the ETC system with the static and the two dynamic event-generators, again with $\xi(0) = (0, 0, 0, 0)$ and disturbance $w$ given by (44). For this disturbance, we find $\|z\|_{\mathcal{L}_2}/\|w\|_{\mathcal{L}_2} = 0.9946$ when using the static event-generator (11), $\|z\|_{\mathcal{L}_2}/\|w\|_{\mathcal{L}_2} = 0.9913$ when using the dynamic event-generator (9) with $\eta_T$ given by (30), and $\|z\|_{\mathcal{L}_2}/\|w\|_{\mathcal{L}_2} = 1.0036$ when using (9) with $\eta_T$ given by (31). As the guaranteed $\mathcal{L}_2$-gain is $\theta = 1.2083$, the level of conservatism in the $\mathcal{L}_2$-gain analysis is at most 20%, which is remarkably small given that (44) is most likely not the worst-case disturbance, and we are dealing with *hybrid* closed-loop systems, for which it is well-known that tight stability and performance bounds are difficult to obtain.




(a)




(b)

Fig. 3. Output $z$ (a) and inter-event times $t_k - t_{k-1}$, $k \in \mathbb{N}_{>0}$ (b).

In Figure 3(b) we see that the static event-generator (11) often generates events directly when $\tau = h$. Both dynamic event-generators do not have this behavior and produce much less events on average than the static event-generator, while the control performance is almost identical. This underlines the benefits of using dynamic event-generators.

## V. Conclusions

We proposed a new method for the design of static and dynamic event-generators with time regularization for linear systems. The proposed method uses design tools tailored to linear systems, which leads to significantly larger inter-event times and tighter $\mathcal{L}_2$-gain estimates than other design methods currently available. In fact, we showed via a numerical example that the conservatism in the guaranteed $\mathcal{L}_2$-gain is small.

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