Chapter 7
Time-Regularized and Periodic Event-Triggered Control for Linear Systems

D. P. Borgers, V. S. Dolk, G. E. Dullerud, A. R. Teel and W. P. M. H. Heemels

Abstract In this chapter, we provide an overview of our recent results for the analysis and design of Event-Triggered controllers that are tailored to *linear systems* as provided in Heemels et al., IEEE Trans Autom Control 58(4):847–861, 2013, Heemels et al., IEEE Trans Autom Control 61(10):2766–2781, 2016, Borgers et al., IEEE Trans Autom Control, 2018. In particular, we discuss two different frameworks for the stability and contractivity analysis and design of (static) periodic Event-Triggered control (PETC) and time-regularized continuous Event-Triggered control (CETC) systems: the lifting-based framework of Heemels et al., IEEE Trans Autom Control 61(10):2766–2781, 2016, which applies to PETC systems, and the Riccati-based framework of Heemels et al., IEEE Trans Autom Control 58(4):847–861, 2013, Borgers et al., IEEE Trans Autom Control (2018), which applies to both PETC systems and time-regularized CETC systems. Moreover, we identify the connections and differences between the two frameworks. Finally, for PETC and time-regularized CETC systems, we show how the Riccati-based analysis leads to new designs for *dynamic* Event-Triggered controllers, which (for identical stability and contractivity guarantees) lead to a significantly reduced consumption of communication and energy resources compared to their static counterparts.

D. P. Borgers (✉) · V. S. Dolk · W. P. M. H. Heemels
Eindhoven University of Technology, Eindhoven, The Netherlands
e-mail: d.p.borgers@tue.nl

V. S. Dolk
e-mail: v.s.dolk@tue.nl

W. P. M. H. Heemels
e-mail: m.heemels@tue.nl

G. E. Dullerud
University of Illinois, Urbana, IL, USA
e-mail: dullerud@illinois.edu

A. R. Teel
University of California, Santa Barbara, CA, USA
e-mail: teel@ece.ucsb.edu

S. Tarbouriech et al. (eds.), *Control Subject to Computational and Communication Constraints*, Lecture Notes in Control and Information Sciences 475, https://doi.org/10.1007/978-3-319-78449-6_7

7.1 Introduction

In most digital control systems, the measured output of the plant is periodically transmitted to the controller, regardless of the state the system is in. This possibly leads to a waste of (e.g., computation, communication, and energy) resources, as many of the transmissions are actually not needed to achieve the desired control performance guarantees. In recent years, many *Event-Triggered control* (ETC) strategies have been proposed, which generate the transmission (event) times based on the current state or output of the system and the most recently transmitted measurement data, thereby bringing feedback into the process of deciding when control tasks are executed and corresponding measurement and control data is transmitted. In contrast, in periodic time-triggered control, the control execution process could be considered as an open-loop mechanism. By using feedback in the control execution process, measurement data is only transmitted to the controller when this is really necessary in order to be able to guarantee the required stability and performance properties of the system. Clearly, in the interconnected world we live in with many networked control applications including cooperative robotics, vehicle platooning, Internet-of-things, and so on, it is important to use the available (computation, communication, and energy) resources of the system carefully in order to avoid congesting the computational devices or communication networks, or draining batteries. The use of ETC can play an important role in achieving this.

A major challenge in the design of ETC strategies is meeting certain *control performance specifications* (quality-of-control), such as global asymptotic stability, bounds on convergence rates, or $\mathscr{L}_p$-gain requirements, while simultaneously satisfying constraints on the resource utilization (required quality-of-service), including a guaranteed positive lower bound on the inter-event times and thus the absence of Zeno behaviour (an infinite number of events in finite time). In [5, 15], it was shown that this combination of quality-of-control and (required) quality-of-service specifications is hard to achieve, especially for *continuous* Event-Triggered control (CETC) schemes, in which the event condition is continuously monitored (which also requires continuous measuring of the state or output of the plant), as proposed in, e.g., [12, 20, 21, 24, 32, 33, 41, 51].

In the recent years, two main solutions were proposed to tackle this problem:

- CETC schemes that adopt a minimal waiting time between two event times ("time-regularization"), see, e.g., [1, 2, 13, 14, 18, 24, 29, 39, 42–44] and the references therein;
- Periodic Event-Triggered control (PETC) schemes that check the event conditions only at periodic sampling times that are equidistantly distributed along the time axis, see, e.g., [24, 25, 28, 29, 36] and the references therein.

In this chapter, we provide an overview of our recent results for the analysis and design of PETC and time-regularized CETC schemes that are tailored to *linear systems* as provided in [8, 25, 26]. In particular, we discuss two different analysis and design frameworks: the framework as developed in [26], which uses ideas from lifting [4, 10, 16, 45, 46, 53], and the framework as developed in [6–8, 25], which exploits matrix Riccati differential equations.

The lifting-based framework of [26] applies to PETC systems, and leads to the important result that the stability and contractivity in $\mathscr{L}_2$-sense (meaning that the $\mathscr{L}_2$-gain is smaller than 1) of PETC closed-loop systems (which are hybrid systems) is equivalent to the stability and contractivity in ℓ_2-sense (meaning that the ℓ_2-gain is smaller than 1) of an appropriate discrete-time piecewise linear system [26]. These new insights are obtained by adopting a lifting-based perspective on this analysis problem, which leads to computable ℓ_2-gain (and thus $\mathscr{L}_2$-gain) conditions, despite the fact that the linearity assumption, which is usually needed in the lifting literature, is not satisfied.

The Riccati-based framework of [6–8, 25] applies both to PETC systems and to time-regularized CETC systems, and exploits matrix Riccati differential equations for the construction of appropriate Lyapunov/storage functions in the stability and performance analysis. For the PETC case, we identify the connections and differences between the Riccati-based and lifting-based approaches. Moreover, for PETC and time-regularized CETC systems, we show how the Riccati-based analysis leads to new designs for *dynamic* Event-Triggered controllers. Interestingly, the inclusion of a dynamic variable in the event-generator can lead to a significantly reduced consumption of communication and energy resources while leading to identical guarantees on stability and performance as their static counterparts, see also [13, 14, 21, 37, 38] in which designs of dynamic ETC schemes for general nonlinear systems were proposed for the first time.

Both frameworks lead to computationally friendly semi-definite programming conditions, and can also be used for applications in many other domains, including reset control, networked control systems, and switching sampled-data controllers [4, 9–11, 19, 45, 46, 53].

The chapter is organized as follows. In Sect. 7.2, we introduce the considered Event-Triggered control setups. For PETC systems, we introduce the lifting-based framework in Sect. 7.3, and the Riccati-based framework in Sect. 7.4. In Sect. 7.5, we show how the Riccati-based framework of Sect. 7.4 can be modified in order to analyze stability and contractivity of time-regularized CETC systems. We illustrate the results by a numerical example in Sect. 7.6, which also shows that our new frameworks tailored to linear systems are much less conservative than our previous results for nonlinear systems in [13, 14], in the sense that tighter performance bounds can be obtained. Finally, we discuss several directions of extensions of the two frameworks in Sect. 7.7, and summarize the chapter in Sect. 7.8.

7.1.1 Notation

By $\mathbb{N}$ we denote the set of natural numbers including zero, i.e., $\mathbb{N} := \{0, 1, 2, \dots\}$. For vectors $x_i \in \mathbb{R}^{n_i}$, $i \in \{1, 2, \dots, N\}$, we denote by $(x_1, x_2, \cdots, x_N)$ the vector $[x_1^\top x_2^\top \cdots x_N^\top]^\top \in \mathbb{R}^n$ with $n = \sum_{i=1}^N n_i$. For a matrix $P \in \mathbb{R}^{n\times n}$, we write $P \succ 0$

($P \succeq 0$) if P is symmetric and positive (semi-)definite, and $P \prec 0$ ($P \preceq 0$) if P is symmetric and negative (semi-)definite. By I and O we denote the identity and zero matrix of appropriate dimensions, respectively. For brevity, we sometimes write symmetric matrices of the form $\left[\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right]$ as $\left[\begin{smallmatrix} A & B \\ \star & C \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} A & \star \\ B^\top & C \end{smallmatrix}\right]$. For a left-continuous signal $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $t \in \mathbb{R}_{\geq 0}$, we use $f(t^+)$ to denote the limit $f(t^+) = \lim_{s \to t, s>t} f(s)$.

For X, Y Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$, respectively, a linear operator $U : X \to Y$ is called isometric if $\langle U x_1, U x_2 \rangle_Y = \langle x_1, x_2 \rangle_X$ for all $x_1, x_2 \in X$. We denote by $U^* : Y \to X$ the (Hilbert) adjoint operator that satisfies $\langle U x, y \rangle_Y = \langle x, U^* y \rangle_X$ for all $x \in X$ and all $y \in Y$. The induced norm of U (provided it is finite) is denoted by $\|U\|_{X,Y} = \sup_{x \in X \setminus \{0\}} \frac{\|Ux\|_Y}{\|x\|_X}$. If the induced norm is finite we say that U is a bounded linear operator. If $X = Y$ we write $\|U\|_X$ and if X, Y are clear from the context we use the notation $\|U\|$. An operator $U : X \to X$ with X a Hilbert space is called self-adjoint if $U^* = U$. A self-adjoint operator $U : X \to X$ is called positive semi-definite if $\langle U x, x \rangle \geq 0$ for all $x \in X$. Given a positive semi-definite U, we say that the bounded linear operator $A : X \to X$ is the square root of U if A is positive semi-definite and $A^2 = U$. This square root exists and is unique, see [31, Theorem 9.4-1]. We denote it by $U^{1/2}$.

To a Hilbert space X with inner product $\langle \cdot, \cdot \rangle_X$, we can associate the Hilbert space $\ell_2(X)$ consisting of infinite sequences $\tilde{x} = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \ldots\}$ with $\tilde{x}_i \in X$, $i \in \mathbb{N}$, satisfying $\sum_{i=0}^{\infty} \|\tilde{x}_i\|_X^2 < \infty$, and the inner product $\langle \tilde{x}, \tilde{y} \rangle_{\ell_2(X)} = \sum_{i=0}^{\infty} \langle \tilde{x}_i, \tilde{y}_i \rangle_X$. We denote $\ell_2(\mathbb{R}^n)$ by ℓ_2 when $n \in \mathbb{N}_{\geq 1}$ is clear from the context. We also use the notation $\ell(X)$ to denote the set of all infinite sequences $\tilde{x} = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \ldots\}$ with $\tilde{x}_i \in X$, $i \in \mathbb{N}$. Note that $\ell_2(X)$ can be considered a subspace of $\ell(X)$. As usual, we denote by $\mathbb{R}^n$ the standard n-dimensional Euclidean space with inner product $\langle x, y \rangle = x^\top y$ and norm $|x| = \sqrt{x^\top x}$ for $x, y \in \mathbb{R}^n$. $\mathscr{L}_2^n([0, \infty))$ denotes the set of square-integrable functions defined on $\mathbb{R}_{\geq 0} := [0, \infty)$ and taking values in $\mathbb{R}^n$ with $\mathscr{L}_2$-norm $\|x\|_{\mathscr{L}_2} = \sqrt{\int_0^\infty |x(t)|^2 dt}$ and inner product $\langle x, y \rangle_{\mathscr{L}_2} = \int_0^\infty x^\top(t) y(t) dt$ for $x, y \in \mathscr{L}_2^n([0, \infty))$. If n is clear from the context we also write $\mathscr{L}_2$. We also use square-integrable functions on subsets $[a, b]$ of $\mathbb{R}_{\geq 0}$ and then we write $\mathscr{L}_2^n([a, b])$ (or $\mathscr{L}_2([a, b])$ if n is clear from context) with the inner product and norm defined analogously. The set $\mathscr{L}_{2,e}^n([0, \infty))$ consists of all locally square-integrable functions, i.e., all functions x defined on $\mathbb{R}_{\geq 0}$, such that for each bounded domain $[a, b] \subset \mathbb{R}_{\geq 0}$ the restriction $x \mid_{[a,b]}$ is contained in $\mathscr{L}_2^n([a, b])$. We also will use the set of essentially bounded functions defined on $\mathbb{R}_{\geq 0}$ or $[a, b] \subset \mathbb{R}_{\geq 0}$, which are denoted by $\mathscr{L}_\infty^n([0, \infty))$ or $\mathscr{L}_\infty^n([a, b])$ with the norm given by the essential supremum denoted by $\|x\|_{\mathscr{L}_\infty}$ for an essentially bounded function x. A function $\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called a $\mathscr{K}$-function if it is continuous, strictly increasing, and $\beta(0) = 0$.

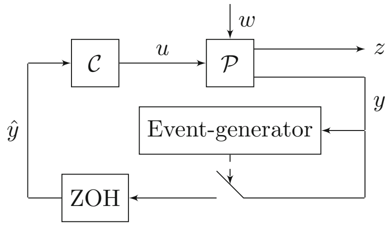

Fig. 7.1 Event-Triggered control setup

7.2 Event-Triggered Control Setup

In this paper, we consider the Event-Triggered control setup as shown in Fig. 7.1, in which the plant $\mathscr{P}$ is given by

$$\mathscr{P}: \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_p = A_p x_p + B_p u + B_{pw} w \\ y = C_y x_p + D_y u \\ z = C_z x_p + D_z u + D_{zw} w \end{cases} \tag{7.1}$$

and the controller $\mathscr{C}$ is given by

$$\mathscr{C}: \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_c = A_c x_c + B_c \hat{y} \\ u = C_u x_c + D_u \hat{y}. \end{cases} \tag{7.2}$$

For ease of exposition, we stick to the configuration of Fig. 7.1, although different control setups are possible as well, see, e.g., [25].

In (7.1) and (7.2), $x_p(t) \in \mathbb{R}^{n_{x_p}}$ denotes the state of the plant $\mathscr{P}$, $y(t) \in \mathbb{R}^{n_y}$ its measured output, $z(t) \in \mathbb{R}^{n_z}$ the performance output, and $w(t) \in \mathbb{R}^{n_w}$ a disturbance at time $t \in \mathbb{R}_{\geq 0}$. Furthermore, $x_c(t) \in \mathbb{R}^{n_{x_c}}$ denotes the state of the controller $\mathscr{C}$, $u(t) \in \mathbb{R}^{n_u}$ is the control input at time $t \in \mathbb{R}_{\geq 0}$, and $\hat{y}(t) \in \mathbb{R}^{n_y}$ denotes the output that is available at the controller, given by

$$\hat{y}(t) = y(t_k), \quad t \in (t_k, t_{k+1}], \tag{7.3}$$

where the sequence $\{t_k\}_{k \in \mathbb{N}}$ denotes the event (or transmission) times, which are generated by the *event-generator*.

In this chapter, we consider *periodic* event-generators, and *continuous* event-generators with *time-regularization*. We will provide their designs in Sects. 7.2.1 and 7.2.2, respectively. In order to do so, we will first define the state $\xi :=(x_p, x_c, \hat{y}) \in \mathbb{R}^{n_\xi}$, with $n_\xi = n_{x_p} + n_{x_c} + n_y$, and the matrix $Y \in \mathbb{R}^{2n_y \times n_\xi}$ as

$$Y := \begin{bmatrix} C_y & D_y C_u & D_y D_u \\ O & O & I \end{bmatrix} \tag{7.4}$$

such that $\zeta := (y, \hat{y}) = Y\xi$.

7.2.1 Periodic Event-Triggered Control

In a periodic Event-Triggered control (PETC) setup, the plant output y is sampled periodically at fixed sample times $s_n = nh$, $n \in \mathbb{N}$, where $h \in \mathbb{R}_{>0}$ is the sample period. At each sample time s_n, $n \in \mathbb{N}$, the event-generator decides whether or not the measured output $y(s_n)$ should be transmitted to the controller. Hence, the sequence of event times $\{t_k\}_{k\in\mathbb{N}}$ is a subsequence of the sequence of sample times $\{s_n\}_{n\in\mathbb{N}}$.

In this work, we consider periodic event-generators of the form

$$t_0 = 0,\ t_{k+1} = \inf\{t > t_k \mid \zeta^\top(t)Q\zeta(t) > 0,\ t = nh,\ n \in \mathbb{N}\}, \tag{7.5}$$

where the scalar $h \in \mathbb{R}_{>0}$ and the matrix $Q \in \mathbb{R}^{2n_y\times 2n_y}$ are design parameters. A possible choice for Q is given by

$$Q = \begin{bmatrix} (1-\sigma^2)I & -I \\ -I & I \end{bmatrix} \tag{7.6}$$

with $\sigma \in (0, 1)$, such that (7.5) reduces to

$$t_0 = 0,\ t_{k+1} = \inf\{t > t_k \mid |\hat{y}(t) - y(t)|^2 > \sigma^2|y(t)|^2,\ t = nh,\ n \in \mathbb{N}\},$$

which can be seen as the digital version of static continuous event-generators [41] of the type

$$t_0 = 0,\ t_{k+1} = \inf\{t \geq t_k \mid |\hat{y}(t) - y(t)|^2 > \sigma^2|y(t)|^2,\ t \in \mathbb{R}_{\geq 0}\}.$$

Other control setups and other choices of Q are also possible, see, e.g., [25].

By introducing a timer variable $\tau \in [0, h]$, which keeps track of the time that has elapsed since the latest sample time, the closed-loop PETC system consisting of (7.1)–(7.3), and (7.5) can be written as the hybrid system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \xi \\ \tau \end{bmatrix} = \begin{bmatrix} A\xi + Bw \\ 1 \end{bmatrix}, \qquad \tau \in [0, h], \tag{7.7a}$$

$$\begin{bmatrix} \xi^+ \\ \tau^+ \end{bmatrix} = \begin{bmatrix} J\xi \\ 0 \end{bmatrix}, \qquad \tau = h \text{ and } \zeta^\top Q\zeta > 0, \tag{7.7b}$$

$$\begin{bmatrix} \xi^+ \\ \tau^+ \end{bmatrix} = \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \qquad \tau = h \text{ and } \zeta^\top Q\zeta \leq 0, \tag{7.7c}$$

$$z = C\xi + Dw, \tag{7.7d}$$

where

$$A = \begin{bmatrix} A_p & B_pC_u & B_pD_u \\ O & A_c & B_c \\ O & O & O \end{bmatrix}, \; B = \begin{bmatrix} B_{pw} \\ O \\ O \end{bmatrix}, \; J = \begin{bmatrix} I & O & O \\ O & I & O \\ C_y & D_yC_u & D_yD_u \end{bmatrix}, \tag{7.8}$$
$$C = \begin{bmatrix} C_z & D_zC_u & D_zD_u \end{bmatrix}, \text{ and } D = D_{zw}.$$

At sample times $s_n = nh, n \in \mathbb{N}$, the reset (7.7b) occurs when an event is triggered by the event-generator, otherwise the state (ξ, τ) jumps according to (7.7c). In between the sample times, the system evolves according to the differential equation (7.7a), where $\big(\xi(s_n^+), \tau(s_n^+)\big)$ given by (7.7b) or (7.7c) denotes the starting point for the solution to (7.7a) in the interval $(s_n, s_{n+1}], n \in \mathbb{N}$. Hence, the solutions are considered to be left-continuous signals.

7.2.2 *Time-Regularized Continuous Event-Triggered Control*

In this chapter, we also consider *continuous* event-generators with *time-regularization*, of the form

$$t_{k+1} = \inf\{t \geq t_k + h \mid \zeta^\top(t) Q \zeta(t) > 0\}, \tag{7.9}$$

where now the scalar $h \in \mathbb{R}_{\geq 0}$ is a timer threshold (a waiting time), which enforces a MIET of (at least) h time units. If we again choose Q as in (7.6), then (7.9) constitutes the time-regularized version of (7.2.1). Note that the practical implementation of (7.9) requires continuous monitoring of the output y, which can be difficult to achieve on digital platforms.

The closed-loop CETC system consisting of (7.1)–(7.3) and (7.9) can be written as the hybrid system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \xi \\ \tau \end{bmatrix} = \begin{bmatrix} A\xi + Bw \\ 1 \end{bmatrix}, \qquad \tau \in [0, h] \text{ or } \zeta^\top Q \zeta \leq 0 \tag{7.10a}$$
$$\begin{bmatrix} \xi^+ \\ \tau^+ \end{bmatrix} = \begin{bmatrix} J\xi \\ 0 \end{bmatrix}, \qquad \tau \in [h, \infty) \text{ and } \zeta^\top Q \zeta > 0 \tag{7.10b}$$
$$z = C\xi + Dw, \tag{7.10c}$$

where the timer variable $\tau \in \mathbb{R}_{\geq 0}$ now keeps track of the time that has elapsed since the latest event time. The matrices A, B, C, D, and J are again given by (7.8).

7.2.3 *Stability and Performance*

As the objective of the paper is to study the $\mathscr{L}_2$-gain and internal stability of the systems (7.7) and (7.10), let us first provide rigorous definitions of these important concepts.

Definition 7.1 The hybrid system (7.7) or (7.10) is said to have an $\mathscr{L}_2$-gain from w to z smaller than γ if there exist a $\gamma_0 \in [0, \gamma)$ and a $\mathscr{K}$-function β such that, for any $w \in \mathscr{L}_2$ and any initial conditions $\xi(0) = \xi_0$ and $\tau(0) = h$, the corresponding solution to (7.7) or (7.10) satisfies $\|z\|_{\mathscr{L}_2} \leq \beta(|\xi_0|) + \gamma_0\|w\|_{\mathscr{L}_2}$. Sometimes, we also use the terminology γ-contractivity (in $\mathscr{L}_2$-sense) if this property holds. Moreover, 1-contractivity is also called contractivity (in $\mathscr{L}_2$-sense).

Definition 7.2 The hybrid system (7.7) or (7.10) is said to be internally stable if there exists a $\mathscr{K}$-function β such that, for any $w \in \mathscr{L}_2$ and any initial conditions $\xi(0) = \xi_0$ and $\tau(0) = h$, the corresponding solution to (7.7) or (7.10) satisfies $\|\xi\|_{\mathscr{L}_2} \leq \beta(\max(|\xi_0|, \|w\|_{\mathscr{L}_2}))$.

A few remarks are in order regarding this definition of internal stability. The requirement $\|\xi\|_{\mathscr{L}_2} \leq \beta(\max(|\xi_0|, \|w\|_{\mathscr{L}_2}))$ is rather natural in this context as we are working with $\mathscr{L}_2$-disturbances and investigate $\mathscr{L}_2$-gains. Indeed, just as in Definition 7.1, where a bound is required on the $\mathscr{L}_2$-norm of the output z (expressed in terms of a bound on $|\xi_0|$ and $\|w\|_{\mathscr{L}_2}$), we require in Definition 7.2 that a similar (though less strict) bound holds on the state trajectory ξ. Apart from internal stability, both design frameworks also lead to global attractivity of the origin (i.e., $\lim_{t\to\infty} \xi(t) = 0$ for all $w \in \mathscr{L}_2$, $\xi(0) = \xi_0$ and $\tau(0) = h$) and Lyapunov stability of the origin, see Proposition 7.1 for the lifting-based framework and [8] for the Riccati-based framework.

Remark 7.1 In this chapter, we focus on the contractivity of the systems (7.7) and (7.10) as γ-contractivity can be studied by proper scaling of the matrices C and D in (7.7), i.e., $C_{\text{scaled}} = \gamma^{-1}C$ and $D_{\text{scaled}} = \gamma^{-1}D$.

7.3 Lifting-Based Static PETC

In this section, we give an overview of our work [26], which provides a framework for the contractivity and internal stability analysis of the static PETC system (7.7) using ideas from lifting [4, 10, 16, 45, 46, 53]. To obtain necessary and sufficient conditions for internal stability and contractivity of (7.7), we use a procedure consisting of three main steps:

- In Sect. 7.3.2, we apply lifting-based techniques to (7.7) (having finite-dimensional input and output spaces) leading to a discrete-time system with infinite-dimensional input and output spaces (see (7.15) below). The internal stability and contractivity of both systems are equivalent.
- In Sect. 7.3.3, we apply a loop transformation to the infinite-dimensional system (7.15) in order to remove the feedthrough term, which is the only operator in the system description having both its domain and range being infinite dimensional. This transformation is constructed in such a manner that the internal stability and contractivity properties of the system are not changed. This step is crucial for

translating the infinite-dimensional system to a finite-dimensional system in the last step.

- In Sect. 7.3.4, the loop-transformed infinite-dimensional system is converted into a discrete-time *finite-dimensional* piecewise linear system (again without changing the stability and the contractivity properties of the system). Due to the finite dimensionality of the latter system, stability and contractivity in ℓ_2-sense can be analyzed, for instance, using well-known Lyapunov-based arguments. We elaborate on these computational aspects (which also exploit semi-definite programming) in Sect. 7.3.5.

These three steps lead to the main result as formulated in Theorem 7.2, which states that the internal stability and contractivity (in $\mathscr{L}_2$-sense) of (7.7) is equivalent to the internal stability and contractivity (in ℓ_2-sense) of a discrete-time finite-dimensional piecewise linear system. To facilitate the analysis, we first introduce the necessary preliminary definitions in Sect. 7.3.1.

7.3.1 Preliminaries

Consider the discrete-time system of the form

$$\xi_{k+1} = \chi(\xi_k, v_k) \tag{7.11a}$$

$$r_k = \psi(\xi_k, v_k) \tag{7.11b}$$

with $v_k \in V$, $r_k \in R$, $\xi_k \in \mathbb{R}^{n_\xi}$, $k \in \mathbb{N}$, with V and R Hilbert spaces, and $\chi : \mathbb{R}^{n_\xi} \times V \to \mathbb{R}^{n_\xi}$ and $\psi : \mathbb{R}^{n_\xi} \times V \to R$.

For this general discrete-time system, we also introduce ℓ_2-gain specifications and internal stability.

Definition 7.3 The discrete-time system (7.11) is said to have an ℓ_2-gain from v to r smaller than γ if there exist a $\gamma_0 \in [0, \gamma)$ and a $\mathscr{K}$-function β such that, for any $v \in \ell_2(V)$ and any initial state $\xi_0 \in \mathbb{R}^{n_\xi}$, the corresponding solution to (7.11) satisfies

$$\|r\|_{\ell_2(R)} \leq \beta(\|\xi_0\|) + \gamma_0 \|v\|_{\ell_2(V)}. \tag{7.12}$$

Sometimes, we also use the terminology γ-contractivity (in ℓ_2-sense) if this property holds. Moreover, 1-contractivity is also called contractivity (in ℓ_2-sense).

Definition 7.4 The discrete-time system (7.11) is said to be internally stable if there is a $\mathscr{K}$-function β such that, for any $v \in \ell_2(V)$ and any initial state $\xi_0 \in \mathbb{R}^{n_\xi}$, the corresponding solution ξ to (7.11) satisfies

$$\|\xi\|_{\ell_2} \leq \beta(\max(|\xi_0|, \|v\|_{\ell_2(V)})). \tag{7.13}$$

Note that this internal stability definition for the discrete-time system (7.11) parallels the continuous-time version in Definition 7.2. Moreover, since $\|\xi\|_{\ell_\infty} \leq \|\xi\|_{\ell_2}$ and $\|\xi\|_{\ell_2} < \infty$ implies $\lim_{k\to\infty} \xi_k = 0$, we also have global attractivity and Lyapunov stability properties of the origin when the discrete-time system is internally stable.

7.3.2 Lifting the System

To study contractivity, we introduce the lifting operator $W : \mathscr{L}_{2,e}[0,\infty) \to \ell(\mathscr{K})$ with $\mathscr{K} = \mathscr{L}_2[0,h]$ given for $w \in \mathscr{L}_{2,e}[0,\infty)$ by $W(w) = \tilde{w} = \{\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, \ldots\}$ with

$$\tilde{w}_k(s) = w(kh+s) \text{ for } s \in [0,h] \tag{7.14}$$

for $k \in \mathbb{N}$. Using this lifting operator, we can rewrite the model in (7.7) as

$$\xi_{k+1} = \hat{A}\xi_k^+ + \hat{B}\tilde{w}_k \tag{7.15a}$$

$$\xi_k^+ = \begin{cases} J\xi_k, & \xi_k^\top Y^\top QY\xi_k > 0 \\ \xi_k, & \xi_k^\top Y^\top QY\xi_k \leq 0 \end{cases} \tag{7.15b}$$

$$\tilde{z}_k = \hat{C}\xi_k^+ + \hat{D}\tilde{w}_k \tag{7.15c}$$

in which ξ_0 is given and $\xi_k = \xi(kh)$, $k \in \mathbb{N}_{\geq 1}$, $\xi_k^+ = \xi(kh^+)$ (assuming that ξ is left-continuous) for $k \in \mathbb{N}$, and $\tilde{w} = \{\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, \ldots\} = W(w) \in \ell_2(\mathscr{K})$ and $\tilde{z} = \{\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, \ldots\} = W(z) \in \ell(\mathscr{K})$. Here we assume in line with Definition 7.1 that $\tau(0) = h$ in (7.7). Moreover,

$$\hat{A} : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_\xi}, \quad \hat{B} : \mathscr{K} \to \mathbb{R}^{n_\xi}, \quad \hat{C} : \mathbb{R}^{n_\xi} \to \mathscr{K}, \quad \text{and } \hat{D} : \mathscr{K} \to \mathscr{K}$$

are given for $x \in \mathbb{R}^{n_\xi}$ and $\omega \in \mathscr{K}$ by

$$\hat{A}x = e^{Ah}x \tag{7.16a}$$

$$\hat{B}\omega = \int_0^h e^{A(h-s)}B\omega(s)ds \tag{7.16b}$$

$$(\hat{C}x)(\theta) = Ce^{A\theta}\xi \tag{7.16c}$$

$$(\hat{D}\omega)(\theta) = \int_0^\theta Ce^{A(\theta-s)}B\omega(s)ds + D\omega(\theta), \tag{7.16d}$$

where $\theta \in [0,h]$.

It follows that (7.15) is contractive if and only if (7.7) is contractive. In fact, we have the following proposition

Proposition 7.1 [26] *The following statements hold:*

- *The hybrid system (7.7) is internally stable if and only if the discrete-time system (7.15) is internally stable.*
- *The hybrid system (7.7) is contractive if and only if the discrete-time system (7.15) is contractive.*
- *In case (7.7) is internally stable, it also holds that* $\lim_{t\to\infty}\xi(t)=0$ *and* $\|\xi\|_{\mathscr{L}_\infty}\leq\beta(\max(|\xi_0|,\|w\|_{\mathscr{L}_2}))$ *for all* $w\in\mathscr{L}_2$, $\xi(0)=\xi_0$ *and* $\tau(0)=h$.

7.3.3 Removing the Feedthrough Term

Following [4], we aim at removing the feedthrough operator $\hat{D}$ as this is the only operator with both its domain and range being infinite dimensional. Removal can be accomplished by using an operator-valued version of Redheffer's lemma, see [4, Lemma 5]. The objective is to obtain a new system (without feedthrough term) and new disturbance inputs $\tilde{v}_k\in\mathscr{K}$, new state $\bar{\xi}_k\in\mathbb{R}^{n_\xi}$, and new performance output $\tilde{r}_k\in\mathscr{K}$, $k\in\mathbb{N}$, given by

$$\bar{\xi}_{k+1}=\bar{A}\bar{\xi}_k^{+}+\bar{B}\tilde{v}_k \tag{7.17a}$$

$$\bar{\xi}_k^{+}=\begin{cases}J\bar{\xi}_k, & \bar{\xi}_k^\top Y^\top QY\bar{\xi}_k>0\\ \bar{\xi}_k, & \bar{\xi}_k^\top Y^\top QY\bar{\xi}_k\leq 0\end{cases} \tag{7.17b}$$

$$\tilde{r}_k=\bar{C}\bar{\xi}_k^{+} \tag{7.17c}$$

such that (7.15) is internally stable and contractive if and only if (7.17) is internally stable and contractive. To do so, we first observe that a necessary condition for the contractivity (7.15) is that $\|\hat{D}\|_{\mathscr{K}}<1$. Indeed, $\|\hat{D}\|_{\mathscr{K}}\geq 1$ would imply that for any $0\leq\gamma_0<1$ there is a $\tilde{w}_0\in\mathscr{K}\setminus\{0\}$ with $\|\hat{D}\tilde{w}_0\|_{\mathscr{K}}\geq\gamma_0\|\tilde{w}_0\|_{\mathscr{K}}$, which, in turn, would lead for the system (7.15) with $\xi_0=0$ and thus $\xi_0^+=0$ and disturbance sequence $\{\tilde{w}_0,0,0,\ldots\}$ to a contradiction with the contractivity of (7.15). We can now find an equivalent system of the form (7.17), with bounded linear operators

$$\bar{A}:\mathbb{R}^{n_\xi}\to\mathbb{R}^{n_\xi},\quad \bar{B}:\mathscr{K}\to\mathbb{R}^{n_\xi},\quad \text{and } \bar{C}:\mathbb{R}^{n_\xi}\to\mathscr{K}.$$

These operators are given by [26, Sect. IV.B]

$$\bar{A}=\hat{A}+\hat{B}\hat{D}^*(I-\hat{D}\hat{D}^*)^{-1}\hat{C}, \tag{7.18a}$$

$$\bar{B}=\hat{B}(I-\hat{D}^*\hat{D})^{-\frac{1}{2}}, \tag{7.18b}$$

$$\bar{C}=(I-\hat{D}\hat{D}^*)^{-\frac{1}{2}}\hat{C}. \tag{7.18c}$$

Hence, we establish the following result.

Theorem 7.1 [26] If $\|\hat{D}\|_{\mathscr{K}} < 1$, then internal stability and contractivity of system (7.15) with $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ as in (7.16) are equivalent to internal stability and contractivity of system (7.17) with $\bar{A}$, $\bar{B}$, and $\bar{C}$ as in (7.18).

7.3.4 From Infinite-Dimensional to Finite-Dimensional Systems

The system (7.17) is still an infinite-dimensional system, although the operators $\bar{A}$, $\bar{B}$, and $\bar{C}$ have finite rank and therefore have finite-dimensional matrix representations. Following (and slightly extending) [4], we now obtain the following result.

Theorem 7.2 [26] Consider system (7.7) and its lifted version (7.15) with $\|\hat{D}\|_{\mathscr{K}} < 1$. Define the discrete-time piecewise linear system

$$\xi_{k+1} = \begin{cases} A_1\xi_k + B_d v_k, & \xi_k^\top Y^\top Q Y \xi_k > 0 \\ A_2\xi_k + B_d v_k, & \xi_k^\top Y^\top Q Y \xi_k \leq 0 \end{cases} \tag{7.19a}$$

$$r_k = \begin{cases} C_1\xi_k, & \xi_k^\top Y^\top Q Y \xi_k > 0 \\ C_2\xi_k, & \xi_k^\top Y^\top Q Y \xi_k \leq 0, \end{cases} \tag{7.19b}$$

$k \in \mathbb{N}$, with $A_1 = A_d J$, $A_2 = A_d$, $C_1 = C_d J$, and $C_2 = C_d$, where A_d is defined by

$$A_d = \hat{A} + \hat{B}\hat{D}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C} \tag{7.20a}$$

and $B_d \in \mathbb{R}^{n_\xi \times n_v}$ and $C_d \in \mathbb{R}^{n_r \times n_\xi}$ are chosen such that

$$\begin{aligned} B_d B_d^\top &= \bar{B}\bar{B}^* = \hat{B}(I - \hat{D}^*\hat{D})^{-1}\hat{B}^* \text{ and} \\ C_d^\top C_d &= \bar{C}^*\bar{C} = \hat{C}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C}. \end{aligned} \tag{7.20b}$$

The system (7.7) is internally stable and contractive if and only if the system (7.19) is internally stable and contractive.

Hence, this theorem states that under the assumption $\|\hat{D}\|_{\mathscr{K}} < 1$ (which is a necessary condition for contractivity of (7.7)) the internal stability and contractivity (in $\mathscr{L}_2$-sense) of (7.7) is equivalent to the internal stability and contractivity (in ℓ_2-sense) of a discrete-time finite-dimensional piecewise linear system given by (7.19). In the next section, we will show how the matrices A_d, B_d, and C_d in (7.19) can be constructed, how the condition $\|\hat{D}\|_{\mathscr{K}} < 1$ can be tested, and how internal stability and contractivity can be tested for the system (7.19).

7.3.5 Computing the Discrete-Time Piecewise Linear System

To explicitly compute the discrete-time system (7.19) provided in Theorem 7.2, we need to determine the operators $\hat{B}\hat{D}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C}$, $\hat{B}(I - \hat{D}^*\hat{D})^{-1}\hat{B}^*$, and $\hat{C}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C}$ to obtain the triple (A_d, B_d, C_d) in (7.19). For the sake of self-containedness, we recall the procedure proposed in [9] to compute this triple, assuming throughout that $\|\hat{D}\|_{\mathscr{K}} < 1$.

First, we verify that $\|\hat{D}\|_{\mathscr{K}} < 1$, which is a necessary condition for the contractivity of (7.7). Define the Hamiltonian matrix

$$H := \begin{bmatrix} A+BMD^\top C & BMB^\top \\ -C^\top LC & -\left(A+BMD^\top C\right)^\top \end{bmatrix} \tag{7.21}$$

in which $L := (I - DD^\top)^{-1}$ and $M := (I - D^\top D)^{-1}$, and the matrix exponential

$$F(\tau) := e^{-H\tau} = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) \\ F_{21}(\tau) & F_{22}(\tau) \end{bmatrix}. \tag{7.22}$$

The condition $\|\hat{D}\|_{\mathscr{K}} < 1$ is equivalent to the following assumption [26].

Assumption 7.3 $\lambda_{\max}(D^\top D) < 1$ and $F_{11}(\tau)$ is invertible for all $\tau \in [0, h]$.

Invertibility of $F_{11}(\tau)$ for all $\tau \in [0, h]$ can always be achieved by choosing h sufficiently small, as $F_{11}(0) = I$ and F_{11} is a continuous function.

The procedure to find A_d, B_d, and C_d boils down to computing $F(h)$, which then leads to

$$A_d = \bar{F}_{11}^{-1}, \tag{7.23}$$

and

$$B_d B_d^\top = -\bar{F}_{11}^{-1}\bar{F}_{12}, \tag{7.24a}$$

$$C_d^\top C_d = \bar{F}_{21}\bar{F}_{11}^{-1}, \tag{7.24b}$$

where we used the notation $\bar{F}_{11} := F_{11}(h)$, $\bar{F}_{12} := F_{12}(h)$, $\bar{F}_{21} := F_{21}(h)$, and $\bar{F}_{22} := F_{22}(h)$.

This provides the matrices needed for explicitly determining the discrete-time piecewise linear system (7.19) for which the internal stability and contractivity tests need to be carried out.

To guarantee the internal stability and contractivity of a discrete-time piecewise linear system as in (7.19) (in order to guarantee these properties for the hybrid system (7.7) using Theorem 7.2), we aim at finding a Lyapunov function $V : \mathbb{R}^{n_\xi} \to \mathbb{R}_{\geqslant 0}$ that satisfies the dissipation inequality [47, 52]

$$V(\xi_{k+1}) - V(\xi_k) < -r_k^\top r_k + v_k^\top v_k,\ k \in \mathbb{N}, \tag{7.25}$$

and require that it holds along the trajectories of the system (7.19). An effective approach is to use versatile piecewise quadratic Lyapunov/storage functions [17, 30] of the form

$$V(\xi) = \begin{cases} \xi^\top P_1^p \xi & \text{with } p = \min\{q \in \{1, \dots, N\} \mid \xi \in \Omega_q\} \text{ when } \xi^\top Y^\top Q Y \xi > 0 \\ \xi^\top P_2^p \xi & \text{with } p = \min\{q \in \{1, \dots, N\} \mid \xi \in \Omega_q\} \text{ when } \xi^\top Y^\top Q Y \xi \leq 0 \end{cases} \tag{7.26}$$

based on the regions

$$\Omega_p := \left\{ \xi \in \mathbb{R}^{n_\xi} \,\middle|\, X_p \xi \geq 0 \right\}, \; p \in \{1, \dots, N\} \tag{7.27}$$

in which the matrices X_p, $p \in \{1, \dots, N\}$, are such that $\{\Omega_1, \Omega_2, \dots, \Omega_N\}$ forms a partition of $\mathbb{R}^{n_\xi}$, i.e., $\cup_{p=1}^N \Omega_p = \mathbb{R}^{n_\xi}$ and the intersection of $\Omega_p \cap \Omega_q$ is of zero measure for all $p, q \in \{1, \dots, N\}$ with $p \neq q$.

This translates into sufficient LMI-based conditions for stability and contractivity using three S-procedure relaxations [30], as formulated next.

Theorem 7.4 If there exist symmetric matrices $P_i^p \in \mathbb{R}^{n_\xi \times n_\xi}$, scalars $a_i^p, c_{ij}^{pq}, d_{ij}^{pq} \in \mathbb{R}_{>0}$, and symmetric matrices $E_i^p, U_{ij}^{pq}, W_{ij}^{pq} \in \mathbb{R}_{\geq 0}^{n_\xi \times n_\xi}$, with $i, j \in \{1, 2\}$, $p, q \in \{1, 2, \dots, N\}$, such that

$$\left[\, P_i^p + (-1)^i a_i^p Y^\top Q Y - X_p^\top E_i^p X_p \,\right] \succ 0 \tag{7.28a}$$

and

$$\begin{aligned} &\begin{bmatrix} P_i^p - C_i^\top C_i - A_i^\top P_j^q A_i & -A_i^\top P_j^q B_d \\ -B_d P_j^q A_i & I - B_d^\top P_j^q B_d \end{bmatrix} \\ &\quad + \begin{bmatrix} (-1)^i c_{ij}^{pq} Y^\top Q Y + (-1)^j d_{ij}^{pq} A_i^\top Y^\top Q Y A_i & (-1)^j d_{ij}^{pq} A_i^\top Y^\top Q Y B_d \\ (-1)^j d_{ij}^{pq} B_d^\top Y^\top Q Y A_i & (-1)^j d_{ij}^{pq} B_d^\top Y^\top Q Y B_d \end{bmatrix} \\ &\quad - \begin{bmatrix} X_p^\top U_{ij}^{pq} X_p + A_i^\top X_q^\top W_{ij}^{pq} X_q A_i & A_i^\top X_q^\top W_{ij}^{pq} X_q B_d \\ B_d^\top X_q^\top W_{ij}^{pq} X_q A_i & B_d^\top X_q^\top W_{ij}^{pq} X_q B_d \end{bmatrix} \prec 0 \end{aligned} \tag{7.28b}$$

hold for all $i, j \in \{1, 2\}$ and all $p, q \in \{1, 2, \dots, N\}$, then the discrete-time piecewise linear system (7.19) is internally stable and contractive.

Two comments are in order regarding this theorem. First, note that due to the strictness of the LMIs (7.28), we guarantee that the ℓ_2-gain is strictly smaller than 1, which can be seen from appropriately including the strictness into the dissipativity inequality (7.25). Moreover, due to the strictness of the LMIs we also guarantee internal stability. Second, the LMI conditions of Theorem 7.4 are obtained by performing a contractivity analysis on the *discrete-time* piecewise linear system (7.19) using three S-procedure relaxations:

(i) require that $\xi^\top P_i^p \xi$ is positive only when $(-1)^i \xi^\top Y^\top Q Y \xi \le 0$ and $X_p \xi \ge 0$ (this corresponds to the terms containing a_i^p and E_i^p in (7.28a), respectively);
(ii) use a relaxation related to the current time instant, i.e., if $V(\xi_k) = \xi_k^\top P_i^p \xi_k$, then it holds that $(-1)^i \xi_k^\top Y^\top Q Y \xi_k \le 0$ and $X_p \xi_k \ge 0$ (this corresponds to the terms containing c_{ij}^{pq} and U_{ij}^{pq} in (7.28b), respectively);
(iii) use a relaxation related to the next time instant, i.e., if $V(\xi_{k+1}) = \xi_{k+1}^\top P_j^q \xi_{k+1}$, then it holds that $(-1)^j \xi_{k+1}^\top Y^\top Q Y \xi_{k+1} \le 0$ and $X_q \xi_{k+1} \ge 0$ (this corresponds to the terms containing d_{ij}^{pq} and W_{ij}^{pq} in (7.28b), respectively).

Theorem 7.4 can be used to guarantee the internal stability and contractivity of (7.19) and hence, the internal stability and contractivity for the hybrid system (7.7). In the next section, we will rigorously show that these results form significant improvements with respect to the earlier conditions for contractivity of (7.7) presented in [11, 22, 25] and [48]. In Sect. 7.6, we also illustrate this improvement using two numerical examples.

7.4 Riccati-Based PETC

In this section, we recall the LMI-based conditions for analyzing the stability and contractivity analysis for the static PETC system (7.7) provided in [11, 25, 48], and show the relationship to the conditions obtained in Sect. 7.3.2. This also reveals that the conditions in Sect. 7.3.2 are (significantly) less conservative.

However, instead of reducing the conservatism in the stability and contractivity analyses of [11, 25, 48], we have shown in [7, 8] that we can also exploit this conservatism in order to reduce the amount of transmissions even further (with the same stability and performance guarantees as the static counterpart). This leads to the design of *dynamic* periodic event-generators, which we also cover in this section.

7.4.1 Static PETC

We follow here the setup discussed in [25], which is based on using a timer-dependent storage function $V : \mathbb{R}^{n_\xi} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, see [47], satisfying

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V \le -z^\top z + w^\top w, \tag{7.29}$$

during the flow (7.7a), and

$$V(J\xi, 0) < V(\xi, h), \text{ for all } \xi \text{ with } \xi^\top Y^\top Q Y \xi > 0, \tag{7.30a}$$

$$V(\xi, 0) < V(\xi, h), \text{ for all } \xi \text{ with } \xi^\top Y^\top Q Y \xi \le 0, \tag{7.30b}$$

during the jumps (7.7b) and (7.7c). From these conditions, we can guarantee that the $\mathscr{L}_2$-gain from w to z is smaller than or equal to 1, see, e.g., [27].

In fact, in [25], $V(\xi, \tau)$ was chosen in the form

$$V(\xi, \tau) = \xi^\top P(\tau)\xi, \quad \tau \in [0, h], \tag{7.31}$$

where $P : [0, h] \to \mathbb{R}^{n_\xi \times n_\xi}$ is a continuously differentiable function with $P(\tau) \succ 0$ for $\tau \in [0, h]$. The function P will be chosen such that (7.31) becomes a storage function [47, 52] for the PETC system (7.7), (7.5) with the supply rate $\theta^{-2}z^\top z - w^\top w$. In order to do so, we select the function $P : [0, h] \to \mathbb{R}^{n_\xi \times n_\xi}$ to satisfy the Riccati differential equation (where we omitted τ for compactness of notation)

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau} P = -A^\top P - PA - C^\top C - (PB + C^\top D)M(D^\top C + B^\top P). \tag{7.32}$$

Note that the solution to (7.32) exists under Assumption 7.3, see also [3, Lemma 9.2]. As shown in the proof of [25, Theorem III.2], this choice for the matrix function P implies the "flow condition" (7.29). The "jump condition" (7.30) is guaranteed in [25] by LMI-based conditions that lead to a proper choice of the boundary value $P_h := P(h)$.

To formulate the result of [25], we again consider the Hamiltonian matrix (7.21) and the matrix exponential (7.22). The function $P : [0, h] \to \mathbb{R}^{n_\xi \times n_\xi}$ is then explicitly defined for $\tau \in [0, h]$ by

$$P(\tau) = (F_{21}(h - \tau) + F_{22}(h - \tau)P(h))\,(F_{11}(h - \tau) + F_{12}(h - \tau)P(h))^{-1}, \tag{7.33}$$

provided that Assumption 7.3 holds.

Before stating the next theorem (which is a slight variation of [25, Theorem III.2]), let us introduce the notation $P_0 := P(0)$, $P_h := P(h)$, and a matrix $\bar{S}$ that satisfies $\bar{S}\bar{S}^\top := -\bar{F}_{11}^{-1}\bar{F}_{12}$. A matrix $\bar{S}$ exists under Assumption 7.3, because this assumption will guarantee that the matrix $-\bar{F}_{11}^{-1}\bar{F}_{12}$ is positive semi-definite.

Theorem 7.5 [7] If there exist matrices $N_T, N_N \in \mathbb{R}^{2n_y \times 2n_y}$ with $N_T, N_N \succeq 0$ and $P_h \in \mathbb{R}^{n_\xi \times n_\xi}$ with $P_h \succ 0$, and scalars $\beta, \mu \in \mathbb{R}_{\geq 0}$, such that

$$\begin{bmatrix} P_h - Y^\top(N_T + \mu Q)Y - J^\top\left(\bar{F}_{11}^{-\top} P_h \bar{F}_{11}^{-1} + \bar{F}_{21}\bar{F}_{11}^{-1}\right)J & J^\top \bar{F}_{11}^{-\top} P_h \bar{S} \\ \star & I - \bar{S}^\top P_h \bar{S} \end{bmatrix} \succ 0, \tag{7.34}$$

$$\begin{bmatrix} P_h - Y^\top(N_N - \beta Q)Y - \left(\bar{F}_{11}^{-\top} P_h \bar{F}_{11}^{-1} + \bar{F}_{21}\bar{F}_{11}^{-1}\right) & \bar{F}_{11}^{-\top} P_h \bar{S} \\ \star & I - \bar{S}^\top P_h \bar{S} \end{bmatrix} \succ 0, \tag{7.35}$$

and Assumption 7.3 hold, then the static PETC system (7.7) is internally stable and contractive.

Here, (7.29) is guaranteed by the choice of the function $P : [0, h] \to \mathbb{R}^{n_\xi \times n_\xi}$, (7.30a) is guaranteed by (7.34), and (7.30b) is guaranteed by (7.35).

In the spirit of Sect. 7.3.5, we can obtain that the LMI-based conditions in this proposition are equivalent to a *conservative* check of the ℓ_2-gain being smaller than or equal to 1 for the discrete-time piecewise linear system (7.19). In particular, the stability and contractivity tests in Theorem 7.5 use a *common* quadratic storage function (although extension towards a piecewise quadratic storage function is possible, see [8]) and only one of the S-procedure relaxations discussed in Sect. 7.3.5 (only (ii) is used). In addition to this new perspective on the results in [11, 22, 25], a strong link can be established between the existing LMI-based conditions described in Theorem 7.5 and the lifting-based conditions obtained in this section, as formalized next.

Theorem 7.6 [26] If the conditions of Theorem 7.5 hold and the regions in (7.27) are chosen such that for each $i = 1, 2, \ldots, N$ there is a $\bar{\xi}_i \in \mathbb{R}^{n_\xi}$ such that $\bar{\xi}_i^\top X_i \bar{\xi}_i > 0$,[1] then $\|\hat{D}\|_{\mathscr{K}} < 1$ and the conditions of Theorem 7.4 hold.

This theorem reveals an intimate connection between the results obtained in [11, 22, 25] and the new lifting-based results obtained in the present paper. Indeed, as already mentioned, the LMI-based conditions in [11, 22, 25] as formulated in Theorem 7.5 boil down to an ℓ_2-gain analysis of a discrete-time piecewise linear system (7.19) based on a *quadratic storage function* using only a part of the S-procedure relaxations possible (only using (7.3.5), while the S-procedure relaxations (i) and (ii) mentioned at the end of Sect. 7.3.5 are not used). Moreover, Theorem 7.6 shows that the lifting-based results using Theorems 7.4 and 7.2 never provide worse estimates of the $\mathscr{L}_2$-gain of (7.7) than the results as formulated in Theorem 7.5. In fact, since the stability and contractivity conditions based on (7.19) can be carried out based on more versatile piecewise quadratic storage functions and more (S-procedure) relaxations (see Theorem 7.4), the conditions in Theorems 7.4 and 7.2 are typically significantly less conservative than the ones obtained in [11, 22, 25].

Remark 7.2 When Q is given by (7.6) with $\sigma = 0$, the static PETC system (7.7) reduces to a sampled-data system. Moreover, in this case the related discrete-time piecewise linear system reduces to a discrete-time LTI system, for which the l_2-gain conditions using a common quadratic Lyapunov/storage function are nonconservative (see [19, Lemma 5.1]). Hence, for sampled-data systems, Theorems 7.5 and 7.4 are equivalent and nonconservative.

7.4.2 Dynamic PETC

Although it is shown above that the stability and contractivity analysis in Theorem 7.5 is conservative, it does provide an explicit Lyapunov/storage function for the PETC

[1] This condition implies that each region has a non-empty interior thereby avoiding redundant regions of zero measure.

system (7.7), which the lifting-based approach does not. Moreover, that this conservatism can be exploited in order to further reduce the amount of communication in the system, while preserving the internal stability and contractivity guarantees [7, 8].

The idea is as follows. First, introduce the buffer variable $\eta \in \mathbb{R}$ (which will be included in the event-generator), and define the signal $\hat{o} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{2n_y} \times [0, h] \times \mathbb{R}$ as

$$\hat{o}(t) := (\zeta(s_n), \tau(t), \eta(t)), \quad t \in (s_n, s_{n+1}], \; n \in \mathbb{N}, \tag{7.36}$$

which is the information that is available to the event-generator at time $t \in \mathbb{R}_{\geq 0}$.

The dynamic variable η will evolve according to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\eta = \Psi(\hat{o}), \qquad t \in (s_n, s_{n+1}), \; n \in \mathbb{N}, \tag{7.37a}$$

$$\eta^+ = \eta_T(\hat{o}), \qquad t \in \{t_k\}_{k\in\mathbb{N}}, \tag{7.37b}$$

$$\eta^+ = \eta_N(\hat{o}), \qquad t \in \{s_n\}_{n\in\mathbb{N}} \setminus \{t_k\}_{k\in\mathbb{N}}, \tag{7.37c}$$

where the functions $\Psi : \mathbb{R}^{2n_y} \times [0, h] \times \mathbb{R} \to \mathbb{R}$, $\eta_T : \mathbb{R}^{2n_y} \times [0, h] \times \mathbb{R} \to \mathbb{R}$ and $\eta_N : \mathbb{R}^{2n_y} \times [0, h] \times \mathbb{R} \to \mathbb{R}$ are to be designed. Note that at transmission times t_k, $k \in \mathbb{N}$, the variable η is updated differently than at the other sample times $s_n \neq t_k$, $n, k \in \mathbb{N}$, at which no transmission occurs.

The Lyapunov/storage function V given by (7.46) is often decreasing more than strictly necessary along jumps (7.7b) and (7.7c). To further reduce the amount of communication, we will store the "unnecessary" decrease of V as much as possible in a dynamic variable η, which acts as a buffer. For contractivity and internal stability, we need that the new Lyapunov/storage function $U(\xi, \tau, \eta) = V(\xi, \tau) + \eta$ satisfies

$$\tfrac{\mathrm{d}}{\mathrm{d}t}U(\xi, \tau, \eta) < w^\top w - z^\top z, \qquad \tau \in (0, h] \tag{7.38a}$$

$$U(\xi^+, \tau^+, \eta^+) \leq U(\xi, \tau, \eta), \qquad \tau = h. \tag{7.38b}$$

When a transmission is necessary according to the static event-generator (7.5), we might choose not to transmit at this sample time. As the state then jumps according to (7.7c), we can no longer guarantee that V does not increase along this jump. However, an increase of V can be compensated by reducing η, and hence we can defer the transmission until the buffer η is no longer large enough. The transmission only needs to occur if the buffer η would become negative otherwise.

First, we choose the flow dynamics (7.37a) of η as

$$\Psi(\hat{o}) = -\rho\eta, \; \text{ for } \tau \in (0, h], \tag{7.39}$$

for any arbitrary decay rate $\rho \in \mathbb{R}_{>0}$. Together with (7.32), this choice of (7.39) implies that (7.38a) holds.

Remark 7.3 As Ψ is given by (7.39), it follows that $\eta(s_{n+1}) = e^{\rho h}\eta(s_n^+)$. Thus, since the event-generator only needs to know the value of η at sample times s_n, $n \in \mathbb{N}$, the variable η does not need to continuously evolve according to (7.39) in the event-generator. Instead, we can use the discrete-time dynamics just described.

For the functions η_T and η_N, we provide the following two designs. Together with the inequalities (7.34) and (7.35), both designs ensure that (7.38b) holds.

(1) State-based dynamic PETC:

$$\eta_T(\hat{o}) = \eta + \xi^\top (P_h - J^\top P_0 J)\xi, \tag{7.40a}$$

$$\eta_N(\hat{o}) = \eta + \xi^\top (P_h - P_0)\xi. \tag{7.40b}$$

(2) Output-based dynamic PETC:

$$\eta_T(\hat{o}) = \eta + \zeta^\top (N_T + \mu Q)\,\zeta, \tag{7.41a}$$

$$\eta_N(\hat{o}) = \eta + \zeta^\top (N_N - \beta Q)\,\zeta. \tag{7.41b}$$

Here, the scalars ρ, μ, and β, and the matrices N_T, N_N, P_0, and P_h follow from the stability analysis of the static PETC system in Theorem 7.5.

The first design requires that the full state $\xi(s_n)$ is known to the event-generator at sample time s_n, $n \in \mathbb{N}$. This is the case when $y = (x_p, x_c)$ (e.g., when $\mathcal{C}$ is a static state-feedback controller in which case $y = x_p$ and $n_{x_c} = 0$), as then $\zeta = \xi$. When $y = x_p$ and $n_{x_c} \neq 0$, a copy of the controller could be included in the event-generator in order to track the controller state x_c.

The second design is more conservative, but can also be used in case the event-generator does not have access to the complete vector (x_p, x_c), in which case $\zeta \neq \xi$. Hence, this choice can be used for output-based dynamic PETC.

Finally, from the definition of U it is clear that for all $\xi \in \mathbb{R}^{n_\xi}$, $\tau \in [0, h]$, and all $\eta \in \mathbb{R}_{\geq 0}$, it holds that

$$c_1|\xi|^2 + |\eta| \leq U(\xi, \tau, \eta) \leqslant c_2|\xi|^2 + |\eta|, \tag{7.42}$$

where c_1 and c_2 are defined by

$$c_1 = \min_{\tau \in [0,h]} \lambda_{min}(P(\tau)), \text{ and} \tag{7.43a}$$

$$c_2 = \max_{\tau \in [0,h]} \lambda_{max}(P(\tau)), \tag{7.43b}$$

and satisfy $c_2 \geq c_1 > 0$. In order to ensure that U is a proper storage function, we now only need to ensure that η does not become negative (i.e., that $\eta(t) \in \mathbb{R}_{\geq 0}$ for all $t \in \mathbb{R}_{\geq 0}$).

First, assume that we start with $\eta(0) \geq 0$. Next, note that in between jumps η evolves according to the differential equation (7.39). When after a jump at sample time s_n, $n \in \mathbb{N}$, we have that $\eta(s_n^+) \geq 0$, then due to (7.39) we have that $\eta(t) \geq 0$ for all $t \in (s_n, s_{n+1}]$. Hence, it only remains to show that η does not become negative due to the jumps at the sample times s_n, $n \in \mathbb{N}$. From (7.35), we know that $\eta_N(\hat{o}(s_n)) \geq 0$

when $\eta(s_n) \geq 0$ and $\zeta(s_n)^\top Q\zeta(s_n) \leq 0$. This implies that, as long as $\eta(s_n) \geq 0$, $\eta_N(\hat{o}(s_n))$ can only become negative when $\zeta(s_n)^\top Q\zeta(s_n) > 0$ (in which case the *static* periodic event-generator (7.5) would trigger a transmission). Moreover, in case $\zeta(s_n)^\top Q\zeta(s_n) > 0$, we know from (7.34) that $\eta_T(\hat{o}(s_n)) \geq 0$ when $\eta(s_n) \geq 0$. In other words, to ensure nonnegativity of η, we only need to trigger a transmission at the sample times s_n, $n \in \mathbb{N}$, at which $\eta_N(\hat{o}(s_n)) < 0$. Hence, we propose to generate the sequence of event/transmission times $\{t_k\}_{k\in\mathbb{N}}$ by a new *dynamic* periodic event-generator of the form

$$t_0 = 0,\ t_{k+1} = \inf\{t > t_k \mid \eta_N(\hat{o}(t)) < 0,\ t = nh,\ n \in \mathbb{N}\}. \tag{7.44}$$

Here, the scalar $h \in \mathbb{R}_{>0}$ and the matrix $Q \in \mathbb{R}^{2n_y\times 2n_y}$ are design parameters, in addition to the functions Ψ, η_T, and η_N. Note that the function η_N appears both in the update dynamics (7.37c), as well as in the triggering condition in (7.44).

The closed-loop dynamic PETC system consisting of (7.1)–(7.3), (7.37), and (7.44) can be written as the hybrid system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\\ \eta\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\\ \Psi(\hat{o})\end{bmatrix}, \qquad \tau \in [0, h] \tag{7.45a}$$

$$\begin{bmatrix}\xi^+\\ \tau^+\\ \eta^+\end{bmatrix} = \begin{bmatrix}J\xi\\ 0\\ \eta_T(\hat{o})\end{bmatrix}, \qquad \tau = h \text{ and } \eta_N(\hat{o}) < 0 \tag{7.45b}$$

$$\begin{bmatrix}\xi^+\\ \tau^+\\ \eta^+\end{bmatrix} = \begin{bmatrix}\xi\\ 0\\ \eta_N(\hat{o})\end{bmatrix}, \qquad \tau = h \text{ and } \eta_N(\hat{o}) \geq 0 \tag{7.45c}$$

$$z = C\xi + Dw. \tag{7.45d}$$

Theorem 7.7 [7] If $\eta(0) \geq 0$ and the conditions of Theorem 7.5 hold, then the dynamic PETC system (7.45) with (7.39) and (7.40a) or (7.41a) is internally stable and contractive.[2] Moreover, if the signal w is uniformly bounded, then also η is uniformly bounded.

While the static periodic event-generator (7.5) only has design parameters h and Q, the state-based dynamic event-generator (7.44) with (7.39) and (7.40a) has design parameters h, Q, ρ, P_0, and P_h, and the output-based dynamic event-generator (7.44) with (7.39) and (7.41a) has design parameters h, Q, ρ, N_T, N_N, μ, and β. However, for fixed h, Q, inequalities (7.34) and (7.35) are LMIs, in which case the parameters P_h, P_0, N_T, N_N, μ, and β can be synthesized (and optimized) numerically via semi-definite programming (e.g., using Yalmip/SeDuMi in MATLAB). Of course, manual tuning of one or more of these parameters is also possible, but can be difficult given the large design space.

[2] In the sense that Definitions 7.1 and 7.2 hold along solutions to the dynamic PETC system (7.45) with (7.39) and (7.40a) or (7.41a).

7.5 Riccati-Based Time-Regularized CETC

In the previous section, we analyzed internal stability and contractivity of the static PETC system (7.7) making use of matrix Riccati differential equations. Similar ideas can also be used to analyze internal stability and contractivity of the static CETC system with time-regularization as in (7.10), which we will discuss in this section. Moreover, just as in the previous section, this analysis also gives rise to a state-based and an output-based *dynamic* continuous event-generator design, which we will also provide here.

7.5.1 Static CETC

To analyze contractivity and stability of the system (7.10), we will now use a Lyapunov/storage function V of the form

$$V(\xi,\tau)=\begin{cases}\xi^\top P(\tau)\xi, & \text{when } \tau\in[0,h)\\ \xi^\top P(h)\xi, & \text{when } \tau\in[h,\infty),\end{cases} \tag{7.46}$$

where we select $P:[0,h]\to\mathbb{R}^{n_\xi\times n_\xi}$ to satisfy the Riccati differential equation (7.32), such that again $P:[0,h]\to\mathbb{R}^{n_\xi\times n_\xi}$ is a continuously differentiable function with $P(\tau)\succ 0$ for $\tau\in[0,h]$.

In order to guarantee contractivity and stability of the system (7.10), we need that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V\leq -z^\top z+w^\top w, \tag{7.47}$$

during flow (7.10a), and

$$V(J\xi,0)<V(\xi,h),\ \text{ for all } \xi \text{ with } \xi^\top Y^\top QY\xi>0, \tag{7.48}$$

during the jumps (7.10b).

Note that (7.48) and (7.30a) are identical, as well as (7.47) and (7.29) as long as $\tau\in[0,h]$. Hence, in contrast to Theorem 7.5, inequality (7.35) is not required, but is replaced by the condition

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V\leq -z^\top z+w^\top w,\ \text{ for all } \xi \text{ with } \xi^\top Y^\top QY\xi\leq 0 \tag{7.49}$$

when $\tau>h$. This leads to the following theorem.

Theorem 7.8 [6] Consider the CETC system (7.10) with (7.9), and $Q\in\mathbb{R}^{2n_y\times 2n_y}$. If there exist matrices $N_N, N_T\in\mathbb{R}^{2n_y\times 2n_y}$, $N_N, N_T\succeq 0$, and $P_h\in\mathbb{R}^{n_\xi\times n_\xi}$, $P_h\succ 0$, and scalars $\beta,\mu\in\mathbb{R}_{\geq 0}$, such that

$$\begin{bmatrix} A^\top P_h + P_h A + C^\top C + Y^\top (N_N - \beta Q) Y & \star \\ B^\top P_h + D^\top C & D^\top D - I \end{bmatrix} \prec 0, \tag{7.50}$$

(7.34) and Assumption 7.3 hold, then the system is internally stable and contractive.

Here, (7.47) is guaranteed by the choice of the function $P : [0, h] \to \mathbb{R}^{n_\xi \times n_\xi}$ when $\tau \in [0, h]$ and by (7.50) when $\tau > h$, and (7.48) is again guaranteed by (7.34).

7.5.2 Dynamic CETC

Similar to the PETC case, by adding a dynamic variable in the continuous event-generator, the conservatism in Theorem 7.8 can be exploited in order to further reduce the amount of communication in the system, while preserving the internal stability and contractivity guarantees.

Again, we introduce the buffer variable $\eta \in \mathbb{R}$ (which will be included in the event-generator). As in the CETC case, the output $y(t)$ can be measured continuously, now

$$o(t) := (\zeta(t), \tau(t), \eta(t)) \tag{7.51}$$

is the information that is available at the event-generator at time $t \in \mathbb{R}_{\geq 0}$,

The variable η will evolve according to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\eta = \Psi(o), \qquad t \in (t_k, t_{k+1}), \tag{7.52a}$$

$$\eta^+ = \eta_T(o), \qquad t = t_k, \tag{7.52b}$$

where $o(t) = (\zeta(t), \tau(t), \eta(t))$ is the information that is available at the event-generator at time $t \in \mathbb{R}_{\geq 0}$, and where the functions $\Psi : \mathbb{R}^{2n_y} \times \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ and $\eta_T : \mathbb{R}^{2n_y} \times \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ are to be designed.

Next, we design the dynamics (7.52) of the variable η with the goal of enlarging the (average) inter-event times compared to the static continuous event-generator (7.9), while maintaining the same stability and performance guarantees.

We now choose the flow dynamics (7.52a) of η as

$$\Psi(o) = \begin{cases} -2\rho\eta, & \text{when } \tau \in [0, h) \quad (7.53\text{a}) \\ -2\rho\eta + \zeta^\top (N_N - \beta Q)\zeta, & \text{when } \tau \in [h, \infty), \quad (7.53\text{b}) \end{cases}$$

for any arbitrary decay rate $\rho \in \mathbb{R}_{>0}$, and we again have two designs for the jump dynamics (7.52b) of η.

(1) State-based dynamic CETC:

$$\eta_T(o) = \eta + \xi^\top \left(P_h - J^\top P_0 J\right) \xi, \tag{7.54}$$

(2) Output-based dynamic CETC:

$$\eta_T(o) = \eta + \zeta^\top (N_T + \mu Q)\,\zeta \tag{7.55}$$

In order to ensure that $U(\xi, \tau, \eta) = V(\xi, \tau) + \eta$ is a proper storage function, we now only need to ensure that η does not become negative (i.e., that $\eta(t) \in \mathbb{R}_{\geq 0}$ for all $t \in \mathbb{R}_{\geq 0}$). Hence, we propose to generate the sequence of jump/event times $\{t_k\}_{k\in\mathbb{N}}$ by a dynamic *continuous* event-generator with time-regularization of the form

$$t_0 = 0,\ t_{k+1} = \inf\{t \geq t_k + h \mid \eta(t) < 0\}. \tag{7.56}$$

The closed-loop dynamic CETC system consisting of (7.1)–(7.3), (7.52), and (7.56) can be written as the hybrid system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\\ \eta\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\\ \Psi(o)\end{bmatrix}, \qquad \tau \in [0, h] \text{ or } \eta \geq 0 \tag{7.57a}$$

$$\begin{bmatrix}\xi^+\\ \tau^+\\ \eta^+\end{bmatrix} = \begin{bmatrix}J\xi\\ 0\\ \eta_T(o)\end{bmatrix}, \qquad \tau > h \text{ and } \eta < 0 \tag{7.57b}$$

$$z = C\xi + Dw. \tag{7.57c}$$

Theorem 7.9 [6] If $\eta(0) \geq 0$ and the conditions of Theorem 7.8 hold, then the dynamic CETC system (7.57) with (7.53) and (7.54) or (7.55) is internally stable and contractive.[3] Moreover, if the signal w is uniformly bounded, then also η is uniformly bounded.

7.6 Numerical Example

Consider the unstable batch reactor of [27, 35, 50], with $n_{x_p} = 4$, $n_{x_c} = 2$, $n_y = n_w = n_u = n_z = 2$, and plant and controller dynamics given by (7.1) and (7.2) with

$$A_p = \begin{bmatrix} 1.3800 & -0.2077 & 6.7150 & -5.6760\\ -0.5814 & -4.2900 & 0.0000 & 0.6750\\ 1.0670 & 4.2730 & -6.6540 & 5.8930\\ 0.0480 & 4.2730 & 1.3430 & -2.1040 \end{bmatrix},\quad B_p = \begin{bmatrix} 0.0000 & 0.0000\\ 5.6790 & 0.0000\\ 1.1360 & -3.1460\\ 1.1360 & 0.0000 \end{bmatrix},$$

$$B_{pw} = \begin{bmatrix} 10 & 0 & 10 & 0\\ 0 & 5 & 0 & 5 \end{bmatrix},\ C_y = C_z = \begin{bmatrix} 1 & 0 & 1 & -1\\ 0 & 1 & 0 & 0 \end{bmatrix},\ D_y = D_z = D_{zw} = \begin{bmatrix} 0 & 0\\ 0 & 0 \end{bmatrix},$$

$$A_c = \begin{bmatrix} 0 & 0\\ 0 & 0 \end{bmatrix},\ B_c = \begin{bmatrix} 0 & 1\\ 1 & 0 \end{bmatrix},\ C_u = \begin{bmatrix} -2 & 0\\ 0 & 8 \end{bmatrix},\ D_u = \begin{bmatrix} 0 & -2\\ 5 & 0 \end{bmatrix}.$$

[3] In the sense that Definitions 7.1 and 7.2 hold along solutions to the dynamic CETC system (7.57) with (7.53) and (7.54) or (7.55).

Note that for this system, the measured output y is not equal to the full plant/controller state (x_p, x_c), and thus we cannot use (7.40), (7.54), but we have to resort to (7.41) for the dynamic PETC case and to (7.55) for the dynamic CETC case.

We choose $h = 0.1$, $\rho = 10^{-3}$, and Q given by (7.6). For each choice of σ, we use a bisection algorithm to minimize the $\mathscr{L}_2$-gain γ (by appropriately scaling the matrices C and D, as discussed in Remark 7.1) based on Theorems 7.4, 7.5, and 7.8. For the lifting-based approach of Theorem 7.4, we use a single region $\Omega_1 = \mathbb{R}^{n_\xi}$, i.e., we choose $p = 1$ and $X_1 = O$. The matrices N_T and N_N and scalars β and μ follow from Theorem 7.5 for the dynamic PETC case, and from Theorem 7.8 for the dynamic time-regularized CETC case.

Figure 7.2a shows the guaranteed $\mathscr{L}_2$-gain γ as a function of σ for both PETC approaches and the time-regularized CETC approach. Here, we see that using the Riccati-based framework, a smaller $\mathscr{L}_2$-gain can be guaranteed by using a time-regularized CETC scheme than by using a PETC scheme. This makes sense intuitively, as after h time units have elapsed, a time-regularized continuous event-generator can trigger an event as soon as its event condition is violated, while a periodic event-generator can only do so at a sample time s_n, $n \in \mathbb{N}$. Figure 7.2a also shows that for a static PETC system, the lifting-based approach of Theorem 7.4 provides better $\mathscr{L}_2$-gain estimates than the Riccati-based approach of Theorem 7.5. However, the $\mathscr{L}_2$-gain estimate of Theorem 7.5 also holds for both *dynamic* PETC strategies, while the $\mathscr{L}_2$-gain estimate of Theorem 7.4 only holds for the *static* PETC strategy.

Figure 7.2b shows τ_{avg} = (total number of events)/(simulation time), the average inter-event times for the static and (output-based) dynamic event-generators, which have been obtained by simulating the systems for 100 time units with $\xi(0) = 0$, $\tau(0) = h$, and $\eta(0) = 0$, and disturbance w given by

$$w(t) = e^{-0.2t} \begin{bmatrix} 5\sin(3.5t) \\ -\cos(3t) \end{bmatrix}. \tag{7.58}$$

Finally, Fig. 7.2c shows the actual ratio $\|z\|_{\mathscr{L}_2}/\|w\|_{\mathscr{L}_2}$ for disturbance w given by (7.58), which has been obtained from the same simulations.

In Fig. 7.2c, we see that the dynamic event-generators exploit (part of) the conservatism in the $\mathscr{L}_2$-gain analysis of Theorems 7.8 and 7.5 to postpone the transmissions. This leads to higher ratios $\|z\|_{\mathscr{L}_2}/\|w\|_{\mathscr{L}_2}$ (but still below the guaranteed bounds in Fig. 7.2b), but also to consistently larger τ_{avg}, as can be seen in Fig. 7.2b.

Based on this example, we can conclude that for PETC systems, the lifting-based, and Riccati-based frameworks each has their own advantages. The lifting-based framework provides tighter $\mathscr{L}_2$-gain guarantees, while the Riccati-based framework allows to extend the transmission intervals by using a dynamic event-generator. For a fixed σ and a given desired performance, the lifting-based framework allows for larger h (hence, for larger *minimum* inter-event times) while the Riccati-based framework may lead to larger *average* inter-event times τ_{avg} by using a dynamic event-generator. Which framework is better thus depends on whether large minimum or average inter-event times are desired.

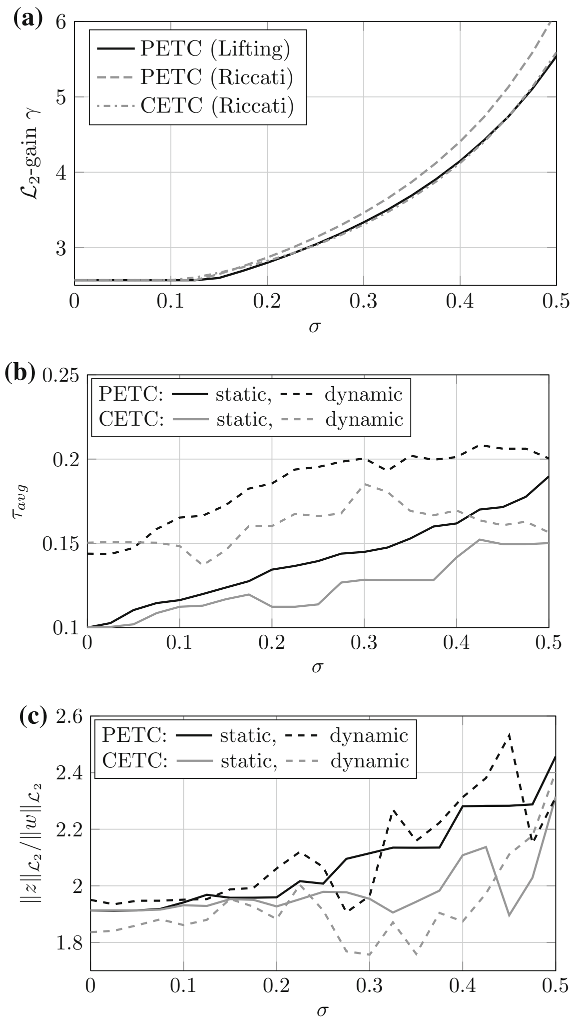

Fig. 7.2 Guaranteed $\mathscr{L}_2$-gain γ for varying σ (**a**), average inter-event times τ_{avg} for disturbance w given by (7.58) and different event-generators (**b**), and actual ratio $\|z\|_{\mathscr{L}_2}/\|w\|_{\mathscr{L}_2}$ for disturbance w given by (7.58) (**c**)

For the time-regularized CETC case, only the Riccati-based framework applies, which for this example yields (almost exactly) the same performance guarantees as the lifting-based PETC approach, with the same minimum inter-event time h, but often larger τ_{avg}. However, the designed continuous event-generator may be difficult to implement on a digital platform, as it requires continuous measuring of the output y.

To compare both frameworks with the (static or dynamic) time-regularized CETC solutions of [13, 14], note that for a given and $\mathscr{L}_2$-gain γ, the waiting time h (or τ_{MIET} in the terminology of [14]) of the continuous event-generator proposed in [13, 14] cannot exceed the maximally allowable transmission interval (MATI) of [27].

Moreover, for the same example in [27, Sect. IV], we can calculate that when using the sampled-data protocol, no notion of stability can be guaranteed for MATI larger than 0.063. In contrast, here we guarantee internal stability and $\mathscr{L}_2$-stability for $h = 0.1$. Hence, our frameworks tailored to linear systems are clearly much less conservative than our previous results for nonlinear systems in [13, 14]. See also [6] for a direct comparison between the static and dynamic continuous event-generators in Sect. 7.5 and the event-generators proposed in [13, 14].

7.7 Extensions

The results presented in this chapter can be extended in several ways.

First of all, the results in Sects. 7.4 and 7.5 can be extended toward the case with communication delays, as long as these delays are upper bounded by the sampling time or time threshold h, see [8]. In order to do so, for each possible delay $d \in [0, h]$, a function $P^d : [0, d] \to \mathbb{R}^{n_\xi \times n_\xi}$ satisfying the Riccati differential equation (7.32) needs to be synthesized. Hence, only a finite number of possible transmission delays can be considered using this approach. However, when the delays can have any value from a continuous interval, this situation can be effectively approximated by using a gridding approach, see also [8, Remark III.7]. Similar ideas can be used to extend the lifting-based approach in Sect. 7.3 toward delays. Moreover, certain Self-Triggered schemes (e.g., [23, 34, 49]) can also be captured in this lifting-based framework, see [40].

Second, our proposed frameworks can be extended toward decentralized setups in a similar manner as in [25, Sect. V] for the static PETC case. However, this requires that the clocks of all local event-generators are synchronized.

These extensions emphasize the usefulness of our new ETC solutions for linear systems, but also uncovers two potential drawbacks. In our earlier work [14], we considered decentralized CETC setups for *nonlinear* systems with transmission delays. These results can also be particularized to linear systems, giving rise to continuous event-generators with time-regularization that are similar (although more conservative) to those proposed in Sect. 7.5. However, the analysis proposed in [14] does not require clock synchronization for all local event-generators, and also directly allows that the transmission delays can have any value from a continuous interval. Hence, although the analysis in [14] (in the case of linear systems) provides less tight performance guarantees than our new results tailored to linear systems that we have proposed in this chapter, it does not suffer from the drawbacks that clocks need to be synchronized (in case of decentralized event-generators) and that only a finite number of possible transmission delays can be allowed.

7.8 Summary

In this chapter, we have provided an overview of our recent results in the design of time-regularized ETC and PETC schemes that are tailored to *linear systems* as provided in [8, 25, 26]. In particular, we have shown that stability and the contractivity in $\mathscr{L}_2$-sense (meaning that the $\mathscr{L}_2$-gain is smaller than 1) of static PETC closed-loop systems (which are hybrid systems) are equivalent to the stability and the contractivity in ℓ_2-sense (meaning that the ℓ_2-gain is smaller than 1) of an appropriate discrete-time piecewise linear system. These new insights are obtained by adopting a lifting-based perspective on this analysis problem, which led to computable ℓ_2-gain (and thus $\mathscr{L}_2$-gain) conditions, despite the fact that the linearity assumption, which is usually needed in the lifting literature, is not satisfied.

We have also reviewed the results in [8] that lead to the design of time-regularized CETC and PETC schemes based on Lyapunov/storage functions exploiting matrix Riccati differential equations. Moreover, we have identified the connections between the two approaches.

Additionally, we have discussed new designs of so-called (time-regularized and periodic) *dynamic* ETC strategies focused on linear systems. Interestingly, the inclusion of a dynamic variable in the event-generator can lead to a significantly reduced consumption of communication and energy resources while leading to identical guarantees on stability and performance as their static counterparts.

Via a numerical example, we have demonstrated that a Riccati-based CETC design, a Riccati-based PETC design, and a lifting-based PETC design each has their own advantages. Hence, which choice of design framework is better depends on the system at hand.

Acknowledgements This work is part of the research programmes "Wireless control systems: A new frontier in automation" with project number 11382 and "Integrated design approach for safety-critical real-time automotive systems" with project number 12698, which are (partly) financed by the Netherlands Organisation for Scientific Research (NWO).

References

1. Abdelrahim, M., Postoyan, R., Daafouz, J., Nešić, D.: Input-to-state stabilization of nonlinear systems using event-triggered output feedback controllers. In: European Control Conference, pp. 2180–2185 (2015)
2. Abdelrahim, M., Postoyan, R., Daafouz, J., Nešić, D.: Stabilization of nonlinear systems using event-triggered output feedback controllers. IEEE Trans. Autom. Control. **61**(9), 2682–2687 (2016)
3. Başar, T., Bernhard, P.: H^{∞}-Optimal Control and Relaxed Minimax Design Problems: A Dynamic Game Approach, 2nd edn. Birkhäuser, Boston (1995)
4. Bamieh, B.A., Pearson, J.B.: A general framework for linear periodic systems with applications to H^{∞}/sampled-data control. IEEE Trans. Autom. Control. **37**(4), 418–435 (1992)
5. Borgers, D.P., Heemels, W.P.M.H.: Event-separation properties of event-triggered control systems. IEEE Trans. Autom. Control. **59**(10), 2644–2656 (2014)

6. Borgers, D.P., Dolk, V.S., Heemels, W.P.M.H.: Dynamic event-triggered control with time-regularization for linear systems. In: 55th IEEE Conference on Decision and Control, pp. 1352–1357 (2016)
7. Borgers, D.P., Dolk, V.S., Heemels, W.P.M.H.: Dynamic periodic event-triggered control for linear systems. In: Hybrid Systems: Computation and Control, pp. 179–186. Pittburgh, PA, USA (2017)
8. Borgers, D.P., Dolk, V.S., Heemels, W.P.M.H.: Riccati-based design of event-triggered controllers for linear systems with delays. IEEE Trans. Autom. Control. (2018). (To appear)
9. Cantoni, M.W., Glover, K.: H_∞ sampled-data synthesis and related numerical issues. Automatica **33**(12), 2233–2241 (1997)
10. Chen, T., Francis, B.A.: Optimal Sampled-Data Control Systems. Communications and Control Engineering, 1st edn. Springer, London (1995)
11. Dai, D., Hu, T., Teel, A.R., Zaccarian, L.: Output feedback synthesis for sampled-data system with input saturation. In: American Control Conference, pp. 1797–1802 (2010)
12. De Persis, C., Sailer, R., Wirth, F.: Parsimonious event-triggered distributed control: a Zeno free approach. Automatica **49**(7), 2116–2124 (2013)
13. Dolk, V.S., Borgers, D.P., Heemels, W.P.M.H.: Event-triggered control: tradeoffs between transmission intervals and performance. In: 53rd IEEE Conference on Decision and Control, pp. 2764–2769 (2014)
14. Dolk, V.S., Borgers, D.P., Heemels, W.P.M.H.: Output-based and decentralized dynamic event-triggered control with guaranteed $\mathscr{L}_p$-gain performance and Zeno-freeness. IEEE Trans. Autom. Control. **62**(1), 34–49 (2017)
15. Donkers, M.C.F., Heemels, W.P.M.H.: Output-based event-triggered control with guaranteed $\mathscr{L}_\infty$-gain and improved and decentralized event-triggering. IEEE Trans. Autom. Control. **57**(6), 1362–1376 (2012)
16. Dullerud, G.E., Lall, S.: Asynchronous hybrid systems with jumps - analysis and synthesis methods. Syst. Control. Lett. **37**(2), 61–69 (1999)
17. Ferrari-Trecate, G., Cuzzola, F.A., Mignone, D., Morari, M.: Analysis of discrete-time piecewise affine and hybrid systems. Automatica **38**(12), 2139–2146 (2002)
18. Forni, F., Galeani, S., Nešić, D., Zaccarian, L.: Event-triggered transmission for linear control over communication channels. Automatica **50**(2), 490–498 (2014)
19. Gahinet, P., Apkarian, P.: A linear matrix inequality approach to H_∞ control. Int. J. Robust Nonlinear Control. **4**(4), 421–448 (1994)
20. Garcia, E., Antsaklis, P.J.: Model-based event-triggered control for systems with quantization and time-varying network delays. IEEE Trans. Autom. Control. **58**(2), 422–434 (2013)
21. Girard, A.: Dynamic triggering mechanisms for event-triggered control. IEEE Trans. Autom. Control. **60**(7), 1992–1997 (2015)
22. Goebel, R., Sanfelice, R.G., Teel, A.R.: Hybrid dynamical systems. IEEE Control. Syst. Mag. **29**(2), 28–93 (2009)
23. Gommans, T.M.P., Antunes, D., Donkers, M.C.F., Tabuada, P., Heemels, W.P.M.H.: Self-triggered linear quadratic control. Automatica **50**(4), 1279–1287 (2014)
24. Heemels, W.P.M.H., Sandee, J.H., van den Bosch, P.P.J.: Analysis of event-driven controllers for linear systems. Int. J. Control. **81**(4), 571–590 (2008)
25. Heemels, W.P.M.H., Donkers, M.C.F., Teel, A.R.: Periodic event-triggered control for linear systems. IEEE Trans. Autom. Control. **58**(4), 847–861 (2013)
26. Heemels, W.P.M.H., Dullerud, G.E., Teel, A.R.: $\mathscr{L}_2$-gain analysis for a class of hybrid systems with applications to reset and event-triggered control: a lifting approach. IEEE Trans. Autom. Control. **61**(10), 2766–2781 (2016)
27. Heemels, W.P.M.H., Teel, A.R., van de Wouw, N., Nešić, D.: Networked control systems with communication constraints: tradeoffs between transmission intervals, delays and performance. IEEE Trans. Autom. Control. **55**(8), 1781–1796 (2010)
28. Heemels, W.P.M.H., Postoyan, R., Donkers, M.C.E., Teel, A.R., Anta, A., Tabuada, P., Nešić, D.: Periodic event-triggered control. In: Miskowicz, M. (ed.) Event-Based Control and Signal Processing. CRC Press/Taylor & Francis, Boca Raton (2015)

29. Henningsson, T., Johannesson, E., Cervin, A.: Sporadic event-based control of first-order linear stochastic systems. Automatica **44**(11), 2890–2895 (2008)
30. Johansson, M., Rantzer, A.: Computation of piecewise quadratic Lyapunov functions for hybrid systems. IEEE Trans. Autom. Control. **43**(4), 555–559 (1998)
31. Kreyszig, E.: Introductory Functional Analysis with Applications. Wiley, New York (1978)
32. Lehmann, D., Lunze, J.: Event-based control with communication delays and packet losses. Int. J. Control. **85**(5), 563–577 (2012)
33. Lunze, J., Lehmann, D.: A state-feedback approach to event-based control. Automatica **46**(1), 211–215 (2010)
34. Mazo Jr., M., Anta, A., Tabuada, P.: An ISS self-triggered implementation of linear controllers. Automatica **46**(8), 1310–1314 (2010)
35. Nešić, D., Teel, A.R.: Input-output stability properties of networked control systems. IEEE Trans. Autom. Control. **49**(10), 1650–1667 (2004)
36. Postoyan, R., Anta, A., Heemels, W.P.M.H., Tabuada, P., Nešić, D.: Periodic event-triggered control for nonlinear systems. In: 52nd IEEE Conference on Decision and Control, pp. 7397–7402 (2013)
37. Postoyan, R., Anta, A., Nešić, D., Tabuada, P.: A unifying Lyapunov-based framework for the event-triggered control of nonlinear systems. In: 50th IEEE Conference on Decision and Control and European Control Conference, pp. 2559–2564 (2011)
38. Postoyan, R., Tabuada, P., Nešić, D., Anta, A.: A framework for the event-triggered stabilization of nonlinear systems. IEEE Trans. Autom. Control. **60**(4), 982–996 (2015)
39. Selivanov, A., Fridman, E.: Event-triggered $\mathscr{H}_\infty$ control: a switching approach. IEEE Trans. Autom. Control. **61**(10), 3221–3226 (2016)
40. Strijbosch, N.W.A., Dullerud, G.E., Teel, A.R., Heemels, W.P.M.H.: $\mathscr{L}_2$-gain analysis of periodic event-triggered and self-triggered control systems with delays using lifting techniques. Submitted
41. Tabuada, P.: Event-triggered real-time scheduling of stabilizing control tasks. IEEE Trans. Autom. Control. **52**(9), 1680–1685 (2007)
42. Tallapragada, P., Chopra, N.: Event-triggered dynamic output feedback control for LTI systems. In: 51st IEEE Conference on Decision and Control, pp. 6597–6602 (2012)
43. Tallapragada, P., Chopra, N.: Decentralized event-triggering for control of nonlinear systems. IEEE Trans. Autom. Control. **59**(12), 3312–3324 (2014)
44. Tarbouriech, S., Seuret, A., Gomes da Silva Jr., J.M., Sbarbaro, D.: Observer-based event-triggered control co-design for linear systems. IET Control. Theory Appl. **10**(18), 2466–2473 (2016)
45. Toivonen, H.T.: Sampled-data control of continuous-time systems with an H_∞ optimality criterion. Automatica **28**(1), 45–54 (1992)
46. Toivonen, H.T., Sågfors, M.F.: The sampled-data H_∞ problem: a unified framework for discretization-based methods and Riccati equation solution. Int. J. Control. **66**(2), 289–310 (1997)
47. van der Schaft, A.: L_2-Gain and Passivity Techniques in Nonlinear Control. Lecture Notes in Control and Information Sciences, vol. 218. Springer, Berlin (1996)
48. van Loon, S.J.L.M., Heemels, W.P.M.H., Teel, A.R.: Improved $\mathscr{L}_2$-gain analysis for a class of hybrid systems with applications to reset and event-triggered control. In: 53rd IEEE Conference on Decision and Control, pp. 1221–1226 (2014)
49. Velasco, M., Fuertes, J.M., Marti, P.: The self triggered task model for real-time control systems. In: 24th IEEE Real-Time Systems Symposium, pp. 67–70 (2003)
50. Walsh, G.C., Ye, H., Bushnell, L.G.: Stability analysis of networked control systems. IEEE Trans. Control. Syst. Technol. **10**(3), 438–446 (2002)
51. Wang, X., Lemmon, M.D.: Event-triggering in distributed networked control systems. IEEE Trans. Autom. Control. **56**(3), 586–601 (2011)
52. Willems, J.C.: Dissipative dynamical systems part I: general theory. Arch. Ration. Mech. Anal. **45**(5), 321–351 (1972)
53. Yamamoto, Y.: New approach to sampled-data control systems-a function space method. In: 29th IEEE Conference on Decision and Control, vol. 3, pp. 1882–1887 (1990)