Hybrid model formulation and stability analysis of a PID-controlled motion system with Coulomb friction ⋆

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Abstract: For a PID-controlled motion system under Coulomb friction described by a differential inclusion, we present a hybrid model comprising logical states indicating whether the closed loop is in stick or in slip, thereby resembling a hybrid automaton. A key step for this description is the addition of a timer exploiting a peculiar semiglobal dwell time of the original dynamics, which then removes defective and unwanted nonconverging Zeno solutions from the hybrid model. Through it, we then revisit an existing proof of global asymptotic stability, which is significantly simplified by way of a smooth weak Lyapunov function. The relevance of the proposed hybrid representation is also illustrated on a novel control strategy resetting the PID integrator and hinging upon the proposed hybrid model.

Keywords: hybrid systems, nonlinear systems, Coulomb friction, Lyapunov methods, global asymptotic stability, PID control.

1. INTRODUCTION

In this paper, we present a hybrid model formulation for a proportional-integral-derivative (PID) controlled single-mass motion system subject to Coulomb friction. Friction is a performance-limiting factor in many high-precision positioning systems in terms of the achievable setpoint accuracy and settling times. Many different control strategies have been developed for frictional systems (e.g., model-based compensation, see Armstrong-Hélouvry et al. (1994), impulsive control, see van de Wouw and Leine (2012), or sliding mode control, see Bartolini et al. (2003)). PID control, however, is still used in the vast majority of industrial motion systems, since the integrator action is able to compensate for unknown static friction. However, it has performance limitations (e.g., long settling times, see Beerens et al. (2018)). The popularity of the PID controller for mechatronics applications in industry, however, motivates the development of *hybrid* add-ons, such as reset control strategies, to complement the classical PID controller and improve its baseline performance.

Coulomb friction is often modeled by a *set-valued* force law, so that the resulting dynamical model of a PID-controlled mass subject to friction yields a *differential inclusion*, see, e.g., (Acary and Brogliato, 2008, Sec. 1.3). When appropriately tuned, the PID control guarantees global asymptotic stability (GAS) of a constant position reference, see Bisoffi et al. (2018).

On the other hand, the PID-controlled frictional system evolves, in some sense, discrete-wise by toggling between the logical states of *stick* and *slip*. The stick phase is characterized by zero velocity and the control force not exceeding the static friction. The integrator action then builds up the control force by integrating the position error and eventually compensating for the (unknown) static friction, so that a slip phase (i.e., a phase with nonzero velocity) occurs. These logical states can be modeled as suitable subsystems of the differential inclusion, see (Åström and Canudas-de-Wit, 2008, p. 106), which motivates the development of an equivalent *hybrid* representation of the differential inclusion model. The main motivation for such a hybrid model representation is the fact that it favors a simplified stability analysis, compared to the one in Bisoffi et al. (2018), as we will show in Section 5. In particular, the hybrid model allows employing a smooth Lyapunov function in order to prove GAS of a constant position reference, instead of the *discontinuous* one used in Bisoffi et al. (2018). Moreover, the hybrid model facilitates a simplified stability analysis of the frictional system complemented with reset control strategies, compared to the analysis framework in Bisoffi et al. (2018). Reset strategies added

⋆ This research is part of the research programme High Tech Systems and Materials (HTSM), which is supported by NWO domain Applied and Engineering Sciences and partly funded by the Dutch Ministry of Economic Affairs and is also supported by ANR via project HANDY, number ANR-18-CE40-0010.

to a classical PID controller for frictional systems have been shown to be promising in Beerens et al. (2018).

This paper has the following contributions. First, we present a hybrid automaton model for a PID-controlled mass subject to Coulomb friction. We show that the solution set of the differential inclusion model is contained in the solution set of the hybrid model, for initial conditions in arbitrarily large compact sets. In particular, we prove that the solutions to the differential inclusion enjoy inherently a semiglobal dwell-time between the stick and slip phases. Second, we prove GAS of the setpoint for the hybrid model through a Lyapunov analysis, and provide, as a corollary, an alternative, simplified proof of GAS for the differential inclusion model, by using the above property of solution sets. Third, we show that the proposed hybrid model is useful for PID-based reset control design and analysis, by presenting a novel reset controller and a simulation example.

After presenting the physical model in Section 2, we show in Section 3 that the solutions to this model enjoy a certain dwell-time property and introduce in Section 4 the hybrid automaton model describing (semiglobally) the same behavior. A stability analysis is in Section 5, and Section 6 illustrates the advantages of the hybrid automaton model for reset design.

Notation. $\mathbb{B}$ is the closed unit ball, of appropriate dimensions, in the Euclidean norm. For $w_1 \in \mathbb{R}^{n_1}$ and $w_2 \in \mathbb{R}^{n_2}$, $(w_1, w_2) := \begin{bmatrix} w_1^\top & w_2^\top \end{bmatrix}^\top$. The function $\mathrm{sign}\colon \mathbb{R} \to [-1, 1]$ is defined as $\mathrm{sign}(0) := 0$, $\mathrm{sign}(x) := x/|x|$ for $x \neq 0$. The set-valued mapping $\mathrm{Sign}\colon \mathbb{R} \rightrightarrows [-1, 1]$ is defined as $\mathrm{Sign}(0) := [-1, 1]$, $\mathrm{Sign}(x) := \{\mathrm{sign}(x)\}$ for $x \neq 0$. For $c > 0$, the function $\mathrm{dz}_c\colon \mathbb{R} \to \mathbb{R}$ is defined as $\mathrm{dz}_c(x) := 0$ for $x \in [-c, c]$, $\mathrm{dz}_c(x) := x - c\,\mathrm{sign}(x)$ for $x \notin [-c, c]$. In particular, $\mathrm{dz}\colon \mathbb{R} \to \mathbb{R}$ is defined as $\mathrm{dz}(x) := \mathrm{dz}_1(x)$ for all x. For a hybrid solution ψ (Goebel et al., 2012, Def. 2.6) with hybrid time domain $\mathrm{dom}\,\psi$ (Goebel et al., 2012, Def. 2.3), the function $j(\cdot)$ is defined as $j(t) := \min_{(t,j)\in \mathrm{dom}\,\psi} j$. For a generic hybrid system $\mathcal{H} = (F, C, G, D)$ in Goebel et al. (2012), $\mathcal{S}_{\mathcal{H}}(x_0)$ denotes the set of all maximal solutions to $\mathcal{H}$ from the initial condition x_0. $\vee$, $\wedge$ denote the logical operators *or*, *and*. For $x \in \mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathbb{R}^n$, $|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |x - y|$.

2. PHYSICAL MODEL AND MAIN RESULT

A PID controller drives a 1-degree-of-freedom mass m on a horizontal plane with position z_1 and velocity z_2, and subject to a friction force taking values from a set-valued force law $z_2 \rightrightarrows \bar{\Psi}(z_2) := -\bar{F}_s \,\mathrm{Sign}(z_2) - \bar{\alpha} z_2$, where $\bar{F}_s$ is the static Coulomb friction and $\bar{\alpha} \geq 0$ is the viscous friction coefficient. With the goal of attaining the constant setpoint $(z_1, z_2) = (r, 0)$, the described model reads

$$\dot{z}_1 = z_2, \tag{1a}$$

$$\dot{z}_2 \in \frac{1}{m}\left(\bar{\Psi}(z_2) - \bar{k}_p(z_1 - r) - \bar{k}_i z_3 - \bar{k}_d z_2\right), \tag{1b}$$

$$\dot{z}_3 = z_1 - r, \tag{1c}$$

where z_3 is the integral state of the PID controller and $\bar{k}_p$, $\bar{k}_i$, $\bar{k}_d$ represent its proportional, integral and derivative gains, respectively. As in Beerens et al. (2018), we introduce mass-normalized parameters to favor clarity

$$k_p := \tfrac{\bar{k}_p}{m}, \quad k_d := \tfrac{\bar{k}_d + \bar{\alpha}}{m}, \quad k_i := \tfrac{\bar{k}_i}{m}, \quad F_s := \tfrac{\bar{F}_s}{m}, \tag{2}$$

and we adopt the convenient change of coordinates

$$\hat{x} := (\hat{\sigma}, \hat{\phi}, \hat{v}) = \begin{bmatrix} -k_i(z_1 - r) \\ -k_p(z_1 - r) - k_i z_3 \\ z_2 \end{bmatrix}, \tag{3}$$

where $\hat{\sigma}$ is a generalized position error, $\hat{\phi}$ corresponds to the proportional and integral actions of the PID controller, and $\hat{v}$ is the velocity. Further details on the model derivation and the reasons for the change of coordinates can be found in Beerens et al. (2018). After the reparametrization in (2) and the change of coordinates in (3), the model in (1) is equivalent to

$$\begin{aligned} \hat{\mathcal{H}}: \begin{bmatrix} \dot{\hat{\sigma}} \\ \dot{\hat{\phi}} \\ \dot{\hat{v}} \end{bmatrix} &\in \begin{bmatrix} -k_i \hat{v} \\ \hat{\sigma} - k_p \hat{v} \\ \hat{\phi} - k_d \hat{v} - F_s \,\mathrm{Sign}(\hat{v}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -k_i \\ 1 & 0 & -k_p \\ 0 & 1 & -k_d \end{bmatrix} \begin{bmatrix} \hat{\sigma} \\ \hat{\phi} \\ \hat{v} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ F_s \end{bmatrix} \mathrm{Sign}(\hat{v}) \\ &=: A\hat{x} - b\,\mathrm{Sign}(\hat{v}), \qquad \hat{x} \in \hat{C} := \mathbb{R}^3. \end{aligned} \tag{4}$$

Although $\hat{\mathcal{H}}$ has an empty jump set in the terminology of Goebel et al. (2012), we call it a hybrid system since we will consider its (truly) hybrid counterpart below (see (21)). We consider $F_s > 0$ and assume that

$$k_i > 0,\ k_p > 0,\ k_p k_d > k_i, \tag{5}$$

which is equivalent, by the Routh-Hurwitz criterion, to the origin being globally asymptotically stable for (4) in the *frictionless* case, i.e., $F_s = 0$. The assumption in (5) corresponds then to tuning a *stabilizing* PID controller and is hence not restrictive. Consider for $\hat{\mathcal{H}}$ in (4) the attractor

$$\hat{\mathcal{A}} := \{\hat{x} = (\hat{\sigma}, \hat{\phi}, \hat{v})\colon \hat{\sigma} = 0,\ |\hat{\phi}| \leq F_s,\ \hat{v} = 0\}, \tag{6}$$

corresponding to the set of all possible equilibria. We state then the next result.

Theorem 1. Under the controller gain selection in (5), $\hat{\mathcal{A}}$ *in* (6) *is globally asymptotically stable for* $\hat{\mathcal{H}}$ *in* (4).

System $\hat{\mathcal{H}}$ in (4) coincides with the model in Bisoffi et al. (2018) where Theorem 1 was proven via a discontinuous Lyapunov function. The motivation of this paper is to provide an alternative hybrid proof of Theorem 1 (see Section 5.2), based on the hybrid model $\mathcal{H}_\delta$ presented below (see (21)). The ingredients of this proof are: *(a)* the solutions to $\hat{\mathcal{H}}$ have a semiglobal dwell time (see Lemma 1) and are bounded (see Lemma 4), *(b)* the solutions to $\hat{\mathcal{H}}$ are semiglobally contained in the solutions to $\mathcal{H}_\delta$ (see Lemma 2), *(c)* the stability analysis of the hybrid model $\mathcal{H}_\delta$ (see Section 5.1) is significantly simplified by the smooth, instead of discontinuous, Lyapunov function. Moreover, Section 6 sketches the benefit of $\mathcal{H}_\delta$ for designing reset laws to improve the PID-control performance.

3. SEMIGLOBAL DWELL TIME OF SOLUTIONS

Given the sets (intuitively associated with incipient slip or stick phases)

$$\hat{S}_1 := \{\hat{x}\colon \hat{v} = 0 \wedge (\hat{\phi} > F_s \vee (\hat{\phi} = F_s \wedge \hat{\sigma} > 0))\} \tag{7}$$

$$\hat{S}_{-1} := \{\hat{x}\colon \hat{v} = 0 \wedge (\hat{\phi} < -F_s \vee (\hat{\phi} = -F_s \wedge \hat{\sigma} < 0))\} \tag{8}$$

$$\hat{S}_0 := \{\hat{x}\colon \hat{v} = 0 \wedge \hat{\sigma} > 0 \wedge \hat{\phi} \in [-F_s, F_s)\}$$

$$\cup \{\hat{x}\colon \hat{v}=0 \wedge \hat{\sigma}<0 \wedge \hat{\phi}\in(-F_s,F_s]\}, \quad (9)$$

we summarize the results from (Bisoffi et al., 2018, Lemma 1 and Claim 1) that we exploit in this paper.

Fact 1. (i) From each initial condition $\hat{x}_0 \in \mathbb{R}^3$, there exists a unique and complete solution $\hat{x}$ to $\hat{\mathcal{H}}$. (ii) For each initial condition $\hat{x}_0=(\hat{\sigma}_0,\hat{\phi}_0,\hat{v}_0)\in\hat{S}_1$ or with $\hat{v}_0>0$ ($\hat{x}_0=(\hat{\sigma}_0,\hat{\phi}_0,\hat{v}_0)\in\hat{S}_{-1}$ or with $\hat{v}_0<0$, respectively), there exists $T>0$ such that the unique solution $\hat{x}$ to $\hat{\mathcal{H}}$ coincides over $[0,T]$ with the unique solution $\hat{x}_l:=(\hat{\sigma}_l,\hat{\phi}_l,\hat{v}_l)$ to

$$\dot{\hat{x}}_l = A\hat{x}_l - b \quad (\dot{\hat{x}}_l = A\hat{x}_l + b, \text{ respectively}), \qquad (10)$$

with A and b in (4). (iii) For each initial condition $\hat{x}_0\in\hat{S}_0$ or $\hat{x}_0=(0,\hat{\phi}_0,0)$ with $|\hat{\phi}_0|\le F_s$, there exists $T>0$ such that the unique solution $\hat{x}$ to $\hat{\mathcal{H}}$ coincides over $[0,T]$ with the unique solution $\bar{x}:=(\bar{\sigma},\bar{\phi},\bar{v})$ to

$$\dot{\bar{x}} := (\dot{\bar{\sigma}},\dot{\bar{\phi}},\dot{\bar{v}}) = (0,\bar{\sigma},0). \qquad (11)$$

The interval $[0,T]$ in Fact 1*(ii)* might not be *uniform* with respect to the initial conditions, and we need this uniformity for building a hybrid model of (4) with desirable stability properties. Indeed, we have the property of solutions to $\hat{\mathcal{H}}$ established in the next lemma, which entails a uniform dwell-time in a semiglobal fashion. Note that the presence of a guaranteed uniform dwell time allows ruling out Zeno phenomena (see Goebel et al. (2012) and references therein) in the following.

Lemma 1. Under selection (5), for each $M>0$, there exists $\delta>0$ such that for each initial condition $\hat{x}_0\in\hat{S}_1\cap M\mathbb{B}$ ($\hat{x}_0\in\hat{S}_{-1}\cap M\mathbb{B}$, respectively), the unique complete solution $\hat{x}=(\hat{\sigma},\hat{\phi},\hat{v})$ to $\hat{\mathcal{H}}$ from the initial condition $\hat{x}_0$ satisfies $\hat{v}(t)\ge 0$ ($\hat{v}(t)\le 0$, respectively) for all $t\in[0,\delta]$.

Proof. We consider only the case $\hat{x}_0\in\hat{S}_1$, because the case $\hat{x}_0\in\hat{S}_{-1}$ is proven with similar arguments. By Fact 1*(ii)*, the solution $\hat{x}$ satisfies $\dot{\hat{x}}=A\hat{x}-b$ on some $[0,T]$. The last equation can be made homogeneous as

$$\dot{\tilde{x}} = A\tilde{x} \quad \text{with} \quad \tilde{x}=(\tilde{\sigma},\tilde{\phi},\tilde{v}) := (\hat{\sigma},\hat{\phi}-F_s,\hat{v}), \qquad (12)$$

so that $\hat{x}\in\hat{S}_1$ if and only if $\tilde{x}\in\tilde{S}_1$, where

$$\tilde{S}_1 := \{\tilde{x}\colon \tilde{v}=0 \wedge \big(\tilde{\phi}>0 \vee (\tilde{\phi}=0\wedge\tilde{\sigma}>0)\big)\}. \qquad (13)$$

Consider an initial condition $\tilde{x}_0\in\tilde{S}_1$ and note that $0\notin\tilde{S}_1$, hence $|\tilde{x}_0|\neq 0$. Because the dynamics in (12) is homogeneous, consider the unitary-norm initial condition $\tilde{n}_0:=\tilde{x}_0/|\tilde{x}_0|$ without loss of generality. We parameterize $\tilde{n}_0$ as $\tilde{n}_0=(\cos(\tilde{\theta}_0),\sin(\tilde{\theta}_0),0)$ with $\tilde{\theta}_0\in[0,\pi)$ (π is excluded because $\tilde{\phi}_0=0$ and $\tilde{\sigma}_0<0$ do not belong to $\tilde{S}_1$). Then, the solution to (12) with initial condition $\tilde{n}_0$ is given by $\tilde{x}(t)=(\tilde{\sigma}(t),\tilde{\phi}(t),\tilde{v}(t))=\exp(At)\tilde{n}_0$, and by expanding the matrix exponential into its powers, straightforward computations yield

$$\tilde{v}(t)=\cos(\tilde{\theta}_0)\big(\tfrac{t^2}{2}+\mathscr{O}(t^3)\big)+\sin(\tilde{\theta}_0)\big(t-\tfrac{k_d t^2}{2}+\mathscr{O}(t^3)\big), \quad (14)$$

where $\mathscr{O}(t^3)$ denotes the terms of order t^3 or higher in the Taylor expansion. Based on (14), we show below for each $\tilde{\theta}_0\in[0,\pi)$ the existence of $T_v>0$ such that

$$\tilde{v}(t)>0 \text{ for all } t\in(0,T_v), \quad \tilde{v}(T_v)=0. \qquad (15)$$

Then, we prove the existence of a uniform δ. For $\tilde{\theta}_0\in[0,\pi/2]$, we are considering a compact set of values of $\tilde{\theta}_0$ where both $\cos(\tilde{\theta}_0)$ and $\sin(\tilde{\theta}_0)$ are nonnegative. Since in (14) the coefficients of their dominant powers ($t^2/2$ and t, respectively) are a positive constant, δ_1 can be found *independently of* θ_0 so that both Taylor expansions are (strictly) positive on $(0,\delta_1)$. As a consequence, $T_v\ge\delta_1$ and the claim of the lemma is proven for $\tilde{\theta}_0\in[0,\pi/2]$. For $\tilde{\theta}_0\in(\pi/2,\pi)$, the existence of $T_v>0$ in (15) follows from the positive coefficient $\sin(\tilde{\theta}_0)$ of the dominant power t in (14), but uniformity of δ might not hold. Note that $\tilde{\theta}_0\in(\pi/2,\pi)$ corresponds to $\tilde{\sigma}_0<0$, which we then assume in the rest of the proof without loss of generality. This implies by (12), (15) and $k_i>0$ in (5) that

$$\tilde{\sigma}(T_v)=\tilde{\sigma}_0+\textstyle\int_0^{T_v}\big(-k_i\tilde{v}(\tau)\big)d\tau \le \tilde{\sigma}_0<0. \qquad (16)$$

Since A is Hurwitz by (5) and $\tilde{x}$ satisfies (12) until $t=T_v$ by Fact 1*(ii)*, there exist positive c and λ such that $|\tilde{x}(t)|=|\exp(At)\tilde{x}_0|\le c\exp(-\lambda t)|\tilde{x}_0|$. Define

$$\delta_2 := F_s/(|A|cM)>0 \qquad (17)$$

where $|A|>0$ is the (induced) 2-norm of matrix A. If $T_v\ge\delta_2$, the claim of the lemma is proven. If $T_v<\delta_2$, we show that the velocity remains zero for a uniform interval. Note first that $\dot{\tilde{v}}(T_v)\le 0$ otherwise $\tilde{v}$ would not cross zero as in (15). Hence $0\ge\dot{\tilde{v}}(T_v)=\tilde{\phi}(T_v)-k_d\tilde{v}(T_v)=\tilde{\phi}(T_v)$, so $\tilde{\phi}(T_v)\le 0$. Also, $\tilde{\phi}(T_v)>-F_s$ as we conclude in (18). Indeed, for $t\in[0,T_v]$ ($\hat{x}_0\in\hat{S}_1\cap M\mathbb{B}$ implies $\tilde{x}_0\in M\mathbb{B}$)

$$\begin{aligned}|\dot{\tilde{\phi}}(t)| &= \left|\tfrac{d}{dt}\big(\left[\begin{smallmatrix}0&1&0\end{smallmatrix}\right]\exp(At)\tilde{x}_0\big)\right| = \left|\left[\begin{smallmatrix}0&1&0\end{smallmatrix}\right]A\exp(At)\tilde{x}_0\right|\\ &\le |A|c\exp(-\lambda t)|\tilde{x}_0| \le |A|cM = F_s/\delta_2\end{aligned}$$

by (17). Hence, for $t\in[0,T_v]$, $\tilde{\phi}(t)=\tilde{\phi}_0+\int_0^t\dot{\tilde{\phi}}(\tau)d\tau\ge\tilde{\phi}_0-\int_0^t|\dot{\tilde{\phi}}(\tau)|d\tau\ge\tilde{\phi}_0-t\frac{F_s}{\delta_2}$. In particular then,

$$\tilde{\phi}(T_v)\ge\tilde{\phi}_0-T_v\tfrac{F_s}{\delta_2}\ge-\tfrac{T_v}{\delta_2}F_s>-F_s \qquad (18)$$

because $\tilde{\phi}_0\ge 0$ and $T_v<\delta_2$. Then, $\tilde{v}(T_v)=0$ by (15), $\tilde{\sigma}(T_v)<0$ by (16) and $-F_s<\tilde{\phi}(T_v)\le 0$, i.e.,

$$\hat{v}(T_v)=0, \quad \hat{\sigma}(T_v)<0, \quad 0<\hat{\phi}(T_v)\le F_s. \qquad (19)$$

We may now continue the solution to $\hat{\mathcal{H}}$ from the "initial" condition in (19). Fact 1*(iii)* shows that the solution $\hat{x}$ to $\hat{\mathcal{H}}$ coincides with the solution $\bar{x}$ to (11) as long as $\bar{\phi}\ge-F_s$. Since, from (11), $\bar{\phi}(t)=\bar{\phi}(T_v)+\bar{\sigma}(T_v)(t-T_v)$ with $t\ge T_v$, $\bar{\phi}(T_v)=\hat{\phi}(T_v)\in(0,F_s]$ and $|\bar{\sigma}(T_v)|=|\hat{\sigma}(T_v)|=|\tilde{\sigma}(T_v)|\le|\tilde{x}(T_v)|\le c\exp(-\lambda T_v)|\tilde{x}_0|\le cM$, we have $\bar{\phi}(t)\in[-F_s,F_s]$ for all $t\in[T_v,T_v+\delta_3]$ with $\delta_3:=\frac{F_s}{cM}>0$. In turn, the solution component $\hat{v}$ to $\hat{\mathcal{H}}$ remains zero over the same interval, whose uniform length is δ_3. Therefore,

$$\delta:=\min\{\delta_1,\delta_2,\delta_3,2F_s/M\}>0 \qquad (20)$$

proves the lemma, where $2F_s/M$ has been introduced without loss of generality for the sake of Lemma 2. $\square$

4. PROPOSED HYBRID MODEL

With the quantity δ from Lemma 1 in mind, we introduce a hybrid model enabling a hybrid-based proof of Theorem 1. Consider $\delta>0$, and for the state

$$x:=(\sigma,\phi,v,\tau,q)\in\mathbb{R}^3\times[0,2\delta]\times\{-1,0,1\}, \qquad (21a)$$

define the hybrid system $\mathcal{H}_\delta$ as

$$\mathcal{H}_\delta:\begin{cases}\dot{x}=f(x), & x\in C:=C_{\rm slip}\cup C_{\rm stick} \qquad (21b)\\ x^+\in G(x), & x\in D:=\bigcup_{i\in\{1,-1,0\}}D_i \qquad (21c)\end{cases}$$

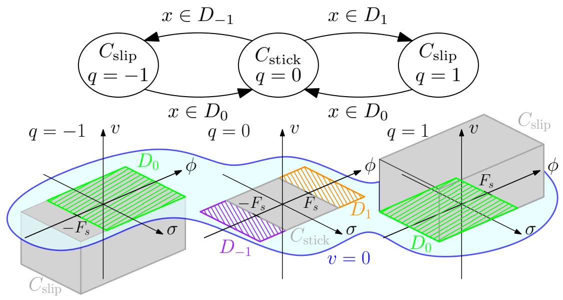

Fig. 1. Top: hybrid automaton underlying (21). Bottom: "projections" to the (σ, ϕ, v) space of the flow and jump sets in (21f)-(21j).

where the flow and jump maps are given by

$$f(x) := \begin{bmatrix} -k_i v \\ \sigma - k_p v \\ -k_d v + |q|\phi - qF_s \\ 1 - \mathrm{dz}(\tau/\delta) \\ 0 \end{bmatrix}, \; G(x) := \bigcup_{i:\, x \in D_i} \{g_i(x)\}, \quad (21d)$$

the different jump maps are given by

$$g_1(x) := \begin{bmatrix} \sigma \\ \phi \\ v \\ 0 \\ 1 \end{bmatrix}, \; g_{-1}(x) := \begin{bmatrix} \sigma \\ \phi \\ v \\ 0 \\ -1 \end{bmatrix}, \; g_0(x) := \begin{bmatrix} \sigma \\ \phi \\ v \\ \tau \\ 0 \end{bmatrix} \quad (21e)$$

and the flow and jump sets are given by

$$C_{\text{slip}} := \{x \colon |q| = 1, qv \geq 0\} \quad (21f)$$
$$C_{\text{stick}} := \{x \colon q = 0, v = 0, |\phi| \leq F_s\} \quad (21g)$$
$$D_1 := \{x \colon q = 0, v = 0, \phi \geq F_s, \tau \in [\delta, 2\delta]\} \quad (21h)$$
$$D_{-1} := \{x \colon q = 0, v = 0, \phi \leq -F_s, \tau \in [\delta, 2\delta]\} \quad (21i)$$
$$D_0 := \{x \colon |q| = 1, v = 0, q\phi \leq F_s\}. \quad (21j)$$

The flow and jump maps for τ ensure the invariance of the set $[0, 2\delta]$ for τ, as per (21a). Since $D_i \cap D_k = \emptyset$ for $i, k \in \{-1, 0, 1\}$ and $i \neq k$, G is actually always a *single-valued* mapping. A pictorial representation of (21) is in Figure 1, which gives a clear hybrid automaton interpretation of (21).

As an important observation, the first three components of the flow map in (21d) coincide in C_{slip} and C_{stick} with the affine right-hand sides in (10) and (11), to which the differential inclusion in (4) has been shown to "reduce" in Fact 1. Then, it is intuitive that a solution to $\mathcal{H}_\delta$ captures the solution to $\hat{\mathcal{H}}$ when the condition $\tau \in [\delta, 2\delta]$ is absent from (21h)-(21i). In such a case, however, (21) would also have nonconverging Zeno solutions (e.g., with $v = 0$, $\phi = F_s$, $\sigma \neq 0$) that would nullify the desired attractivity property of Section 5.1. The timer in $\mathcal{H}_\delta$ prevents these Zeno solutions, and translates the inherent dwell-time property of solutions to (4) in Lemma 1. Indeed, after solutions to $\mathcal{H}_\delta$ exit a stick phase and enter a slip phase jumping from D_1 or D_{-1}, the timer is reset to zero via g_1 or g_{-1} and enforces that a time δ passes before solutions exit a stick phase again (due to the condition $\tau \in [\delta, 2\delta]$), which corresponds to the property of solutions to (4) in Lemma 1. For the sake of Lemma 2, note that due to Fact 1 *(i)*, for each $\hat{x}_0 \in \mathbb{R}^3$, the set of maximal solutions $\mathcal{S}_{\hat{\mathcal{H}}}(\hat{x}_0)$ contains one single element $\hat{x}$.

Lemma 2. Under selection (5)*, for each $M > 0$, there exists $\delta > 0$ such that for each initial condition $\hat{x}_0 \in \mathbb{R}^3 \cap M\mathbb{B}$, there exist τ_0 and q_0 such that, for some solution $x = (\sigma, \phi, v, \tau, q) \in \mathcal{S}_{\mathcal{H}_\delta}((\hat{x}_0, \tau_0, q_0))$, for all $t \geq 0$*

$$\sigma(t, j(t)) = \hat{\sigma}(t), \phi(t, j(t)) = \hat{\phi}(t), v(t, j(t)) = \hat{v}(t), \quad (22)$$

where $\hat{x} = (\hat{\sigma}, \hat{\phi}, \hat{v}) \in \mathcal{S}_{\hat{\mathcal{H}}}(\hat{x}_0)$.

The intuition behind Lemma 2 is that there exists one solution to (21) that can jump so as to reproduce the solution to (4), although there might be other solutions to $\mathcal{H}_\delta$ that are not complete. The proof of Lemma 2 is in Appendix A.

5. STABILITY ANALYSIS

In this section, we first provide a stability analysis for $\mathcal{H}_\delta$ and then, as a result of the relationship between the solution sets of $\hat{\mathcal{H}}$ and $\mathcal{H}_\delta$ shown in Section 4, we obtain a hybrid proof of Theorem 1.

5.1 Stability analysis of $\mathcal{A}$ for $\mathcal{H}_\delta$

We study the global asymptotic stability of the following attractor for $\mathcal{H}_\delta$ in (21)

$$\mathcal{A} := \{(0, \phi, 0, \tau, q) \colon \tau \in [0, 2\delta], q \in \{-1, 0, 1\}, \phi \in F_s \operatorname{Sign}(q)\}, \quad (23)$$

which is compact. Note that $\hat{\mathcal{A}}$ in (6) is essentially the "projection" of $\mathcal{A}$ for $\hat{\mathcal{H}}$ in the directions $\hat{\sigma}$, $\hat{\phi}$, $\hat{v}$. Consider the smooth Lyapunov function

$$V(x) := \begin{bmatrix} \sigma \\ v \end{bmatrix}^\top \begin{bmatrix} \frac{k_d}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} + |q|(\phi - qF_s)^2 + (1 - |q|)(\mathrm{dz}_{F_s}(\phi))^2. \quad (24)$$

From (5) and x as in (21a), $\mathcal{A}$ corresponds to the zero-level set of V. We can prove the following relations.

Lemma 3. Consider the hybrid system $\mathcal{H}_\delta$ in (21) *with $\delta > 0$. Then, V in* (24) *satisfies:*

$$\langle \nabla V(x), f(x) \rangle = -2(k_p k_d - k_i)v^2 \quad \forall x \in C_{\text{slip}} \cup C_{\text{stick}} \quad (25a)$$
$$V(g_1(x)) - V(x) = 0 \quad \forall x \in D_1 \quad (25b)$$
$$V(g_{-1}(x)) - V(x) = 0 \quad \forall x \in D_{-1} \quad (25c)$$
$$V(g_0(x)) - V(x) = (\mathrm{dz}_{F_s}(\phi))^2 - (\phi - qF_s)^2 \leq 0 \quad \forall x \in D_0. \quad (25d)$$

Proof. As for (25a), some computations yield

$$\langle \nabla V(x), f(x) \rangle = 2 \begin{bmatrix} \sigma \\ v \end{bmatrix}^\top \begin{bmatrix} \frac{k_d}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} -k_i v \\ -k_d v + |q|\phi - qF_s \end{bmatrix} + 2\Big(|q|(\phi - qF_s) + (1 - |q|)\,\mathrm{dz}_{F_s}(\phi)\Big)(\sigma - k_p v) \quad (26)$$

from (24) and (21d). (25a) follows readily from (26) by considering separately the cases of $|q| = 1$ (C_{slip}) and $q = 0$ (C_{stick}). Next, consider (25b). D_1 in (21h) and g_1 in (21e) yield $V(g_1(x)) - V(x) = (\phi - F_s)^2 - (\mathrm{dz}_{F_s}(\phi))^2$, which is zero for $\phi \geq F_s$. (25c) follows analogously. (25d) follows from (21j) and g_0 in (21e) updating q from $|q| = 1$ to 0. ☐

The next proposition establishes GAS of $\mathcal{A}$ for $\mathcal{H}_\delta$.

Proposition 1. Under selection (5)*, consider $\mathcal{H}_\delta$ in* (21) *with $\delta > 0$. $\mathcal{A}$ in* (23) *is globally asymptotically stable for $\mathcal{H}_\delta$.*

Proof. The proof exploits (Seuret et al., 2019, Thm. 1). The set $\mathcal{A}$ in (23) is compact and $\mathcal{H}_\delta$ in (21) satisfies the hybrid basic conditions (Goebel et al., 2012, Assumption 6.5). $G(D \cap \mathcal{A}) \subset \mathcal{A}$ follows from the fact that σ, ϕ, v remain constant across jumps, and the Lyapunov nonincrease conditions hold due to Lemma 3.

Hence, we only need to prove that no complete solution $x_{\rm bad}$ to $\mathcal{H}_\delta$ keeps V constant and nonzero. To this end, first note that the dwell time property enforced by τ implies that complete solutions exhibit an infinite amount of flow (cf. Fig. 1, top). Due to the decrease property in (25a), and since v remains constant across jumps, the only possibility for $x_{\rm bad}$ to exist is that it flows outside $\mathcal{A}$ (where V is nonzero) with velocity v identically equal to zero (otherwise V would decrease from (25a) in any arbitrarily small interval of variation of v). Such flowing (with $v \equiv 0$) is impossible in $C_{\rm slip}\backslash\mathcal{A}$ due to the linear representation in (10) and (12), and the pair $([\,0\ 0\ 1\,], A)$ being observable. Hence, $x_{\rm bad}$ could flow indefinitely only in $C_{\rm stick}$. However, this is also impossible because the components (σ, ϕ, v) of any solution flowing in $C_{\rm stick}\backslash\mathcal{A}$ obey dynamics (11) and, since the σ component of $x_{\rm bad}$ must be nonzero (otherwise $x_{\rm bad}$ would be in $\mathcal{A}$), the ϕ component of $x_{\rm bad}$ would grow unbounded, thereby contradicting the fact that $x_{\rm bad}$ keeps flowing in $C_{\rm stick}\backslash\mathcal{A}$. hence, such solution $x_{\rm bad}$ cannot exist and the proof is completed. $\square$

5.2 Hybrid proof of Theorem 1

The proof of Proposition 1 is significantly simplified because the automaton model $\mathcal{H}_\delta$ allows using the smooth Lyapunov function V in (24) (rather than the discontinuous Lyapunov function used in Bisoffi et al. (2018)). The proof of Theorem 1 is completed here through Lemma 2, Proposition 1, and the next straightforward property of $\hat{\mathcal{H}}$.

Lemma 4. Under selection (5)*, solutions to $\hat{\mathcal{H}}$ in* (4) *are bounded.*

Proof. By (5) (i.e., A is Hurwitz), the system $\dot{\hat{x}}_l = A\hat{x}_l - b\nu$ is exponentially stable, hence bounded-input-bounded-output stability of this linear system implies that solutions to $\hat{\mathcal{H}}$ are bounded (consider (4) with input $|\nu| \le 1$). $\square$

Proof of Theorem 1. Preliminarily, define $\hat{\mathcal{A}}_5 := \{(\hat\sigma, \hat\phi, \hat v, \tau, q)\colon \hat\sigma = 0, |\hat\phi| \le F_s, \hat v = 0, \tau \in [0, 2\delta], q \in \{-1, 0, 1\}\}$, i.e., $\hat{\mathcal{A}}$ in (6) written as a subset of $\mathbb{R}^5$ (instead of $\mathbb{R}^3$). Clearly $\hat{\mathcal{A}}_5 \supset \mathcal{A}$ with $\mathcal{A}$ in (23). Then,

$$|x|_{\mathcal{A}} \ge \inf_{y \in \hat{\mathcal{A}}_5} |x - y| = \inf_{y \in \hat{\mathcal{A}}_5} |(\hat x, \tau, q) - y| = |\hat x|_{\hat{\mathcal{A}}}. \qquad (27)$$

For the proof, we need to show stability and global attractivity of $\hat{\mathcal{A}}$, where the latter entails by (Goebel et al., 2012, Def. 7.1) that for each solution $\hat x$ with $\hat x(0) \in \mathbb{R}^3$, $\hat x$ is bounded and satisfies $\lim_{t\to\infty} |\hat x(t)|_{\hat{\mathcal{A}}} = 0$ (because we know that solutions are complete from Fact 1*(i)*). Boundedness of solutions is guaranteed by Lemma 4. Indeed, for each $M > 0$, there exists $M' \ge M$, such that $\hat x(0) \in M\mathbb{B}$ implies, by Lemma 4, $\operatorname{rge} \hat x \subset M'\mathbb{B}$. Then, Lemma 2 guarantees that there exists $\delta' > 0$ so that the solution $\hat x$ (with $\operatorname{rge} \hat x \subset M'\mathbb{B}$) coincides with the first three components of one solution x to $\mathcal{H}_{\delta'}$, as in (22). Global asymptotic stability of $\mathcal{A}$ for $\mathcal{H}_{\delta'}$ (Proposition 1) and completeness of such x (due to (22)) ensure $\lim_{t\to\infty} |x(t, j(t))|_{\mathcal{A}} = 0$ and hence, by (27), $\lim_{t\to\infty} |\hat x(t)|_{\hat{\mathcal{A}}} = 0$. Since both $\mathcal{H}_\delta$ and $\hat{\mathcal{H}}$ satisfy the hybrid basic conditions (Goebel et al., 2012, Ass. 6.5), global asymptotic stability of $\mathcal{A}$ for $\mathcal{H}_\delta$ in Proposition 1 actually implies uniform global asymptotic stability (Goebel et al., 2012, Thm. 7.12), and uniform global attractivity. Hence, also $\hat{\mathcal{A}}$ is uniformly globally attractive. By (4) and Fact 1*(iii)*, $\hat{\mathcal{A}}$ is strongly forward invariant because for every solution $\hat x \in \mathcal{S}_{\hat{\mathcal{H}}}(\hat{\mathcal{A}})$, $\operatorname{rge} \hat x \subset \hat{\mathcal{A}}$. Then $\hat{\mathcal{A}}$ is stable by (Goebel et al., 2012, Prop. 7.5). $\square$

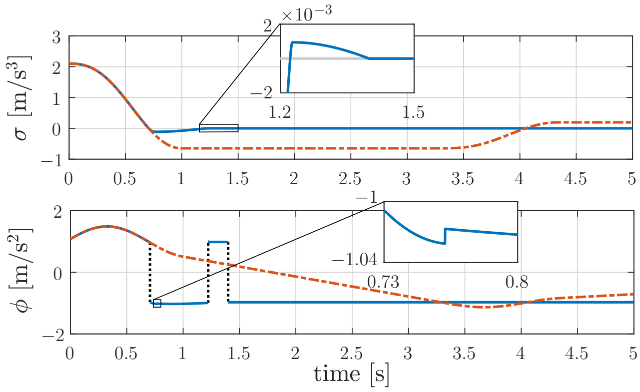

Fig. 2. Simulated response of σ (top) and ϕ (bottom), for $\mathcal{H}_{\rm r}$ (blue, solid) and $\mathcal{H}_\delta$ (red, dash-dotted).

6. AUTOMATON MODEL $\mathcal{H}_\delta$ FOR RESET DESIGN

Motivated by improving performance of mechatronics systems through PID-based reset controllers (see, e.g., Beerens et al. (2018)), we design from (21) a reset integrator to reduce the overshoot of its linear counterpart. In particular, we change sign to the controller state ϕ when the generalized position error σ crosses zero. Such a design ensures that the PI control force always points in the direction of the setpoint. Secondly, a scaling of ϕ by a factor $\alpha \in [0, 1)$ is required when the mass hits zero velocity, after overshooting the reference, to leave intact the global asymptotic stability properties.

As a baseline, we consider system $\mathcal{H}_\delta$ in (21), and we augment it with suitable jump sets and maps, corresponding to the described controller resets. Let us introduce a boolean state $b \in \{-1, 1\}$, characterizing whether the mass moves towards the reference ($b = 1$), or away from it ($b = -1$), when initialized properly. Consider the augmented state $x_{\rm r} := (x, b)$, and the resulting hybrid system including the controller resets:

$$\mathcal{H}_{\rm r}\colon \begin{cases} \dot x_{\rm r} = f_{\rm r}(x_{\rm r}), & x_{\rm r} \in C_{\rm r} \qquad (28a) \\ x_{\rm r}^+ \in \displaystyle\bigcup_{p\colon x_{\rm r} \in D_{p,{\rm r}}} \{g_{p,{\rm r}}(x_{\rm r})\}, & x_{\rm r} \in \displaystyle\bigcup_{p \in \{1,-1,0,\phi,\alpha\}} D_{p,{\rm r}} \qquad (28b) \end{cases}$$

with flow map $f_{\rm r}(x_{\rm r}) := (f(x), 0)$, and jump maps $g_{1,{\rm r}}(x_{\rm r}) := (g_1(x), b)$, $g_{-1,{\rm r}}(x_{\rm r}) := (g_{-1}(x), b)$, $g_{0,{\rm r}}(x_{\rm r}) := (g_0(x), b)$, $g_{\phi,{\rm r}}(x_{\rm r}) := (\sigma, -\phi, v, \tau, q, -1)$, and $g_{\alpha,{\rm r}}(x_{\rm r}) := (\sigma, \alpha\phi, v, \tau, q, 1)$. The flow and jump sets are given by $C_{\rm r} := \{(x, b)\colon x \in C,\ b \in \{-1, 1\}\}$, $D_{1,{\rm r}} := \{(x, b)\colon x \in D_1,\ b = 1\}$, $D_{-1,{\rm r}} := \{(x, b)\colon x \in D_{-1},\ b = 1\}$, $D_{0,{\rm r}} := \{(x, b)\colon x \in D_0,\ b = 1\}$, $D_{\phi,{\rm r}} := \{(x, b)\colon |q| = 1,\ qv \ge 0,\ b = 1,\ \sigma = 0\}$, $D_{\alpha,{\rm r}} := \{(x, b)\colon x \in D_0, b = -1\}$, where $D_{\phi,{\rm r}}$ and $D_{\alpha,{\rm r}}$ are the jump sets implementing the resets. We leave the robust detection of $\sigma = 0$ and $v = 0$ in $D_{\phi,{\rm r}}$ and $D_{\alpha,{\rm r}}$ as future work (with an approach similar to (Beerens et al., 2018, Remark 1) and drawing from the ideas in (Goebel et al., 2012, Example 4.18 and Chap. 4)).

Computer simulations of $\mathcal{H}_\delta$ (i.e., with a classical PID controller) and of $\mathcal{H}_{\rm r}$ in (28) (i.e., with a reset enhancement) are performed with δ sufficiently small. For a unitary mass, we consider $F_s = 0.981$ m/s^2, and select $k_p = 18$ s^{-2}, $k_i = 35$ s^{-3}, $k_d = 2.5$ s^{-1} (satisfying (5)), $\alpha = 0.95$. The generalized position error σ, and the PI control force ϕ (illustrating the controller resets) are depicted in Fig. 2, where the red dash-dotted line corresponds to the solution to $\mathcal{H}_\delta$. Such solution enjoys the stability properties of Theorem 1, but its settling time and overshoot are significantly reduced by using the reset controller, while maintaining global asymptotic stability. Indeed, the developments in this paper pave the way for an analysis that shows 1) global asymptotic stability for $\mathcal{H}_{\rm r}$, and 2) that global asymptotic stability for $\mathcal{H}_{\rm r}$ implies global asymptotic stability for the corresponding inclusion model, augmented with the proposed reset controller. In particular, the smooth Lyapunov function in (24) can be extended to incorporate also b.

7. CONCLUSIONS

For PID-controlled motion systems subject to Coulomb friction, we introduced a hybrid model exploiting some intrinsic semiglobal dwell-time between stick and slip enjoyed by the original solutions. This new model allows to simplify the proof of GAS, which can now rely on a smooth Lyapunov function, rather than the discontinuous one used in previous work. The model and the smooth function are important preliminary steps towards proving the effectiveness of certain hybrid resetting rules for the integrator state, providing improved closed-loop performance in mechatronics systems. While simulations illustrate this fact here, a complete proof of GAS for the PID scheme with resets is left as future work.

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Appendix A. PROOF OF LEMMA 2

If a generic solution $x = (\sigma, \phi, v, \tau, q) \in \mathcal{S}_{\mathcal{H}_\delta}$ flows, its components (σ, ϕ, v) satisfy also the differential inclusion in (4). Indeed, flow according to (21d) is possible only on a subset of $C_{\rm slip} \cup C_{\rm stick}$ given by $S_a \cup S_b \cup S_c$ defined as

$$\begin{aligned}
S_a &:= \{x\colon (q = 1 \wedge v > 0) \vee (q = 1 \wedge v = 0 \\
&\quad \wedge \phi > F_s) \vee (q = 1 \wedge v = 0 \wedge \phi = F_s \wedge \sigma \geq 0)\} \\
S_b &:= \{x\colon (q = 0 \wedge v = 0 \wedge \phi \in [-F_s, F_s) \wedge \sigma > 0) \\
&\quad \vee (q = 0 \wedge v = 0 \wedge \phi \in [-F_s, F_s] \wedge \sigma = 0) \\
&\quad \vee (q = 0 \wedge v = 0 \wedge \phi \in (-F_s, F_s] \wedge \sigma < 0)\} \\
S_c &:= \{x\colon (q = -1 \wedge v < 0) \vee (q = -1 \wedge v = 0 \\
&\quad \wedge \phi < -F_s) \vee (q = -1 \wedge v = 0 \wedge \phi = -F_s \wedge \sigma \leq 0)\}.
\end{aligned} \tag{A.1}$$

Define $\hat{f}$ from $f(x) =: (\hat{f}(x), 1 - \mathrm{dz}(\tau/\delta), 0)$. On S_a, S_b, S_c, f in (21d) is such that, respectively, $\hat{f}(x) = A \begin{bmatrix} \sigma \\ \phi \\ v \end{bmatrix} - b$, $\hat{f}(x) = \begin{bmatrix} 0 \\ \sigma \\ 0 \end{bmatrix}$, $\hat{f}(x) = A \begin{bmatrix} \sigma \\ \phi \\ v \end{bmatrix} + b$ (with A and b in (4)). By Fact 1, the solution to (4) coincides with the unique solution of the affine systems given by (10) and (11), which coincide with the vector fields $\hat{f}$ obtained from (21d). Since solutions to $\mathcal{H}_\delta$ are locally absolutely continuous by (Goebel et al., 2012, Defs. 2.6 and 2.4), this is sufficient to conclude that the components (σ, ϕ, v) of $x \in \mathcal{S}_{\mathcal{H}_\delta}$ satisfy the differential inclusion in (4) when flowing. Moreover, since the jump map in (21e) is such that $(\sigma^+, \phi^+, v^+) = (\sigma, \phi, v)$, (22) holds for $x \in \mathcal{S}_{\mathcal{H}_\delta}$ for all $t \in [0, \sup_t \operatorname{dom} x]$.

It now remains to prove that there exists a *complete* solution to $\mathcal{H}_\delta$. To this end, the following observations are made based on the above discussion on the sets in (A.1). Whenever the solution $x = (\sigma, \phi, v, \tau, q)$ cannot flow with the given q (i.e., $x(t, j) \in (C_{\rm stick} \cup C_{\rm slip}) \backslash (S_a \cup S_b \cup S_c)$), the designed jump sets in (21h)-(21j) *without* the condition $\tau \in [\delta, 2\delta]$ would ensure that x can jump and, after at most two jumps, q takes a value such that the flow map f corresponds to the affine system $\hat{f}$ according to which (4) evolves; in particular, the complete solution to (4) augmented with suitable components τ and q would be a solution to (21), and (22) would hold. Without the condition $\tau \in [\delta, 2\delta]$, however, (21) would also have solutions with Zeno behavior nullifying the desired stability property in Proposition 1. Then, it remains to prove that for each $\hat{x}_0 \in M\mathbb{B}$ such that $x_0 = (\hat{x}_0, \tau_0, q_0) \in D_1 \cup D_{-1}$, a solution $x \in \mathcal{S}_{\mathcal{H}_\delta}(x_0)$ evolves without jumping from $D_1 \cup D_{-1}$ for at least $t \in (0, \delta)$, so that after that time, jumps from $D_1 \cup D_{-1}$ are enabled again thanks to the flow map for τ. We address the case $x_0 \in D_1$ because $x_0 \in D_{-1}$ follows from similar arguments. Define the set $L_1 := \{x = (\sigma, F_s, 0, \tau, 0)\colon \sigma \leq 0\}$. Consider *first* $x_0 = (\hat{x}_0, \tau_0, 0) \in D_1 \cap L_1, \hat{x}_0 \in M\mathbb{B}$. A solution $x \in \mathcal{S}_{\mathcal{H}_\delta}(x_0)$ can flow in $C_{\rm stick}$ for at least $[0, 2F_s/M]$, and actually $x(t, 0) \in C_{\rm stick} \backslash (D_1 \cup D_{-1})$ for $t \in (0, 2F_s/M)$ with $2F_s/M \geq \delta$ by (20) in the proof of Lemma 1, and this shows that x can evolve without jumping from $D_1 \cup D_{-1}$ for at least $t \in (0, \delta)$. Consider *second* $x_0 = (\hat{x}_0, \tau_0, 0) \in D_1 \backslash L_1, \hat{x}_0 \in M\mathbb{B}$. By selecting $\tau_0 = \delta$ and $q_0 = 0$, we have $\tau(0, 1) = 0$ and $q(0, 1) = 1$. By using T_v defined in (15) in the proof of Lemma 1, we have $q(t, 1) = 1$ for $t \in [0, T_v]$ and $q(T_v, 2) = 0$ corresponding to a jump from D_0. Then, $x(t, 2) \in C_{\rm stick}$ for $t \in [T_v, T_s]$ with some time T_s. Lemma 1 guarantees that $T_v \geq \delta$ or $T_s \geq \delta$, which shows that x can evolve without jumping from $D_1 \cup D_{-1}$ for at least $t \in (0, \delta)$.