# 5

# Further switched systems

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*Mixed logical dynamical systems and linear complementarity systems are representations of switched systems, which under the conditions described here are equivalent to the model used in Chapter 4. They are particularly useful for model-predictive control. The equivalences of several hybrid system models show that different models, which are suitable for specific analysis and design problems and have been investigated in detail, cover the same class of hybrid systems. The analysis of the well-posedness of the models leads to conditions on the model equations under which a unique solution exists.*

**Chapter contents**



*Handbook of Hybrid Systems Control: Theory, Tools, Applications*, ed. Jan Lunze and Françoise Lamnabhi-Lagarrigue. Published by Cambridge University Press.

## 5.1 Model-predictive control of hybrid systems

*Model-predictive control* (MPC) is a widely used technology in industry for control design of highly complex multivariable processes. The idea behind MPC is to start with a model of the open-loop process that explains the dynamical relations among system's variables (command inputs, internal states, and measured outputs). Then, constraint specifications on system variables are added, such as input limitations (typically due to actuator saturation) and desired ranges where states and outputs should remain. Desired performance specifications complete the control problem setup and are expressed through different weights on tracking errors and actuator efforts (as in classical linear quadratic regulation). At each sampling time, an open-loop optimal control problem based on the given model, constraints, weights, and with initial condition set at the current (measured or estimated) state, is repeatedly solved through numerical optimization. The result of the optimization is an optimal sequence of future control moves. Only the first sample of such a sequence is actually applied to the process; the remaining moves are discarded. At the next time step, a new optimal control problem based on new measurements is solved over a shifted prediction horizon.

After quickly reviewing the basics of MPC based on linear models, in this section we introduce two hybrid model classes useful for MPC design, discrete hybrid automata (DHA) and mixed logical dynamical systems, and review the main ideas of hybrid MPC.

This section is based on the paper [58] for reviewing the basics of model-predictive control (MPC), and on [632] for DHA and MLD models used in MPC of hybrid systems.

### 5.1.1 Linear model-predictive control

The simplest MPC algorithm is based on the linear discrete-time prediction model

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) \tag{5.1}$$

of the open-loop process, where $\boldsymbol{x}(k) \in \mathbb{R}^n$ is the state vector at time $k$, and $\boldsymbol{u}(k) \in \mathbb{R}^m$ is the vector of manipulated variables to be determined by the controller, and on the solution of the finite-time-optimal control problem

$$\min_{U} \quad \boldsymbol{x}_N^{\mathrm{T}} \boldsymbol{P} \boldsymbol{x}_N + \sum_{k=0}^{N-1} \boldsymbol{x}^{\mathrm{T}}(k) \boldsymbol{Q} \boldsymbol{x}(k) + \boldsymbol{u}^{\mathrm{T}}(k) \boldsymbol{R} \boldsymbol{u}(k), \tag{5.2a}$$

$$\text{s.t.} \quad \boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k),\ k = 0, \ldots, N-1, \tag{5.2b}$$

$$\boldsymbol{x}_0 = \boldsymbol{x}^(k), \tag{5.2c}$$

$$\boldsymbol{u}_{\min} \le \boldsymbol{u}(k) \le \boldsymbol{u}_{\max},\ k = 0, \ldots, N-1, \tag{5.2d}$$

$$\boldsymbol{y}_{\min} \le \boldsymbol{C}\boldsymbol{x}(k) \le \boldsymbol{y}_{\max},\ k = 1, \ldots, N, \tag{5.2e}$$

where $N$ is the prediction horizon, $\boldsymbol{U} \triangleq \left[\, \boldsymbol{u}^{\mathrm{T}}(0) \cdots \boldsymbol{u}^{\mathrm{T}}(N-1) \,\right]^{\mathrm{T}} \in \mathbb{R}^{Nm}$ is the sequence of manipulated variables to be optimized, $\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}} \ge 0$, $\boldsymbol{R} = \boldsymbol{R}^{\mathrm{T}} > 0$, and

$\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} \geq 0$ are weight matrices of appropriate dimensions defining the performance index, $\boldsymbol{u}_{\min}, \boldsymbol{u}_{\max} \in \mathbb{R}^m$, $\boldsymbol{y}_{\min}, \boldsymbol{y}_{\max} \in \mathbb{R}^p, \boldsymbol{C} \in \mathbb{R}^{p\times n}$ define constraints on input and state variables, respectively, and "$\leq$" denotes component-wise inequalities. By substituting $\boldsymbol{x}(k) = \boldsymbol{A}^k\boldsymbol{x}(k) + \sum_{j=0}^{k-1} \boldsymbol{A}^j\boldsymbol{B}\boldsymbol{u}(k-1-j)$, (5.2) can be recast as the quadratic programming (QP) problem

$$\boldsymbol{U}^*(\boldsymbol{x}(k)) \triangleq \arg\min_{\boldsymbol{U}} \quad \frac{1}{2}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{U} + \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{C}^{\mathrm{T}}\boldsymbol{U} + \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{Y}\boldsymbol{x}(k), \tag{5.3a}$$

$$\text{s.t.} \quad \boldsymbol{G}\boldsymbol{U} \leq \boldsymbol{W} + \boldsymbol{S}\boldsymbol{x}(k), \tag{5.3b}$$

where $\boldsymbol{U}^*(\boldsymbol{x}(k)) = [\, \boldsymbol{u}^{\mathrm{T}*}(0)(\boldsymbol{x}(k)) \; \ldots \; \boldsymbol{u}^{\mathrm{T}*}(N-1)(\boldsymbol{x}(k)) \,]^{\mathrm{T}}$ is the optimal solution, $\boldsymbol{H} = \boldsymbol{H}^{\mathrm{T}} > 0$ and $\boldsymbol{C}$, $\boldsymbol{Y}$, $\boldsymbol{G}$, $\boldsymbol{W}$, $\boldsymbol{S}$ are matrices of appropriate dimensions [57, 67, 69]. Note that $\boldsymbol{Y}$ is not needed to compute $\boldsymbol{U}^*(\boldsymbol{x}(k))$, as it only affects the optimal value of (5.3a).

The MPC control algorithm is based on the following iterations: at time $k$, measure or estimate the current state $\boldsymbol{x}(k)$, solve the QP problem (5.3) to get the optimal sequence of future input moves $\boldsymbol{U}^*(\boldsymbol{x}(k))$, apply

$$\boldsymbol{u}(k) = \boldsymbol{u}_0^*(\boldsymbol{x}(k)) \tag{5.4}$$

to the process, discard the remaining optimal moves, and repeat the procedure again at time $k+1$.

In the absence of constraints (5.2d)–(5.2e), for $N \to \infty$ (or, equivalently, for $N < \infty$ and by choosing $\boldsymbol{P}$ as the solution of the algebraic Riccati equation associated with matrices $(\boldsymbol{A}, \boldsymbol{B})$ and weights $(\boldsymbol{Q}, \boldsymbol{R})$), the MPC control law (5.3)–(5.4) coincides with the linear quadratic regulator (LQR) [67]. From a design viewpoint, the MPC setup (5.2) can therefore be thought of as a way of bringing the LQR methodology to systems with constraints.

The basic MPC setup (5.2) can be extended in many ways. In particular in tracking problems usually one has to make a certain output vector $\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) \in \mathbb{R}^p$ track a reference signal $\boldsymbol{r}(k) \in \mathbb{R}^p$ under constraints (5.2d)–(5.2e). In order to do so, the cost function (5.2a) is replaced by

$$\sum_{k=0}^{N-1} (\boldsymbol{y}(k) - \boldsymbol{r}(k))^{\mathrm{T}}\boldsymbol{Q}_y(\boldsymbol{y}(k) - \boldsymbol{r}(k)) + \Delta\boldsymbol{u}^{\mathrm{T}}(k)\boldsymbol{R}\Delta\boldsymbol{u}(k), \tag{5.5}$$

where $\boldsymbol{Q}_y = \boldsymbol{Q}_y^{\mathrm{T}} \geq 0 \in \mathbb{R}^{p\times p}$ is a matrix of output weights, and the increments of command variables $\Delta\boldsymbol{u}(k) \triangleq \boldsymbol{u}(k) - \boldsymbol{u}(k-1)$ are the new optimization variables, possibly further constrained by $\Delta\boldsymbol{u}_{\min} \leq \Delta\boldsymbol{u}(k) \leq \Delta\boldsymbol{u}_{\min}$. In the above tracking setup vector $[\boldsymbol{x}^{\mathrm{T}}(k) \; \boldsymbol{r}^{\mathrm{T}}(k) \; \boldsymbol{u}^{\mathrm{T}}(t-1)]^{\mathrm{T}}$ replaces $\boldsymbol{x}(k)$ in (5.3b) and the control law (5.4) becomes $\boldsymbol{u}(k) = \boldsymbol{u}(k-1) + \Delta\boldsymbol{u}_0^*(\boldsymbol{x}(k), \boldsymbol{r}(k), \boldsymbol{u}(k-1))$.

The standard way of computing the linear MPC control action, which is implemented in most commercial MPC packages, is to solve the QP problem (5.3) on-line at each time $k$ (for example in the *MPC Toolbox for MATLAB* [69]).

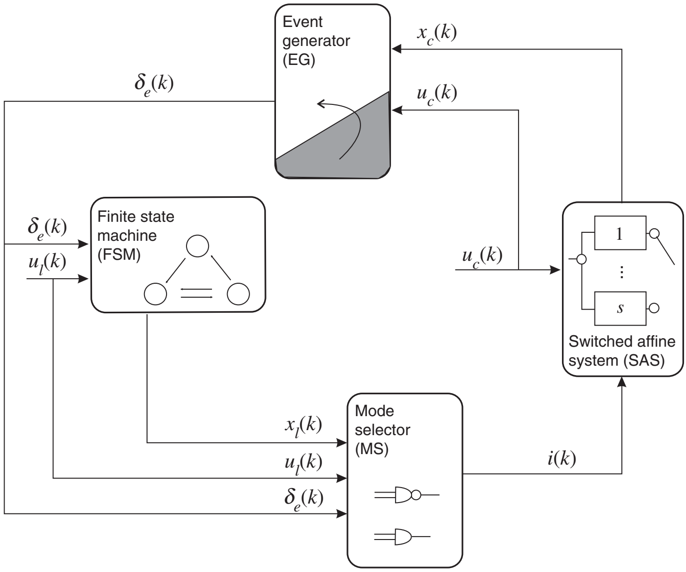


Fig. 5.1 Discrete hybrid automaton (DHA) as the connection of a finite-state machine (FSM) and a switched affine system (SAS), through a mode selector (MS) and an event generator (EG). The output signals are omitted for clarity.

Besides MPC schemes based on linear prediction models, several formulations of MPC based on general smooth nonlinear prediction models (as well as on uncertain linear models) exist. Most of them rely on nonlinear optimization methods for generic nonlinear functions/constraints to compute the control actions, and are therefore more rarely deployed in practical applications.

MPC based on *hybrid* dynamical models has emerged as a very promising approach to handle switching linear dynamics, on/off inputs, logic states, as well as logic constraints on input and state variables [62]. Here below we review a modeling framework for hybrid systems that is tailored to the synthesis of MPC controllers.

### 5.1.2 Discrete hybrid automata

*Discrete hybrid automata* (DHA) [632] are the interconnection of a finite-state machine and a switched linear dynamical system through a mode selector and an event generator (Fig. 5.1).

In the following we will use the fact that any discrete variable $\alpha \in \{\alpha_1, \ldots, \alpha_j\}$, admits a Boolean encoding $a \in \{0,1\}^{d(j)}$, where $d(j)$ is the number of bits used to represent $\alpha_1, \ldots, \alpha_j$. From now on we will refer to either the variable or its encoding with the same name.

**Switched affine system (SAS)** A switched affine system is a collection of linear affine systems:

$$\boldsymbol{x}_c(k+1) = \boldsymbol{A}_{i(k)}\boldsymbol{x}_c(k) + \boldsymbol{B}_{i(k)}\boldsymbol{u}_c(k) + \boldsymbol{f}_{i(k)}, \tag{5.6a}$$

$$\boldsymbol{y}_c(k) = \boldsymbol{C}_{i(k)}\boldsymbol{x}_c(k) + \boldsymbol{D}_{i(k)}\boldsymbol{u}_c(k) + \boldsymbol{g}_{i(k)}, \tag{5.6b}$$

where $k \in \mathbb{Z}^+$ is the time indicator, $\boldsymbol{x}_c \in \mathcal{X}_c \subseteq \mathbb{R}^{n_c}$ is the continuous state vector, $\boldsymbol{u}_c \in \mathcal{U}_c \subseteq \mathbb{R}^{m_c}$ is the exogenous continuous input vector, $\boldsymbol{y}_c \in \mathcal{Y}_c \subseteq \mathbb{R}^{p_c}$ is the continuous output vector, $\{\boldsymbol{A}_i, \boldsymbol{B}_i, \boldsymbol{f}_i, \boldsymbol{C}_i, \boldsymbol{D}_i, \boldsymbol{g}_i\}_{i\in\mathcal{Q}}$ is a collection of matrices of opportune dimensions, and the mode $i(k) \in \mathcal{Q} \triangleq \{1,\ldots,s\}$ is an input signal that chooses the affine state update dynamics. An SAS can be rewritten as the combination of linear terms and *if-then-else* rules: the state-update equation (5.6a) is equivalent to

$$\boldsymbol{z}_1(k) = \begin{cases} \boldsymbol{A}_1\boldsymbol{x}_c(k) + \boldsymbol{B}_1\boldsymbol{u}_c(k) + \boldsymbol{f}_1, & \text{if } (i(k)=1), \\ \boldsymbol{0}, & \text{otherwise}, \end{cases} \tag{5.7a}$$

$$\vdots$$

$$\boldsymbol{z}_s(k) = \begin{cases} \boldsymbol{A}_s\boldsymbol{x}_c(k) + \boldsymbol{B}_s\boldsymbol{u}_c(k) + \boldsymbol{f}_s, & \text{if } (i(k)=s), \\ \boldsymbol{0}, & \text{otherwise}, \end{cases} \tag{5.7b}$$

$$\boldsymbol{x}_c(k+1) = \sum_{i=1}^{s} \boldsymbol{z}_i(k), \tag{5.7c}$$

where $\boldsymbol{z}_i(k) \in \mathbb{R}^{n_c}, i = 1,\ldots,s$, and (5.6b) admits a similar transformation.

**Event generator (EG)** An event generator is a mathematical object that generates a logic signal according to the satisfaction of a linear affine constraint

$$\delta_e(k) = \boldsymbol{f}_{\mathrm{H}}(\boldsymbol{x}_c(k), \boldsymbol{u}_c(k), k), \tag{5.8}$$

where $\boldsymbol{f}_{\mathrm{H}} : \mathcal{X}_c \times \mathcal{U}_c \times \mathbb{Z}_{\geq 0} \to \mathcal{D} \subseteq \{0,1\}^{n_e}$ is a vector of descriptive functions of a linear hyperplane, and $\mathbb{Z}_{\geq 0} \triangleq \{0,1,\ldots\}$ is the set of nonnegative integers. In particular *threshold events* are modeled as $[\delta_e^i(k) = 1] \leftrightarrow [\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}_c(k) + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{u}_c(k) \leq c]$, where the superscript $i$ denotes the $i$-th component of a vector. *Time events* can be also modeled as: $[\delta_e^i(k) = 1] \leftrightarrow [t(k) \geq t_0]$, where $t(k+1) = t(k) + T_s$ denotes time, $T_s$ is the sampling time, and $t_0$ is a given time.

**Finite state machine (FSM)** A finite-state machine (or automaton) is a discrete dynamical process that evolves according to a logic state update function:

$$\boldsymbol{x}_\ell(k+1) = \boldsymbol{f}_{\mathrm{B}}(\boldsymbol{x}_\ell(k), \boldsymbol{u}_\ell(k), \delta_e(k)), \tag{5.9a}$$

where $\boldsymbol{x}_\ell \in \mathcal{X}_\ell \subseteq \{0,1\}^{n_\ell}$ is the Boolean state, $\boldsymbol{u}_\ell \in \mathcal{U}_\ell \subseteq \{0,1\}^{m_\ell}$ is the exogenous Boolean input, $\delta_e(k)$ is the endogenous input coming from the EG, and $\boldsymbol{f}_{\mathrm{B}} : \mathcal{X}_\ell \times \mathcal{U}_\ell \times \mathcal{D} \to \mathcal{X}_\ell$ is a deterministic logic function. (Here we will only refer to synchronous finite-state machines, where the transitions may happen only at sampling times. The adjective "synchronous" will be omitted for brevity.) An FSM can be conveniently represented using an oriented graph. An FSM may also have an associated Boolean output

$$\boldsymbol{y}_\ell(k) = \boldsymbol{g}_\mathrm{B}(\boldsymbol{x}_\ell(k), \boldsymbol{u}_\ell(k), \delta_e(k)), \tag{5.9b}$$

where $\boldsymbol{y}_\ell \in \mathcal{Y}_\ell \subseteq \{0,1\}^{p_\ell}$ and $\boldsymbol{g}_\ell : \mathcal{X}_c \times \mathcal{U}_c \times D \to \mathcal{Y}_\ell$.

**Mode selector (MS)** The logic state $\boldsymbol{x}_\ell(k)$, the Boolean inputs $\boldsymbol{u}_\ell(k)$, and the events $\delta_e(k)$ select the dynamical mode $i(k)$ of the SAS through a Boolean function $f_\mathrm{M} : \mathcal{X}_\ell \times \mathcal{U}_\ell \times \mathcal{D} \to \mathcal{Q}$, which is therefore called a *mode selector*. The output of this function

$$i(k) = f_\mathrm{M}(\boldsymbol{x}_\ell(k), \boldsymbol{u}_\ell(k), \delta_e(k)) \tag{5.10}$$

is called the *active mode*. We say that a *mode switch* occurs at step $k$ if $i(k) \neq i(k-1)$. Note that, in contrast to continuous-time hybrid models, where switches can occur at any time, in our discrete-time setting a mode switch can only occur at sampling instants.

DHA are related to *hybrid automata* (HA) [15], the main difference is in the time model: DHA admit time in the natural numbers, while in HA the time is a real number. Moreover, DHA models do not allow instantaneous transitions, and are deterministic, as opposed to HA where any enabled transition may occur in zero time. This has two consequences: (i) DHA do not admit live-locks (infinite switches in zero time), (ii) DHA do not admit Zeno behaviors (infinite switches in finite time). Finally, in DHA models, guards, reset maps, and continuous dynamics are limited to linear affine functions. Moreover, contrarily to HA, in DHA the continuous dynamics is not a property of the state of the automaton but is selected by the mode selector (MS) according also to discrete inputs and events. For equivalence results between linear hybrid automata and continuous-time piecewise affine systems see [136]. Reset maps in DHA can be dealt with as described in [632].

### 5.1.3 Mixed logical dynamical systems

This section describes how to transform a DHA into an equivalent hybrid model described by linear mixed-integer equations and inequalities, by generalizing several results that have already appeared in the literature [62, 331, 455, 553, 664]. The hybrid systems modeling language *HYSDEL* introduced in [632] and also described in Chapter 10 was developed to describe DHA and to automatically operate the transformations.

**Logical functions** Boolean functions can be equivalently expressed by inequalities [165].

In order to introduce our notation, we recall here some basic definitions of Boolean algebra. A variable $X$ is a *Boolean variable* if $X \in \{0,1\}$. A *Boolean expression* is inductively defined (for the sake of simplicity, we will neglect precedence) by the grammar

$$\begin{aligned} \phi ::= X|\neg\phi_1|\phi_1 \vee \phi_2|\phi_1 \oplus \phi_2|\phi_1 \wedge \phi_2| \\ \phi_1 \leftarrow \phi_2|\phi_1 \rightarrow \phi_2|\phi_1 \leftrightarrow \phi_2|(\phi_1), \end{aligned} \tag{5.11}$$

where $X$ is a Boolean variable, and the logic operators $\neg$ (not), $\vee$ (or), $\wedge$ (and), $\leftarrow$ (implied by), $\rightarrow$ (implies), $\leftrightarrow$ (iff) have the usual semantics. A Boolean expression is in *conjunctive normal form* (CNF) or *product of sums* if it can be written according to the following grammar:

$$\phi ::= \psi | \phi \wedge \psi, \tag{5.12}$$

$$\psi ::= \psi_1 \vee \psi_2 | \neg X | X, \tag{5.13}$$

where $\psi$ are called the *terms of the product*, and $X$ are the *terms of the sum* $\psi$. A CNF is minimal if it has the minimum number of terms of product and each term has the minimum number of terms of sum. Every Boolean expression can be rewritten as a minimal CNF.

A Boolean expression $f$ will be also called a *Boolean function* when is used to define a literal $X_n$ as a function of $X_1, \ldots, X_{n-1}$:

$$X_n = f(X_1, X_2, \ldots, X_{n-1}). \tag{5.14}$$

In general, we can define relations among Boolean variables $X_1, \ldots, X_n$ through a *Boolean formula*

$$F(X_1, \ldots, X_n) = 1, \tag{5.15}$$

where $X_i \in \{0, 1\}$, $i = 1, \ldots, n$. Note that each Boolean function is also a Boolean formula, but not vice versa. Boolean formulas can be equivalently translated into a set of integer linear inequalities. For instance, $X_1 \vee X_2 = 1$ is equivalent to $X_1 + X_2 \geq 1$ [664]. The translation can be performed either using an *symbolical* method or a *geometrical* method (see details in [632]). In particular, the symbolical method consists of first converting (5.14) or (5.15) into its CNF

$$\bigwedge_{j=1}^{m} \left( \bigvee_{i \in P_j} X_i \bigvee_{i \in N_j} \neg X_i \right),$$

with $N_j, P_j \subseteq \{1, \ldots, n\}$ $\forall j = 1, \ldots, m$. Then, the corresponding set of integer linear inequalities is

$$\begin{cases} 1 \leq \sum_{i \in P_1} X_i + \sum_{i \in N_1} (1 - X_i), \\ \vdots \\ 1 \leq \sum_{i \in P_m} X_i + \sum_{i \in N_m} (1 - X_i). \end{cases} \tag{5.16}$$

**Continuous-logic interfaces** By using the so-called "big-M" technique, events of the form (5.8) can be equivalently expressed as

$$f_{\mathrm{H}}^i(x_c(k), u_c(k), k) \leq M^i (1 - \delta_e^i), \tag{5.17a}$$

$$f_{\mathrm{H}}^i(x_c(k), u_c(k), k) > m^i \delta_e^i, \qquad i = 1, \ldots, n_e, \tag{5.17b}$$

where $M^i$, $m^i$ are upper and lower bounds, respectively, on $f_{\mathrm{H}}^i(x_c(k), u_c(k), k)$. As we will point out in Section 5.19e, sometimes, from a computational point of view, it may be convenient to have a system of inequalities without strict inequalities. In this case we will follow the common practice [664] of replacing the strict inequality (5.17b) by

$$f_{\mathrm{H}}^i(x_c(k), u_c(k), k) \geq \epsilon + (m^i - \epsilon)\delta_e^i, \tag{5.17c}$$

where $\epsilon$ is a small positive scalar, e.g. the machine precision, although the equivalence does not hold for $0 < f_{\mathrm{H}}^i(x_c(k), u_c(k), k) < \epsilon$, as the numbers in the interval $(0, \epsilon)$ cannot be represented in a computer.

The most common *logic to continuous* interface is the if-then-else construct

$$\text{IF } \delta \text{ THEN } z = a_1^{\mathrm{T}} x + b_1^{\mathrm{T}} u + f_1 \text{ ELSE } z = a_2^{\mathrm{T}} x + b_2^{\mathrm{T}} u + f_2, \tag{5.18}$$

which can be translated into [66]

$$(m_2 - M_1)\delta + z \leq a_2 x + b_2 u + f_2, \tag{5.19a}$$
$$(m_1 - M_2)\delta - z \leq -a_2 x - b_2 u - f_2, \tag{5.19b}$$
$$(m_1 - M_2)(1 - \delta) + z \leq a_1 x + b_1 u + f_1, \tag{5.19c}$$
$$(m_2 - M_1)(1 - \delta) - z \leq -a_1 x - b_1 u - f_1, \tag{5.19d}$$

where $M_i$, $m_i$ are upper and lower bounds on $a_i x + b_i u + f_i$, $i = 1, 2$, $\delta \in \{0, 1\}$, $z \in \mathbb{R}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. Note that when $a_2$, $b_2$, $f_2$ are zero, (5.18)–(5.19) coincide with the product $z = \delta \cdot (ax + bu + f)$ described in [664].

**Continuous dynamics** As already mentioned, we will deal with dynamics described by linear affine difference equations

$$x_c(k+1) = \sum_{i=1}^{s} z_i(k). \tag{5.20}$$

**Mixed logical dynamical systems** In [62] the authors proposed discrete-time hybrid systems denoted as mixed logical dynamical (MLD) systems. An MLD system is described by the following relations:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}_1\boldsymbol{u}(k) + \boldsymbol{B}_2\delta(k) + \boldsymbol{B}_3\boldsymbol{z}(k) + \boldsymbol{B}_5, \tag{5.21a}$$
$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}_1\boldsymbol{u}(k) + \boldsymbol{D}_2\delta(k) + \boldsymbol{D}_3\boldsymbol{z}(k) + \boldsymbol{D}_5, \tag{5.21b}$$
$$\boldsymbol{E}_2\delta(k) + \boldsymbol{E}_3\boldsymbol{z}(k) \leq \boldsymbol{E}_1\boldsymbol{u}(k) + \boldsymbol{E}_4\boldsymbol{x}(k) + \boldsymbol{E}_5, \tag{5.21c}$$
$$\tilde{\boldsymbol{E}}_2\delta(k) + \tilde{\boldsymbol{E}}_3\boldsymbol{z}(k) < \tilde{\boldsymbol{E}}_1\boldsymbol{u}(k) + \tilde{\boldsymbol{E}}_4\boldsymbol{x}(k) + \tilde{\boldsymbol{E}}_5. \tag{5.21d}$$

where $\boldsymbol{x} \in \mathbb{R}^{n_c} \times \{0, 1\}^{n_\ell}$ is a vector of continuous and binary states, $\boldsymbol{u} \in \mathbb{R}^{m_c} \times \{0, 1\}^{m_\ell}$ are the inputs, $\boldsymbol{y} \in \mathbb{R}^{p_c} \times \{0, 1\}^{p_\ell}$ the outputs, $\delta \in \{0, 1\}^{r_\ell}$, $\boldsymbol{z} \in \mathbb{R}^{r_c}$ represent auxiliary binary and continuous variables, respectively, and $\boldsymbol{A}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{B}_3$, $\boldsymbol{C}$, $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $\boldsymbol{D}_3$, $\boldsymbol{E}_1$,…,$\boldsymbol{E}_5$, and $\tilde{\boldsymbol{E}}_1$,…,$\tilde{\boldsymbol{E}}_5$ are matrices of suitable dimensions. Given the current state $\boldsymbol{x}(k)$ and input $\boldsymbol{u}(k)$, the time-evolution of (5.21) is determined by solving $\delta(k)$ and $\boldsymbol{z}(k)$ from (5.21c)–(5.21d), and then updating $\boldsymbol{x}^{\mathrm{T}}(k)$ and $\boldsymbol{y}(k)$ from (5.21a)–(5.21b). Since the problems of synthesis and analysis of MLD models are tackled using optimization techniques, we have replaced strict inequalities as in (5.17b) by non-strict inequalities as in (5.17c). (One may also explicitly include in (5.21) strict inequalities, as well as equalities.) A formal definition of well-posedness for MLD systems and a test to assess the well-posedness have been presented in [62].

For equivalence results between MLD systems and PWA systems, see Section 5.3.

### 5.1.4 Hybrid model-predictive control

MPC based on hybrid dynamical models has emerged in recent years as a very promising approach to operate switching linear dynamics, on/off inputs, and logic states, subject to combinations of linear and logical constraints on input and state variables [62]. Hybrid dynamics are often so complex that a satisfactory feedback controller cannot be synthesized by using analytical tools, and heuristic design procedures usually require trial and error sessions and extensive testing, and are time consuming, costly, and often inadequate to deal with the complexity of the hybrid control problem properly.

As for the linear MPC case, hybrid MPC design is a systematic approach to meet performance and constraint specifications in spite of the aforementioned switching among different linear dynamics, logical state transitions, and more complex logical constraints on system's variables. The approach consists of modeling the switching open-loop process and constraints as a discrete hybrid automaton using the language *HYSDEL* [632], and then automatically transforming the model into the MLD form (5.21).

The associated finite-horizon optimal control problem based on quadratic costs takes the form (5.3) with

$$\boldsymbol{U} = \left[\, \boldsymbol{u}^{\mathrm{T}}(0) \ldots \boldsymbol{u}^{\mathrm{T}}(N-1)\ \delta^{\mathrm{T}}(0) \ldots \delta^{\mathrm{T}}(N-1)\ \boldsymbol{z}^{\mathrm{T}}(0) \ldots \boldsymbol{z}^{\mathrm{T}}(N-1) \right]^{\mathrm{T}},$$

subject to the further restriction that some of the components of $\mathbf{U}$ must be either 0 or 1. The problem is therefore a mixed-integer quadratic programming (MIQP) problem, for which both commercial [198, 334] and public domain solvers (such as the one in [61]) are available. When infinity norms $\|\boldsymbol{Q}\boldsymbol{x}(k)\|_\infty$, $\|\boldsymbol{R}\boldsymbol{u}(k)\|_\infty$, $\|\boldsymbol{P}\boldsymbol{x}(k)\|_\infty$ are used in (5.2a) in place of quadratic costs, the optimization problem becomes a mixed-integer linear programming (MILP) problem [57, 63], which can be also handled by efficient public domain solvers such as [434], as well as by commercial solvers [198, 334].

Unfortunately MIPs are NP-complete problems. However, the state of the art in solving MIP problems is growing constantly, and problems of relatively large size can be solved quite efficiently. While MIP problems can always be solved to the global optimum, closed-loop stability properties can be guaranteed as long as the optimum value in (5.3) decreases at each time step. Usually, MIP solvers provide good feasible solutions within a relatively short time compared to the total time required to find and certify the global optimum. In the worst-case the complexity of optimally computing the control action $\boldsymbol{u}(k)$ on-line at each time $k$ depends exponentially on the number of integer variables [553]. In principle, this limits the scope of application of the proposed method to relatively slow systems, since the sampling time should be large enough for real-time implementation to allow the worst-case computation.

In general, an MIP solver provides the solution after solving a sequence of relaxed standard linear (or quadratic) problems (LP, QP). A potential drawback of MIP is (1) the need for converting the discrete/logic part of the hybrid problem into mixed-integer inequalities, therefore losing most of the original discrete structure, and (2) the fact that its efficiency mainly relies upon the tightness of the continuous LP/QP relaxations. Such drawbacks are not suffered by techniques for solving constraint satisfaction problems (CSP), i.e. the problem of determining whether a set of constraints over discrete variables can be satisfied. Under the class of CSP solvers we mention constraint logic programming (CLP) [439] and satisfiability (SAT) solvers [287], the latter specialized for the satisfiability of Boolean formulas. The approach of [60] combines MIP and CSP techniques in a co-operative way. In particular, convex programming for optimization over real variables, and SAT solvers for determining the satisfiability of Boolean formulas (or logic constraints), are combined in a single branch and bound solver.

Another approach for reducing the complexity of on-line computations is to look for suboptimal solutions. For instance in [337] the authors propose to suitably constrain the mode sequence over the prediction horizon, so that on-line optimization is solved more quickly. Although closed-loop stability is still guaranteed by this approach, clearly in general the overall tracking performance of the feedback loop gets deteriorated.

In the last decade, *explicit model-predictive control* has been proposed as a way to completely get rid of the need of on-line solvers (see [11] for a survey on explicit MPC).

For linear MPC, to get rid of on-line QP an approach to evaluate the MPC law (5.4) was proposed in [67]. Rather then solving the QP problem (5.3) on-line for the current vector $\boldsymbol{x}(k)$, the idea is to solve (5.3) off-line for all vectors $\boldsymbol{x}$ within a given range and make the dependence of $\boldsymbol{u}$ on $\boldsymbol{x}$ *explicit* (rather than implicitly defined by the optimization procedure (5.3)). The key idea is to treat (5.3) as a *multi-parametric* quadratic programming problem, where $\boldsymbol{x}(k)$ is the vector of parameters. It turns out that the optimizer $\boldsymbol{U}^* : \mathbb{R}^n \to \mathbb{R}^{mN_u}$ is a piecewise affine and continuous function, and consequently the MPC controller defined by (5.4) can be represented explicitly as

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{cases} \boldsymbol{F}_1\boldsymbol{x} + \boldsymbol{g}_1 \text{ if } \boldsymbol{H}_1\boldsymbol{x} \leq \boldsymbol{k}_1 \\ \quad\vdots \quad \vdots \\ \boldsymbol{F}_M\boldsymbol{x} + \boldsymbol{g}_M \text{ if } \boldsymbol{H}_M\boldsymbol{x} \leq \boldsymbol{k}_M. \end{cases} \tag{5.22}$$

It turns also out that the set of states $\mathcal{X}^*$ for which problem (5.3) admits a solution is a polyhedron, and that the optimum value in (5.3) is a piecewise quadratic, convex, and continuous function of $\boldsymbol{x}(k)$. The controller structure (5.22) is simply a look-up table of linear gains $(\boldsymbol{F}_i, \boldsymbol{g}_i)$, where the $i$-th gain is selected according to the set of linear inequalities $\boldsymbol{H}_i\boldsymbol{x} \leq \boldsymbol{k}_i$ that the state vector satisfies. Hence, the evaluation of the MPC controller (5.4), once put in the form (5.22), can be carried out by a very simple piece of control code. In the most naive implementation, the number of operations depends linearly in the worst case on the number $M$ of partitions, or even logarithmically if the partitions are properly stored [630].

An alternative way of solving MIP problems on-line is to extend explicit MPC ideas to the hybrid case. For hybrid MPC problems based on infinity norms, [63] showed that an equivalent piecewise affine explicit reformulation–possibly discontinuous, due to binary variables–can be obtained through off-line multiparametric mixed-integer linear programming techniques.

Thanks to the possibility of converting hybrid models (such as those designed through *HYSDEL*) to an equivalent piecewise affine (PWA) form [56], an explicit hybrid MPC approach dealing with quadratic costs was proposed in [105], based on dynamical programming (DP) iterations. Multiparametric quadratic programs (mpQP) are solved at each iteration, and quadratic value functions are compared to possibly eliminate regions that are proved to never be optimal. A different approach still exploiting the PWA structure of the hybrid model was proposed in [446], where all possible switching sequences are enumerated, an mpQP is solved for each sequence, and quadratic costs are compared on-line to determine the optimal input (in this respect, one could define the approach semi-explicit). To overcome the problem of enumerating all switching sequences and storing all the corresponding mpQP solutions, backwards reachability analysis is exploited in [10] (and implemented in the *Hybrid Toolbox*). A procedure to post-process the mpQP solutions and eliminate all polyhedra (and their associated control gains) that never provide the lowest cost was suggested in [10]. Typically the DP approach provides simpler explicit solutions when long horizons $N$ are chosen, but on the contrary tends to subdivide the state space in a larger number of polyhedra than the enumeration approach for short horizons.

For closed-loop convergence results of hybrid MPC the reader is referred to [62, 138, 386, 387, 388] and to the PhD thesis [385]. Extensions of hybrid MPC to stochastic hybrid systems was proposed in [59], and to event-based continuous-time hybrid systems in [71].

The *Hybrid Toolbox for MATLAB* [57] provides a nice development environment for hybrid and explicit MPC design. Hybrid dynamical systems described in *HYSDEL* are automatically converted to *MATLAB* MLD and PWA objects. MLD and PWA objects can be validated in open-loop simulation, either from the command line or through their corresponding *Simulink* blocks. Hybrid MPC controllers based on MILP/MIQP optimization can be designed and simulated, either from the command line or in *Simulink*, and can be converted to their explicit form for deployment. Several demos are available in the *Hybrid Toolbox* distribution. The toolbox can be freely downloaded from http://www.dii.unisi.it/hybrid/toolbox.

Similar functionalities are also included in the *Multi Parametric Toolbox* [375]. The reader is referred to Chapter 10 for a more detailed description of these tools.

In conclusion, hybrid MPC control can deal with very complex specifications in terms of models and constraints by using mixed-integer programming solvers. Explicit versions of hybrid MPC are possible, but still limited to small systems with few binary variables. Examples of applications of hybrid MPC to industrial control problems arising in the automotive domain are reported in Chapter 15.

## 5.2 Complementarity systems

### 5.2.1 Modeling aim

In many areas, especially in the domain of physical systems or in economic applications, continuous-time hybrid systems usually arise in specific forms. The continuous-time dynamics corresponding to the different modes, as well as their location invariants and guards, are often closely related. Indeed, in many cases the dynamics corresponding to the different modes all share a part that can be called the *core dynamics* of the system.

The theory of complementarity hybrid systems, as originally put forward in [572, 573], aims at providing a compact representation of many of such systems. It combines location invariants and guards in the form of *complementarity conditions* such as $0 \leq \boldsymbol{z} \perp \boldsymbol{w} \geq 0$, where $\boldsymbol{z}$ and $\boldsymbol{w}$ are equal-dimensioned vectors, and the inequalities hold componentwise. It is not without reason that many hybrid systems can be formulated in this manner, since complementarity conditions are closely related with variational and optimal formulations, which are known to be underlying many systems in physics and economical applications. Furthermore, it can be shown that, roughly speaking, all piecewise-linear characteristics can be modeled by complementarity conditions.

In addition to the rather broad applicability of complementarity modeling there are two other important advantages of complementarity models. First, complementarity models often provide a very compact description of hybrid systems, especially in comparison with hybrid automata. Furthermore, the complementarity model usually remains to the physics of the system, and physical system properties (such as passivity) are naturally reflected in the representation. Secondly, complementarity modeling offers powerful methods for analysis. Using the well-developed theory of the linear complementarity problem (LCP) from optimization theory [189] one may prove strong results concerning well-posedness (existence and uniqueness of solutions), stability and controllability. Also the theory of the LCP offers a wealth of computational methods, e.g. for the efficient computation of the next location at an event time. We refer to, e.g., [149, 300] for a detailed description of these results, especially for linear complementarity systems.

In this section we will mainly concentrate on indicating the modeling power of complementarity hybrid systems by discussing a list of appealing examples from different application areas, including the running examples in this handbook. Furthermore, we will briefly sketch some of the main results which have been obtained on the well-posedness, stability, controllability, and stabilizability of linear complementarity systems.

### 5.2.2 Definition

Complementarity systems can be constructed as follows [572, 573]. Start from a nonlinear input/output system, with $k$ inputs and $k$ outputs:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{5.23a}$$

$$\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{5.23b}$$

where $\boldsymbol{x}(t)$ is an $n$-dimensional state variable, $\boldsymbol{u}(t) \in \mathbb{R}^k$ is the input vector and $\boldsymbol{y}(t) \in \mathbb{R}^k$ is the output vector. To the system (5.23a)–(5.23(b)), add the relation

$$0 \leq \boldsymbol{y}(t) \perp \boldsymbol{u}(t) \geq 0. \tag{5.23c}$$

A relation of the form (5.23c) are called a *complementarity relation* in mathematical programming; whence the name *complementarity systems* for dynamical systems of the form (5.23). Note that (5.23c) is equivalent to the componentwise requirement that, for each $i = 1, \ldots, k$, the following holds: $y_i(t) \geq 0$, $u_i(t) \geq 0$, and at least one of these two inequalities is satisfied with equality.

In view of the particular role of the input and output variables in the formulation of complementarity systems, the notations $\boldsymbol{y}$ and $\boldsymbol{u}$ are sometimes replaced by $\boldsymbol{w}$ and $\boldsymbol{z}$, to steer away from the interpretation of the input as a control and the output as an observation and also to be in line with notational conventions in mathematical programming. In addition, the formulation in (5.23) can be made more general by allowing the functions $\boldsymbol{f}$ and $\boldsymbol{h}$ to depend directly on time.

Implicit in (5.23c) is the choice of an "active index set" $\alpha(t) \subset \{1, \ldots, k\}$ which is such that $y_i(t) = 0$ for $i \in \alpha(t)$ and $u_i(t) = 0$ for $i \notin \alpha(t)$. Any such index set is said to represent a *mode* of operation. In a fixed mode, the system above behaves as the dynamical system described by the differential equation (5.23a) and the algebraic relations (5.23b) together with the equalities that follow from the choice of the active index set in (5.23c). A change of mode occurs when continuation within a given mode would violate the nonnegativity constraints associated with this mode. The description format of complementarity systems is such that the nonsmoothness is made canonical, and specific properties, therefore, must be expressible in terms of the functions $\boldsymbol{f}(\cdot,\cdot)$ and $\boldsymbol{h}(\cdot,\cdot)$ occurring in (5.23a) and (5.23b), and possibly in terms of an initial condition.

A subclass of particular interest arises when the functions $f$ and $h$ in (5.23) are required to be linear; the resulting *linear complementarity systems* [304] are described by relations of the form

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), && (5.24a)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t), && (5.24b)\end{aligned}$$

$$0 \le \boldsymbol{y}(t) \perp \boldsymbol{u}(t) \ge 0, \tag{5.24c}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are linear mappings. In some applications it is natural to allow an external input (forcing term) in a complementarity system. The equations (5.23a) and (5.23b) are then replaced by equations of the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{v}(t)), \tag{5.25a}$$

$$\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{v}(t)), \tag{5.25b}$$

where $\boldsymbol{v}(t)$ denotes the forcing term; the equation (5.23c) is unchanged. In linear complementarity systems we require that the forcing term also enters linearly, so that the system (5.24) is replaced by

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{v}(t), && (5.26a)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) + \boldsymbol{F}\boldsymbol{v}(t), && (5.26b)\end{aligned}$$

$$0 \le \boldsymbol{y}(t) \perp \boldsymbol{u}(t) \ge 0, \tag{5.26c}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$, and $\boldsymbol{F}$ are linear mappings. Another useful generalization of (5.23) is obtained when the complementarity relation (5.23c) is replaced by the relation

$$\mathcal{C} \ni \boldsymbol{y}(t) \perp \boldsymbol{u}(t) \in \mathcal{C}^*, \tag{5.27}$$

where $\mathcal{C}$ is a cone in $\mathbb{R}^k$ and $\mathcal{C}^*$ is the dual cone defined by

$$\mathcal{C}^* = \{\boldsymbol{u} \mid \langle \boldsymbol{y}, \boldsymbol{u} \rangle \ge 0 \text{ for all } \boldsymbol{y} \in \mathcal{C}\}.$$

In particular, this format allows the incorporation of both equality and inequality constraints; a typical choice of the cone $\mathcal{C}$ is $\mathbb{R}_+^{k_1} \times \{0\}$, which implies $\mathcal{C}^* = \mathbb{R}_+^{k_1} \times \mathbb{R}^{k_2}$.

### 5.2.3 Examples

This section shows by means of several examples how complementarity relations can be obtained when modeling physical systems of different nature.

**Example 5.1** *DC-DC converter*

The DC-DC converter shown in Fig. 1.17 consists of an inductor $L$, a capacitor $C$, resistances $R_L$, $R_C$, and a resistance load $R$, together with a diode $D$ and an ideal switch $S$. The diode is modeled as an ideal diode, and its constitutive relation can be succinctly expressed by the complementarity condition

$$0 \le v_D \perp i_D \ge 0, \tag{5.28}$$

with $i_D$, respectively $v_D$, the current through the diode and the voltage across the diode. Furthermore, the constitutive relation of the ideal switch $S$ can be simply expressed as

$$v_S \perp i_S, \tag{5.29}$$

with $i_S, v_S$ the current through the switch and the voltage across the switch.

Taking as continuous state variables the voltage $v_C$ across the capacitor and the current $i_L$ through the inductor, we obtain the following dynamical equations of the DC-DC converter:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{C(R+R_C)} & 0 \\ 0 & -\frac{R_L}{L} \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{R}{C(R+R_C)} & 0 \\ 0 & \frac{1}{L} \end{bmatrix}\begin{bmatrix} i_D \\ v_S \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} E, \tag{5.30a}$$

$$\begin{bmatrix} v_D \\ i_S \end{bmatrix} = \begin{bmatrix} \frac{R}{R+R_C} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{RR_C}{R+R_C} & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} i_D \\ v_S \end{bmatrix}. \tag{5.30b}$$

Here, $E$ is the voltage of the input source.

Defining the *switching function* $\pi$ as

$$\pi(t) = \begin{cases} -1 & \text{if the switch } S \text{ is off at time } t, \\ 1 & \text{if the switch } S \text{ is on at time } t, \end{cases}$$

and the cones

$$\mathcal{C}_{-1} = \mathbb{R}_+ \times \{0\} \qquad \mathcal{C}_1 = \mathbb{R}_+ \times \mathbb{R},$$

one can represent the relations (5.28) and (5.29) as

$$\mathcal{C}_{\pi(t)} \ni \begin{bmatrix} v_D \\ i_S \end{bmatrix} \perp \begin{bmatrix} i_D \\ v_S \end{bmatrix} \in \mathcal{C}^*_{\pi(t)}. \tag{5.30c}$$

Systems of the form (5.30) are called *switched cone complementarity systems*. As shown in [157, 158, 307], this class provides a compact representation of any type of power converters.

Note that certain properties of the DC-DC converter are immediately obtained as a direct consequence of the complementarity modeling. For example, it is readily verified that the total energy $H(v_c, i_L) := \frac{1}{2}Cv_C^2 + \frac{1}{2}Li_L^2$ stored in the circuit satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}H = -R_L i_L^2 - R_C i_C^2 - RI^2 + EI_E,$$

where $I$ denotes the current though the resistive load $R$, and $I_E$ is the current through the voltage source with voltage $E$. Hence *passivity* of the obtained model is directly established. (In fact, this becomes even more transparant by writing (5.30) into a port-Hamiltonian form, thus obtaining a *port-Hamiltonian complementarity system*.) □

**Example 5.2** *Two-tank system*

The discrete states of the two-tank system, introduced in Section 1.3.1 are determined by inequalities involving the continuous states and by external switches. The system can be modeled in a switched complementarity framework. The main issues are: (i) modeling of the mode-dependent flow through the valve $\mathrm{V}_1$, and (ii) modeling of the opening and closing of the valves.

Consider first item (i). The flow equations can be described in terms of the positive-part operator which is defined, for $x \in \mathbb{R}$, by

$$x^+ = \max(x, 0).$$

Indeed, we can write

$$Q_{12}^{V_1}(t) = u_1(t)T\big((h_1(t) - h_0)^+ - (h_2(t) - h_0)^+\big), \tag{5.31}$$

where $T(x)$ is the Torricelli characteristic

$$T(x) = c\,\mathrm{sgn}(x)\sqrt{2g|x|}, \tag{5.32}$$

which may be considered to be a smooth function even though its derivative at $x = 0$ is infinity. The positive-part operator in its turn can be described in terms of a complementary characteristic, since the relations

$$w = x^+, \qquad z = x^- = (-x)^+$$

are equivalent to

$$x = w - z, \qquad 0 \le w \perp z \ge 0.$$

Therefore, the relation (5.31) can alternatively be formulated as

$$Q_{12}^{V_1}(t) = u_1(t)\,T(w_1(t) - w_2(t)), \tag{5.33}$$

together with the complementarity relations

$$0 \le w_1(t) \perp z_1(t) \ge 0, \tag{5.34}$$
$$0 \le w_2(t) \perp z_2(t) \ge 0, \tag{5.35}$$

and the algebraic relations

$$w_1(t) = h_1(t) - h_0 + z_1(t), \tag{5.36}$$
$$w_2(t) = h_2(t) - h_0 + z_2(t). \tag{5.37}$$

The Torricelli characteristic (5.32) could be replaced by a nonsmooth function; for instance a relay characteristic might be an alternative. Complementarity modeling is then still possible by using the techniques described below in the discussion of relay systems.

The switching of the valves can be modeled as in (5.31) by means of a multiplicative factor, but an alternative is to use the setting of switched cone complementarity systems as proposed in [157]. One then introduces to the flow variable $Q_{12}$ another variable $\lambda_{12}$, which relates to pressure drop across the valve $\mathrm{V}_1$. The modeling (5.31) (or equivalently (5.33)) is then replaced by

$$Q_{12}^{V_1}(t) = T(w_1(t) - w_2(t)) + \lambda_{12}(t), \tag{5.38}$$

together with the complementarity relation

$$\mathcal{C}^*_{u_1(t)} \ni \lambda_{12}(t) \perp Q_{12}(t) \in \mathcal{C}_{u_1(t)}, \tag{5.39}$$

where $\mathcal{C}$ is the cone-valued function defined by

$$\mathcal{C}_0 = \{0\}, \qquad \mathcal{C}_1 = \mathbb{R}.$$

This models an on/off switch. One could also describe switches that admit flow in one direction only by making use of the cones $\mathbb{R}_+$ and $\mathbb{R}_-$. □

**Example 5.3** *Relay systems*

Consider a dynamical system of the form

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), u(t)), \tag{5.40}$$

together with the following nonsmooth relation that specifies the dependence of the input variable $u(t)$ on the state $\boldsymbol{x}(t)$:

$$u(t) = 1 \text{ if } h(\boldsymbol{x}(t)) > 0, \tag{5.41a}$$

$$0 \leq u(t) \leq 1 \text{ if } h(\boldsymbol{x}(t)) = 0, \tag{5.41b}$$

$$u(t) = 0 \text{ if } h(\boldsymbol{x}(t)) < 0. \tag{5.41c}$$

Such a system is called a *differential equation with discontinuous right-hand side* or a *relay system*. The latter terminology is derived from the fact that, if one introduces an output variable $y(t) = h(\boldsymbol{x}(t))$, the relation between $u(t)$ and $y(t)$ given by (5.41) is known as a *relay characteristic*. The system can of course also be viewed as a hybrid system with three different modes.

Relay systems can be modeled as cone complementarity systems. For this purpose, introduce two input variables, say $v(t)$ and $z(t)$, in addition to $u(t)$. Corresponding to the three input variables $u(t)$, $v(t)$, and $z(t)$, introduce three output variables $p(t)$, $q(t)$, and $r(t)$, which are defined in terms of the state and input variables by

$$\begin{aligned} p(t) &= z(t), \\ q(t) &= z(t) + h(\boldsymbol{x}(t)), \\ r(t) &= u(t) + v(t) - 1. \end{aligned}$$

A cone complementarity system is formed by taking the equation (5.40) together with the following cone complementarity relations:

$$\begin{aligned} 0 \leq p(t) &\perp u(t) \geq 0, \\ 0 \leq q(t) &\perp v(t) \geq 0, \\ 0 = r(t) &\perp z(t) \in \mathbb{R}. \end{aligned}$$

This is of the form (5.27) with $\mathcal{C} = \mathbb{R}^2_+ \times \{0\}$. The third relation is equivalent to the requirement that $u(t) + v(t) = 1$ for all $t$. As a consequence, the number of modes implied by the first two relations is reduced from four to three, since the mode corresponding to $u(t) = 0$ and $v(t) = 0$ is eliminated. The three remaining votes can be described as follows:

1. $u(t) = 0$, $p(t) \geq 0$, $q(t) = 0$, $v(t) \geq 0$. The relations $p(t) \geq 0$ and $q(t) = 0$ imply that in this case we must have $h(\boldsymbol{x}(t)) \leq 0$.

2. $u(t) \geq 0$, $p(t) = 0$, $q(t) = 0$, $v(t) \geq 0$. The relations $p(t) = 0$ and $q(t) = 0$ imply that $h(\boldsymbol{x}(t)) = 0$. From the relations $u(t) \geq 0$, $v(t) \geq 0$, and $u(t) + v(t) = 1$ it follows that $0 \leq u(t) \leq 1$.

3. $u(t) \geq 0$, $p(t) = 0$, $q(t) \geq 0$, $v(t) = 0$. The relations $p(t) \geq 0$ and $q(t) = 0$ imply that in this case we must have $h(\boldsymbol{x}(t)) \geq 0$. From $v(t) = 0$ and $u(t) + v(t) = 1$ it follows that $u(t) = 1$.

It is seen that the cone complementarity system describes the same dynamics as the relay system. In this specific case, where we have a single relay characteristic, the reformulation in cone complementarity form may appear to be artificial and perhaps even awkward. However, the cone complementarity form simplifies substantially the description of multi-regime dynamics in more complicated situations. This is demonstrated below. □

**Example 5.4** *Filippov systems*

Consider the dynamics

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{k} \lambda_i(t) f_i(\boldsymbol{x}(t)), \qquad \boldsymbol{w}(t) = H(\boldsymbol{x}(t)), \tag{5.42a}$$

where $H$ is a smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$, together with the linear constraint

$$\sum_{i=1}^{k} \lambda_i(t) = 1 \tag{5.42b}$$

and the complementarity conditions

$$0 \le \boldsymbol{\lambda}(t) \perp \boldsymbol{w}(t) - a(t)\mathbb{1} \ge 0, \tag{5.42c}$$

where $a(t)$ is an additional unknown, and where $\mathbb{1}$ denotes the $k$-vector whose entries are all equal to 1. To explain the meaning of these equations, consider a time $t$ at which the state vector $x(t)$ is located in the open set $\mathcal{H}_i$ defined by

$$\mathcal{H}_i = \{\boldsymbol{x} \mid H_i(\boldsymbol{x}) < H_j(\boldsymbol{x}), \text{ for all } j \neq i\}. \tag{5.43}$$

To ensure that the $i$-th component of $\boldsymbol{w}(t) - a(t)\mathbb{1}$ is nonnegative, we must have $a(t) \le w_i(t) = H_i(\boldsymbol{x}(t))$. If the strict inequality $a(t) < w_i(t)$ was valid, then all components of the vector $\boldsymbol{w}(t) - a(t)\mathbb{1}$ would be strictly positive, which by the complementarity condition would imply that all coefficients of the vector $\boldsymbol{\lambda}$ would be zero. This would contradict the constraint (5.42b). It follows that, for $\boldsymbol{x}(t) \in \mathcal{H}_i$, we must have $a(t) = w_i(t)$. Since $w_j(t) - a(t) > 0$ for $j \neq i$, the complementarity conditions implies then that $\lambda_j = 0$ for $j \neq i$, and from the constraint (5.42b) it follows that $\lambda_i = 1$. Therefore, we find that for all $i = 1, \ldots, k$,

$$\dot{\boldsymbol{x}}(t) = f_i(\boldsymbol{x}(t)), \quad \text{if } \boldsymbol{x}(t) \in \mathcal{H}_i. \tag{5.44}$$

In this way we see that the equations (5.42) describe a multi-regime system with state-dependent switching. Moreover, the equations define a convex relaxation on the boundaries between these regions. Systems of this type have been studied extensively [237].

To write the system in the cone complementarity form (5.23a)–(5.23b)–(5.27), define

$$\mathcal{C} = \mathbb{R}_+^k \times \{0\}, \quad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{\lambda} \\ a \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} H(\boldsymbol{x}) - a\mathbb{1} \\ \mathbb{1}^{\mathrm{T}}\boldsymbol{\lambda} - 1 \end{bmatrix}. \;\square \tag{5.45}$$

**Example 5.5** *A Leontiev economy*

A model for a continuous-time Leontiev economy may be constructed as follows. Let $x_i(t)$, $u_i(t)$, and $v_i(t)$ respectively denote the inventory, production rate, and net exogenous demand associated with commodity $i$ at time $t$. Furthermore, let $q_{ij}$ denote the amount of commodity $i$ required for the production of one unit of commodity $j$. A balance equation for the evolution of the inventory may then be written in the form

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{I} - \boldsymbol{Q})\boldsymbol{u}(t) - \boldsymbol{v}(t), \tag{5.46a}$$

where $\boldsymbol{Q}$ is the matrix formed from the elements $q_{ij}$. It is natural to impose that inventory should be nonnegative, but this is by no means sufficient to determine a solution uniquely.

However, if we furthermore impose that the economy is efficient in the sense that it produces the lowest amounts of commodities that are sufficient to meet demand, then commodities are not produced when there is still a positive inventory, and are otherwise produced in just sufficient amounts to prevent inventory from becoming negative. In other words, the complementarity relation

$$0 \leq \boldsymbol{x}(t) \perp \boldsymbol{u}(t) \geq 0 \tag{5.46b}$$

must hold for all $t$. The system (5.46) is in the form of the forced linear complementarity system (5.26) with $\boldsymbol{A} = \boldsymbol{O}$, $\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{Q}$, $\boldsymbol{C} = \boldsymbol{I}$, $\boldsymbol{D} = \boldsymbol{O}$, $\boldsymbol{E} = -\boldsymbol{I}$, and $\boldsymbol{F} = \boldsymbol{O}$. □

**Example 5.6** *A user-resource model*

Many models for network usage can be described in terms of users who have access to several resources. For instance, users may be origin-destination pairs in a traffic network model, and in this case resources are the links between crossings. In the context of production planning, users may be products and resources may be machines. The use of a given resource generates a certain cost for the user, for instance in terms of incurred delay; this cost depends in general on the load that is placed on the resource by all users. A typical purpose of modeling is to describe the behavior of users in determining their demand for services from the resources available to them.

To set up a general model in mathematical terms, suppose that we have $p$ users and $m$ resources. Introduce the following quantities:

- $l_{ij}(t)$ = load per unit of time placed by user $i$ on resource $j$ at time $t$;
- $q_{ij}(t)$ = cost incurred at time $t$ by user $i$ when applying to resource $j$;
- $d_i(t)$ = total demand of user $i$ at time $t$;
- $a_i(t)$ = cost accepted by user $i$ at time $t$.

The above quantities are summarized in a *load matrix* $\boldsymbol{L}(t) \in \mathbb{R}_+^{p\times m}$ (load is taken to be nonnegative), a *cost matrix* $\boldsymbol{Q}(t) \in \mathbb{R}^{p\times m}$, a *demand vector* $\boldsymbol{d}(t) \in \mathbb{R}^p$, and an *accepted cost vector* $\boldsymbol{a}(t) \in \mathbb{R}^p$. Moreover we introduce a *state vector* $\boldsymbol{x}(t) \in \mathbb{R}^n$ in terms of which the dynamics of the system is described, and which moreover determines the cost matrix:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(\boldsymbol{x}(t), \boldsymbol{L}(t)), \tag{5.47a}$$

$$\boldsymbol{Q}(t) = h(\boldsymbol{x}(t), \boldsymbol{L}(t)). \tag{5.47b}$$

To describe the behavior of users, we assume that the Wardrop principle holds at every time instant $t$. In other words, given a demand level, each user distributes its load over resources in such a way that all resources that are used generate the same cost (this is the accepted cost), and there is no resource that is not used and that would generate a lesser cost. This behavioral principle, together with the nonnegativity of the load, can be expressed in matrix terms by

$$0 \leq \boldsymbol{L}(t) \perp \boldsymbol{Q}(t) - \boldsymbol{a}(t) \cdot \mathbb{1}^{\mathrm{T}} \geq 0, \tag{5.47c}$$

where the "perp" relation is understood in the sense of the inner product $\langle A, B\rangle = \mathrm{tr}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B})$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{p\times m}$. To close the model, we furthermore need the accounting relation

$$\boldsymbol{L}(t)\mathbb{1} = \boldsymbol{d}(t) \tag{5.47d}$$

as well as a "constitutive relation" between the demand and the accepted cost, which we take to be of the form

$$R(\boldsymbol{d}, \boldsymbol{a}) = 0, \tag{5.47e}$$

where $R$ is a mapping from $\mathbb{R}^p \times \mathbb{R}^p$ to $\mathbb{R}^p$. The system (5.47) can be rendered as a cone complementarity system (5.23a) – (5.23b) – (5.27) by means of the identifications

$$\begin{aligned} \boldsymbol{u} &= (\boldsymbol{L}, \boldsymbol{a}), \\ \boldsymbol{y} &= \big(h(\boldsymbol{x}, \boldsymbol{L}) - \boldsymbol{a} \cdot \mathbb{1}^{\mathrm{T}}, R(\boldsymbol{L}\mathbb{1}, \boldsymbol{a})\big), \\ \mathcal{C} &= \mathbb{R}_{+}^{p\times m} \times \{0\} \subset \mathbb{R}^{p\times(m+1)}. \end{aligned} \tag{5.48}$$

As a specific case, consider a situation where the resources consist of $m$ noninteracting queues and the state variables are the queue lengths. Ignoring the situations in which buffers are empty or full (which in fact could be naturally modeled in a complementarity framework), we write simple queue dynamics

$$\frac{\mathrm{d}x_j}{\mathrm{d}t}(t) = (\mathbb{1}^{\mathrm{T}}\boldsymbol{L})_j - c_j \tag{5.49}$$

where $c_j$ is a constant that represents the processing speed of queue $j$. A possible expression for cost is

$$Q_{ij} = k_j x_j + m_{ij},$$

where $k_j$ is a proportionality constant, and the constants $m_{ij}$ represent a fixed cost that may be user-specific. Finally assume that demand is constant, say, $\boldsymbol{d}(t) = \boldsymbol{d}_0$ irrespective of the actual cost $a(t)$. We then arrive at the following dynamical model:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{L}^{\mathrm{T}}(t)\mathbb{1} - \boldsymbol{c}, \tag{5.50a}$$

$$\boldsymbol{Q}(t) = \mathbb{1} \cdot (\boldsymbol{K}\boldsymbol{x}(t))^{\mathrm{T}} + \boldsymbol{M}, \tag{5.50b}$$

$$\boldsymbol{L}(t)\mathbb{1} = \boldsymbol{d}_0, \tag{5.50c}$$

$$0 \le \boldsymbol{L}(t) \perp \boldsymbol{Q}(t) - \boldsymbol{a}(t) \cdot \mathbb{1}^{\mathrm{T}} \ge 0. \tag{5.50d}$$

This is a linear (actually affine) cone complementarity system. The constant terms can be treated as external inputs, analogously to (5.26). □

### 5.2.4 Preliminaries

For the sake of completeness, we review the linear complementarity problem of mathematical programming and the notion of passivity of systems theory.

**Linear complementarity problem** Given an $m$-vector $\boldsymbol{q}$ and an $m \times m$ matrix $\boldsymbol{M}$, the linear complementarity problem LCP($\boldsymbol{q}$, $\boldsymbol{M}$) is to find an $m$-vector $\boldsymbol{z}$ such that

$$\boldsymbol{z} \ge 0, \tag{5.51a}$$

$$\boldsymbol{q} + \boldsymbol{M}\boldsymbol{z} \ge 0, \tag{5.51b}$$

$$\boldsymbol{z}^{\mathrm{T}}(\boldsymbol{q} + \boldsymbol{M}\boldsymbol{z}) = 0. \tag{5.51c}$$

If such a vector $\boldsymbol{z}$ exists, we say that $\boldsymbol{z}$ *solves* (is a *solution of*) LCP($\boldsymbol{q}$, $\boldsymbol{M}$). We say that the LCP($\boldsymbol{q}$, $\boldsymbol{M}$) is *feasible* if there exists $\boldsymbol{z}$ satisfying (5.51a) and (5.51b).

We define the sets

$$\text{LCP} - \text{Im}(\boldsymbol{M}) := \{\boldsymbol{q} \in \mathbb{R}^m \mid LCP(\boldsymbol{q}, \boldsymbol{M}) \text{ admits a solution}\} \tag{5.52}$$

and

$$\text{LCP} - \ker(\boldsymbol{M}) := \{\boldsymbol{z} \in \mathbb{R}^m \mid \boldsymbol{z} \text{ solves } LCP(0, \boldsymbol{M})\}. \tag{5.53}$$

The LCP is a well-studied subject of mathematical programming. An excellent survey of the topic can be found in the book [189]. For the sake of completeness, we quote the following two theorems. The first one can be considered as the *fundamental theorem* of LCP theory. It states necessary and sufficient conditions for the unique solvability of the LCP for all vectors $\boldsymbol{q}$.

**Theorem 5.1** *[189] The LCP($\boldsymbol{q}, \boldsymbol{M}$) has a unique solution for all $\boldsymbol{q}$ if and only if all the principal minors of the matrix $\boldsymbol{M}$ are positive.*

Matrices with the above-mentioned property are known as P-matrices. It is well-known that every positive definite matrix is in this class. Besides positive definite matrices, the nonnegative definite matrices will appear in the LCP context in the sequel. If the $\boldsymbol{M}$ matrix is nonnegative definite then the LCP does not necessarily have solutions for all vectors $\boldsymbol{q}$. For example, the LCP($\boldsymbol{q}, 0$) admits solutions only if $\boldsymbol{q} \geq 0$.

The following theorem characterizes the conditions under which an LCP with a nonnegative definite matrix $\boldsymbol{M}$ has solutions:

**Theorem 5.2** *[189] Let $M$ be a nonnegative definite matrix. Then,*

$$\text{LCP} - \text{Im}(\boldsymbol{M}) = (\text{LCP} - \ker(\boldsymbol{M}))^*. \tag{5.54}$$

**Linear passive systems** Having roots in circuit theory, passivity is a concept that has always played a central role in systems theory. A system is passive if for any time interval the difference between the stored energy at the end of the interval and at the beginning is less than or equal to the supplied energy during the interval.

**Definition 5.1 (Passive system)** [663] *A linear system $\Sigma(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ given by*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{z}(t), \tag{5.55a}$$

$$\boldsymbol{w}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{z}(t) \tag{5.55b}$$

*is called* passive *if there exists a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that for all $t_0 \leq t_1$ and all trajectories $(z, x, w)$ of the system (5.55) the following inequality holds:*

$$V(\boldsymbol{x}(t_0)) + \int_{t_0}^{t_1} \boldsymbol{z}^{\mathrm{T}}(t)\boldsymbol{w}(t)\,\mathrm{d}t \geq V(\boldsymbol{x}(t_1)). \tag{5.56}$$

*If it exists the function $V$ is called a* storage function.

Passivity property can be characterized in terms of the state space representation or the transfer matrix of the system as follows.

**Proposition 5.1** *Consider the following statements:*

1. *The system $\Sigma(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ is passive.*
2. *The linear matrix inequalities*

$$\boldsymbol{K} = \boldsymbol{K}^{\mathrm{T}} \geq 0 \quad \textit{and} \quad \begin{bmatrix} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{K} + \boldsymbol{K}\boldsymbol{A} & \boldsymbol{K}\boldsymbol{B} - \boldsymbol{C}^{\mathrm{T}} \\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{K} - \boldsymbol{C} & -(\boldsymbol{D} + \boldsymbol{D}^{\mathrm{T}}) \end{bmatrix} \leq 0 \tag{5.57}$$

   *have a solution $\boldsymbol{K}$.*
3. *The function $V(x) = \frac{1}{2}x^{\mathrm{T}}\boldsymbol{K}x$ defines a storage function.*
4. *The transfer matrix $\boldsymbol{G}(s) = \boldsymbol{D} + \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$ is positive real, i.e., $u^*[\boldsymbol{G}(\lambda) + \boldsymbol{G}^*(\lambda)]u \geq 0$ for all complex vectors $\boldsymbol{u}$ and all complex numbers $\lambda$ such that $Re(\lambda) > 0$ and $\lambda$ is not an eigenvalue of $\boldsymbol{A}$.*
5. *The triple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is minimal.*
6. *The pair $(\boldsymbol{C}, \boldsymbol{A})$ is observable.*
7. *The matrix $\boldsymbol{K}$ is positive definite.*

*The following implications hold:*

*(i). 1 $\Leftrightarrow$ 2 $\Leftrightarrow$ 3.*
*(ii). 2 $\Rightarrow$ 4.*
*(iii). 4 and 5 $\Rightarrow$ 2.*
*(iv). 2 and 6 $\Rightarrow$ 7.*

### 5.2.5 Existence and uniqueness of solutions

Consider the system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{z}(t) + \boldsymbol{E}\boldsymbol{u}(t), \tag{5.58a}$$
$$\boldsymbol{w}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{z}(t) + \boldsymbol{F}\boldsymbol{u}(t), \tag{5.58b}$$
$$0 \leq \boldsymbol{z}(t) \perp \boldsymbol{w}(t) \geq 0, \tag{5.58c}$$

where the state $\boldsymbol{x}$ takes values from $\mathbb{R}^n$, the input $\boldsymbol{u}$ from $\mathbb{R}^k$, the complementarity variables $(\boldsymbol{z}, \boldsymbol{w})$ from $\mathbb{R}^{m+m}$. We call these systems *linear complementarity systems* and denote (5.58) by LCS$(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F})$. When the sextuple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F})$ is clear from the context, we use only LCS.

We say that a triple $(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{w})$, where $\boldsymbol{x}$ is absolutely continuous and $(\boldsymbol{z}, \boldsymbol{w})$ is locally integrable:

- is a *Carathéodory solution* of (5.58) for the initial state $\boldsymbol{x}_0$ and the input $\boldsymbol{u}$ if $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and (5.58a) is satisfied for almost all $t \geq 0$ and (5.58b)–(5.58c) are satisfied for all $t \geq 0$;

- is a *forward solution* of (5.58) for the initial state $\boldsymbol{x}_0$ and the input $\boldsymbol{u}$ if $(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{w})$ is a solution and for each $\bar{t} \geq 0$ there exist an index set $\alpha(\bar{t}) \subseteq \{1, 2, \ldots, m\}$, and a positive number $\varepsilon_{\bar{t}}$ such that

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{z}(t) + \boldsymbol{E}\boldsymbol{u}(t), \tag{5.59a}$$
$$\boldsymbol{w}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{z}(t) + \boldsymbol{F}\boldsymbol{u}(t), \tag{5.59b}$$
$$\boldsymbol{z}_{\alpha(\bar{t})}(t) \geq 0 \quad \boldsymbol{w}_{\alpha(\bar{t})}(t) = 0, \tag{5.59c}$$
$$\boldsymbol{z}_{\alpha^c(\bar{t})}(t) = 0 \quad \boldsymbol{w}_{\alpha^c(\bar{t})}(t) \geq 0 \tag{5.59d}$$

holds for all $t \in (\bar{t}, \bar{t} + \varepsilon)$. Here $\alpha^c$ denotes the complement of the set $\alpha$ in $\{1, 2, \ldots, m\}$.

Throughout the chapter, we will be mainly interested in Bohl-type inputs. A function $f : \mathbb{R}_+ \to \mathbb{R}^p$ is said to be a *Bohl function* if $f(t) = \boldsymbol{Z}\exp(\boldsymbol{X}t)\boldsymbol{Y}$ holds for all $t \geq 0$ and for some matrices $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ with appropriate sizes.

**Existence and uniqueness of forward solutions** The following theorem provides sufficient conditions for the existence and uniqueness of forward solutions:

**Theorem 5.3** *[153] Let $\boldsymbol{G}(s) = \boldsymbol{D} + \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$. Suppose that*

- *for all $\pi \subseteq \{1, 2, \ldots, m\}$, $\boldsymbol{G}_{\pi\pi}(s)$ is invertible as a rational matrix and $s^{-1}\boldsymbol{G}^{-1}{}_{\pi\pi}(s)$ is proper; and*
- *$\boldsymbol{G}(\sigma)$ is a P-matrix for all sufficiently large real numbers $\sigma$.*

*Then, the following statements are equivalent:*

1. *There exists a forward solution of the LCS (5.58) for the initial state $\boldsymbol{x}_0$ and the Bohl input $\boldsymbol{u}$.*
2. $\boldsymbol{C}\boldsymbol{x}_0 + \boldsymbol{F}\boldsymbol{u}(0) \in \mathrm{LCP} - \mathrm{Im}(\boldsymbol{D})$.

*Moreover, if a forward solution exists it is unique.*

**Existence and uniqueness of Carathéodory solutions** Theorem 5.3 presents conditions for the existence and uniqueness of forward solutions. However, the uniqueness of Carathéodory solutions is not guaranteed by those conditions in general as illustrated by the following example.

**Example 5.7** *Complementarity system with multiple solutions*

The LCS$(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, 0, 0, 0)$ with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

has multiple Carathéodory solutions for the initial state $\boldsymbol{x}_0 = \mathrm{col}(0, 0, 0, 1)$ [78, 584]. Note that $\boldsymbol{C}\boldsymbol{B}$ is a $P$-matrix and hence all the conditions of Theorem 5.3 are satisfied. Consequently, there exists a unique forward solution. □

The following theorem provides conditions for uniqueness of Carathéodory solutions. It follows from the standard existence and uniqueness results of ordinary differential equations with Lipschitzian right-hand sides.

**Theorem 5.4** *Suppose that* $\boldsymbol{D}$ *is a* $P$*-matrix. Then, the following statements are equivalent:*

1. *There exists a Carathéodory solution of the LCS (5.58) for any initial state* $\boldsymbol{x}_0$ *and any locally integrable input* $\boldsymbol{u}$.
2. *There exists a forward solution of the LCS (5.58) for any initial state* $\boldsymbol{x}_0$ *and any locally integrable input* $\boldsymbol{u}$.

*Moreover, if* $(\boldsymbol{z}^i, \boldsymbol{x}^i, \boldsymbol{w}^i)$ $i = 1, 2$ *are solutions with the initial state* $\boldsymbol{x}_0$*, and the input* $\boldsymbol{u}$*, then* $(\boldsymbol{z}^1, \boldsymbol{x}^1, \boldsymbol{w}^1) = (\boldsymbol{z}^2, \boldsymbol{x}^2, \boldsymbol{w}^2)$.

The $P$-matrix condition of this theorem is somewhat restrictive. It turns out that passivity of the underlying linear system is sufficient in order to guarantee uniqueness of Carathéodory solutions as stated next.

**Theorem 5.5** *[161] Suppose that the system* $\Sigma(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ *is passive and the LMIs (5.57) have a positive definite solution. Then, the following statements are equivalent for a given positive real number* $T$*, an initial state* $\boldsymbol{x}_0$*, and an input* $\boldsymbol{u}$*:*

1. *There exists a Carathéodory solution of the LCS (5.58) for the initial state* $\boldsymbol{x}_0$ *and the Bohl input* $\boldsymbol{u}$.
2. *There exists a forward solution of the LCS (5.58) for the initial state* $\boldsymbol{x}_0$ *and the Bohl input* $\boldsymbol{u}$.
3. *The relations*

$$\boldsymbol{F}\boldsymbol{u}(t) \in (\mathrm{LCP} - \ker(\boldsymbol{D}))^* + \mathrm{Im}\,\boldsymbol{C}, \quad \text{for all } t \geq 0, \tag{5.60a}$$

$$\boldsymbol{C}\boldsymbol{x}_0 + \boldsymbol{F}\boldsymbol{u}(0) \in (\mathrm{LCP} - \ker(\boldsymbol{D}))^* \tag{5.60b}$$

*hold.*

*Moreover, if* $(\boldsymbol{z}^i, \boldsymbol{x}^i, \boldsymbol{w}^i)$ $i = 1, 2$ *are solutions with the initial state* $\boldsymbol{x}_0$*, and the input* $\boldsymbol{u}$*, then the relations*

1. $\boldsymbol{x}^1 - \boldsymbol{x}^2 = 0$;
2. $\boldsymbol{z}^1 - \boldsymbol{z}^2 \in \ker \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{D} + \boldsymbol{D}^{\mathrm{T}} \end{bmatrix}$;
3. $\boldsymbol{w}^1 - \boldsymbol{w}^2 \in \boldsymbol{D}\ker \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{D} + \boldsymbol{D}^{\mathrm{T}} \end{bmatrix}$

*hold.*

**Zeno phenomena** Consider the input-free LCS

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{z}(t), \tag{5.61a}$$

$$\boldsymbol{w}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{z}(t), \tag{5.61b}$$

$$0 \leq \boldsymbol{z}(t) \perp \boldsymbol{w}(t) \geq 0. \tag{5.61c}$$

Let $(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{w})$ be a Carathéodory solution of the LCS (5.61). Define the index sets

$$\alpha(t) = \{i \mid \boldsymbol{z}_i(t) > 0 = \boldsymbol{w}_i(t)\}, \tag{5.62}$$
$$\beta(t) = \{i \mid \boldsymbol{z}_i(t) = 0 = \boldsymbol{w}_i(t)\}, \tag{5.63}$$
$$\gamma(t) = \{i \mid \boldsymbol{z}_i(t) = 0 < \boldsymbol{w}_i(t)\}, \tag{5.64}$$

for $t \geq 0$. We say that a time instant $t^* > 0$ is:

- a *nonswitching time instant* with respect to the solution $(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{w})$ if there exist a positive real number $\varepsilon$ and index sets $(\alpha_*, \beta_*, \gamma_*)$ such that $(\alpha(t), \beta(t), \gamma(t)) = (\alpha_*, \beta_*, \gamma_*)$ for all $t \in (t^* - \varepsilon, t^*) \cup (t^*, t^* + \varepsilon)$;
- a *switching time instant* if it is not a nonswitching time instant.

Let $\Gamma$ be the set of all switching time instants with respect to the solution $(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{w})$. We say that the solution $(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{w})$ is

- *left-Zeno free* if the set $\Gamma$ has no left accumulation points, i.e. for each $t \geq 0$ there exists a positive real number $\varepsilon$ such that $\Gamma \cap (t, t + \varepsilon) = \emptyset$;
- *right-Zeno free* if the set $\Gamma$ has no right accumulation points, i.e. for each $t > 0$ there exists a positive real number $\varepsilon$ such that $\Gamma \cap (t - \varepsilon, t) = \emptyset$;
- *Zeno free* if it is both left- and right-Zeno free.

Four theorems that provide sufficient conditions that exclude certain types of Zeno behavior are in order. The first one rules out both left and right Zeno behavior under a restrictive condition:

> **Theorem 5.6** *[583] Suppose that $\boldsymbol{D}$ is a $P$-matrix. Then, all solutions of the LCS (5.61) are Zeno free.*

The second rules out only left-Zenoness under a less restrictive condition, namely the passivity assumption:

**Theorem 5.7** *[306] Suppose that the system $\Sigma(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ is passive and* $\mathrm{col}(\boldsymbol{B}, \boldsymbol{D} + \boldsymbol{D}^{\mathrm{T}})$ *is of full column rank. Then, all solutions of the LCS (5.61) are left-Zeno free.*

The third result rules out Zeno behavior in case the underlying system is passive and $\boldsymbol{D}$ matrix satisfies certain conditions:

**Theorem 5.8** *[149, 152] Suppose that the system $\Sigma(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ is passive and* $\mathrm{col}(\boldsymbol{B}, \boldsymbol{D} + \boldsymbol{D}^{\mathrm{T}})$ *is of full column rank. If there exists an index set $\alpha \subseteq \{1, 2, \ldots, m\}$ such that*
- *$\boldsymbol{D}_{\alpha\alpha}$ is positive definite;*
- *$\boldsymbol{D}_{\alpha\alpha^c} = \boldsymbol{O}$ and $\boldsymbol{D}_{\alpha^c\alpha} = \boldsymbol{O}$; and*
- *$\boldsymbol{D}_{\alpha^c\alpha^c}$ is skew-symmetric.*

*Then all solutions of the LCS (5.61) are Zeno free.*

The final result relaxes the passivity requirement:

**Theorem 5.9** *[162] Suppose that* $m = 1$, $\boldsymbol{D} = \boldsymbol{O}$, *and* $\boldsymbol{CB} > 0$. *Then, all solutions of the LCS (5.61) are Zeno free.*

**Stability** To study stability, we introduce a stronger version of passivity:

**Definition 5.2 (Strictly passive system)** *The system* $\Sigma(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ *is called* strictly passive *, if the matrix inequalities*

$$\boldsymbol{K} = \boldsymbol{K}^{\mathrm{T}} > 0 \text{ and } \begin{bmatrix} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{K} + \boldsymbol{K}\boldsymbol{A} + \varepsilon\boldsymbol{K} & \boldsymbol{K}\boldsymbol{B} - \boldsymbol{C}^{\mathrm{T}} \\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{K} - \boldsymbol{C} & -(\boldsymbol{D} + \boldsymbol{D}^{\mathrm{T}}) \end{bmatrix} \leq 0 \tag{5.65}$$

*have a solution* $\boldsymbol{K}$ *for some* $\varepsilon > 0$.

Lyapunov stability of linear complementarity systems is established by the following theorem under the passivity assumption:

**Theorem 5.10** *[155] Consider the LCS (5.61). Suppose that the linear system* $\Sigma(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ *is strictly passive. Then the LCS (5.61) is globally exponentially stable. In case* $\Sigma(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ *is passive only, then the system is Lyapunov stable.*

In general, obtaining necessary and sufficient conditions for stability is a hard task. Only in the planar case, one can provide such conditions as stated in the next theorem:

**Theorem 5.11** *[156] Consider the LCS (5.61) with* $m = 1$, $n = 2$, *and* $(\boldsymbol{C}, \boldsymbol{A})$ *is an observable pair. The following statements hold:*

1. *Suppose that* $\boldsymbol{D} > 0$. *The LCS (5.61) is asymptotically stable if and only if*
   (a) *neither* $\boldsymbol{A}$ *nor* $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}$ *has a real nonnegative eigenvalue, and*
   (b) *if both* $\boldsymbol{A}$ *and* $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}$ *have nonreal eigenvalues then* $\sigma_1/\omega_1 + \sigma_2/\omega_2 < 0$ *where* $\sigma_1 \pm i\omega_1$ $(\omega_1 > 0)$ *are the eigenvalues of* $\boldsymbol{A}$ *and* $\sigma_2 \pm i\omega_2$ $(\omega_2 > 0)$ *are the eigenvalues of* $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}$.
2. *Suppose that* $\boldsymbol{D} > 0$. *The LCS (5.61) has a nonconstant periodic solution if and only if both* $\boldsymbol{A}$ *and* $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}$ *have nonreal eigenvalues, and* $\sigma_1/\omega_1 + \sigma_2/\omega_2 = 0$ *where* $\sigma_1 \pm i\omega_1$ *are the eigenvalues of* $\boldsymbol{A}$ *and* $\sigma_2 \pm i\omega_2$ *are the eigenvalues of* $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}$. *Moreover, if there is one periodic solution, then all other solutions are also periodic. And,* $\pi/\omega_1 + \pi/\omega_2$ *is the period of any solution.*
3. *Suppose that* $\boldsymbol{D} = \boldsymbol{O}$ *and* $\boldsymbol{CB} > 0$. *The LCS (5.61) is asymptotically stable if and only if* $\boldsymbol{A}$ *has no real nonnegative eigenvalue and* $[\boldsymbol{I} - \boldsymbol{B}(\boldsymbol{CB})^{-1}\boldsymbol{C}]\boldsymbol{A}$ *has a real negative eigenvalue (note that one eigenvalue is already zero).*

**Controllability and stabilizability** Let $(\boldsymbol{z}^{\boldsymbol{x}_0,\boldsymbol{u}}, \boldsymbol{x}^{\boldsymbol{x}_0,\boldsymbol{u}}, \boldsymbol{w}^{\boldsymbol{x}_0,\boldsymbol{u}})$ denote the solution of the LCS (5.58) for the initial state $\boldsymbol{x}_0$ and the input $\boldsymbol{u}$. We say that the LCS (5.58) is

- *controllable* if for any pair of states $(\boldsymbol{x}_0, \boldsymbol{x}_\mathrm{f}) \in \mathbb{R}^{n+n}$ there exists a locally integrable input $\boldsymbol{u}$ such that the trajectory $\boldsymbol{x}^{\boldsymbol{x}_0,\boldsymbol{u}}$ satisfies $\boldsymbol{x}^{\boldsymbol{x}_0,\boldsymbol{u}}(T) = \boldsymbol{x}_\mathrm{f}$ for some $T > 0$;
- *stabilizable* if for any initial state $\boldsymbol{x}_0$ there exists a locally integrable input $\boldsymbol{u}$ such that $\lim_{t\uparrow\infty} \boldsymbol{x}^{\boldsymbol{x}_0,\boldsymbol{u}} = 0$.

The following theorem presents algebraic necessary and sufficient conditions for the controllability of an LCS:

**Theorem 5.12** *[150] Suppose that $\boldsymbol{D}$ is a $P$-matrix and the transfer matrix $\boldsymbol{F}+\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{E}$ is invertible as a rational matrix. Then, the LCS (5.58) is controllable if, and only if, the following two conditions hold:*
1. *The pair $(\boldsymbol{A}, [\boldsymbol{B}\ \boldsymbol{E}])$ is controllable.*
2. *The system of inequalities*

$$\boldsymbol{\eta} \geq 0, \tag{5.66a}$$

$$\begin{bmatrix}\boldsymbol{\xi}^\mathrm{T} & \boldsymbol{\eta}^\mathrm{T}\end{bmatrix} \begin{bmatrix}\boldsymbol{A}-\lambda\boldsymbol{I} & \boldsymbol{E} \\ \boldsymbol{C} & \boldsymbol{F}\end{bmatrix} = 0, \tag{5.66b}$$

$$\begin{bmatrix}\boldsymbol{\xi}^\mathrm{T} & \boldsymbol{\eta}^\mathrm{T}\end{bmatrix} \begin{bmatrix}\boldsymbol{B} \\ \boldsymbol{D}\end{bmatrix} \leq 0 \tag{5.66c}$$

*admits no solution $\lambda \in \mathbb{R}$ and $0 \neq (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^{n+m}$.*

It turns out that stabilizability can also be characterized in the same way:

**Theorem 5.13** *Suppose that $\boldsymbol{D}$ is a $P$-matrix and the transfer matrix $\boldsymbol{F}+\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{E}$ is invertible as a rational matrix. Then, the LCS (5.58) is stabilizable if, and only if, the following two conditions hold:*
1. *The pair $(\boldsymbol{A}, [\boldsymbol{B}\ \boldsymbol{E}])$ is stabilizable.*
2. *The system of inequalities*

$$\boldsymbol{\eta} \geq 0, \tag{5.67a}$$

$$\begin{bmatrix}\boldsymbol{\xi}^\mathrm{T} & \boldsymbol{\eta}^\mathrm{T}\end{bmatrix} \begin{bmatrix}\boldsymbol{A}-\lambda\boldsymbol{I} & \boldsymbol{E} \\ \boldsymbol{C} & F\end{bmatrix} = 0, \tag{5.67b}$$

$$\begin{bmatrix}\boldsymbol{\xi}^\mathrm{T} & \boldsymbol{\eta}^\mathrm{T}\end{bmatrix} \begin{bmatrix}\boldsymbol{B} \\ \boldsymbol{D}\end{bmatrix} \leq 0 \tag{5.67c}$$

*admits no solution $0 \leq \lambda \in \mathbb{R}$ and $0 \neq (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^{n+m}$.*

## 5.3 Equivalence of piecewise affine systems, mixed logical dynamical systems, and linear complementarity systems

In this section we discuss equivalences among five classes of discrete-time hybrid systems, viz. mixed logical dynamical (MLD) systems, linear complementarity (LC) systems, extended linear complementarity (ELC) systems, piecewise affine (PWA) systems, and max-min-plus-scaling (MMPS) systems. Some of the equivalences can be established under (rather mild) additional assumptions. These results are of paramount importance for transferring theoretical properties and tools from one class to another, with the consequence that for the study of a particular hybrid system that belongs to any of these classes, one can choose the most convenient hybrid modeling framework. The proofs of all the equivalence results reported in this section can be found in [305].

### 5.3.1 Summary of the five classes of hybrid models

In the previous chapters of this handbook it has already been indicated that, as tractable methods to analyze general hybrid systems are not available, several authors have focussed on special subclasses of hybrid dynamical systems for which analysis and/or control design techniques are currently being developed. Some examples of such subclasses are: linear complementarity (LC) systems, mixed logical dynamical (MLD) systems, first-order linear hybrid systems with saturation, and piecewise affine (PWA) systems. Each subclass has its own advantages over the others. For instance, stability criteria were proposed for PWA systems (Section 4.4), control and verification techniques for MLD hybrid models (Section 5.1), and conditions of existence and uniqueness of solution trajectories (well-posedness) for LC systems (Section 5.4).

In this section we will show that several of such subclasses of hybrid systems are equivalent when considered in their discrete-time formulation. Some of the equivalences are obtained under additional assumptions related to well-posedness and boundedness of input, state, output, or auxiliary variables. These results allow to transfer all the above analysis and synthesis tools to any of the equivalent subclasses of hybrid systems.

First we briefly recapitulate the five classes of hybrid systems considered in this section. The variables $\boldsymbol{u}(k) \in \mathbb{R}^m$, $\boldsymbol{x}(k) \in \mathbb{R}^n$ and $\boldsymbol{y}(k) \in \mathbb{R}^l$ denote the input, state and output, respectively, at time $k$.

**Piecewise affine (PWA) systems** PWA systems are described by

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}_i \boldsymbol{x}(k) + \boldsymbol{B}_i \boldsymbol{u}(k) + \boldsymbol{f}_i \\ \boldsymbol{y}(k) &= \boldsymbol{C}_i \boldsymbol{x}(k) + \boldsymbol{D}_i \boldsymbol{u}(k) + \boldsymbol{g}_i \end{aligned} \quad \text{for} \begin{bmatrix} \boldsymbol{x}(k) \\ \boldsymbol{u}(k) \end{bmatrix} \in \Omega_i, \tag{5.68}$$

where $\Omega_i$ are convex polyhedra (i.e. given by a finite number of linear inequalities) in the input/state space. PWA systems form the "simplest" extension of linear systems that can still model nonlinear and non-smooth processes with arbitrary accuracy and are capable of handling hybrid phenomena.

**Mixed logical dynamical (MLD) systems** As introduced in Section 5.1.3, an integration of logic, dynamics, and constraints results in the description

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}_1\boldsymbol{u}(k) + \boldsymbol{B}_2\boldsymbol{\delta}(k) + \boldsymbol{B}_3\boldsymbol{z}(k), \tag{5.69}$$

$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}_1\boldsymbol{u}(k) + \boldsymbol{D}_2\boldsymbol{\delta}(k) + \boldsymbol{D}_3\boldsymbol{z}(k), \tag{5.70}$$

$$\boldsymbol{E}_2\boldsymbol{\delta}(k) + \boldsymbol{E}_3\boldsymbol{z}(k) \leqslant \boldsymbol{E}_1\boldsymbol{u}(k) + \boldsymbol{E}_4\boldsymbol{x}(k) + \boldsymbol{E}_5, \tag{5.71}$$

where $\boldsymbol{x}(k) = [\boldsymbol{x}_{\mathrm{c}}^{\mathrm{T}}(k)\ \boldsymbol{x}_{\mathrm{b}}^{\mathrm{T}}(k)]^{\mathrm{T}}$ with $\boldsymbol{x}_{\mathrm{c}}(k) \in \mathbb{R}^{n_{\mathrm{c}}}$ and $\boldsymbol{x}_{\mathrm{b}}(k) \in \{0,1\}^{n_{\mathrm{b}}}$. $\boldsymbol{z}(k) \in \mathbb{R}^{r_{\mathrm{c}}}$ and $\boldsymbol{\delta}(k) \in \{0,1\}^{r_{\mathrm{b}}}$ are auxiliary variables. The inequalities (5.71) have to be interpreted componentwise.

*Remark 5.1* It is assumed that for all $\boldsymbol{x}(k)$ with $\boldsymbol{x}_{\mathrm{b}}(k) \in \{0,1\}^{n_{\mathrm{b}}}$, all $\boldsymbol{u}(k)$ with $\boldsymbol{u}_{\mathrm{b}}(k) \in \{0,1\}^{m_{\mathrm{b}}}$, all $\boldsymbol{z}(k) \in \mathbb{R}^{r_{\mathrm{c}}}$ and all $\boldsymbol{\delta}(k) \in \{0,1\}^{r_{\mathrm{b}}}$ satisfying (5.71) it holds that $\boldsymbol{x}(k+1)$ and $\boldsymbol{y}(k)$ determined from (5.69) – (5.70) are such that $\boldsymbol{x}_{\mathrm{b}}(k+1) \in \{0,1\}^{n_{\mathrm{b}}}$ and $\boldsymbol{y}_{\mathrm{b}}(k) \in \{0,1\}^{l_{\mathrm{b}}}$. This is without loss of generality, as we can take binary components of states and outputs (if any) to be auxiliary variables as well (see the proof of Proposition 1 of [65]). Indeed, if, for instance, $\boldsymbol{y}_{\mathrm{b}}(k) \in \{0,1\}^{l_{\mathrm{b}}}$ is not directly implied by the (in)equalities, we introduce an additional binary vector variable $\boldsymbol{\delta}_y(k) \in \{0,1\}^{l_{\mathrm{b}}}$ and the inequalities

$$[\boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}_1\boldsymbol{u}(k) + \boldsymbol{D}_2\boldsymbol{\delta}(k) + \boldsymbol{D}_3\boldsymbol{z}(k)]_{\mathrm{b}} - \boldsymbol{\delta}_y(k) \leq 0,$$
$$[-\boldsymbol{C}\boldsymbol{x}(k) - \boldsymbol{D}_1\boldsymbol{u}(k) - \boldsymbol{D}_2\boldsymbol{\delta}(k) - \boldsymbol{D}_3\boldsymbol{z}(k)]_{\mathrm{b}} + \boldsymbol{\delta}_y(k) \leq 0,$$

which sets $\boldsymbol{\delta}_y(k)$ equal to $\boldsymbol{y}_{\mathrm{b}}(k)$. The notation $[\ ]_{\mathrm{b}}$ is used to select the rows of the expression (5.70) that correspond to the binary part of $\boldsymbol{y}(k)$. Hence, $\boldsymbol{y}_{\mathrm{b}}(k) = \boldsymbol{\delta}_y(k) \in \{0,1\}^{l_{\mathrm{b}}}$. Similarly, we can deal with $\boldsymbol{u}_{\mathrm{b}}(k)$ and $\boldsymbol{x}_{\mathrm{b}}(k+1)$.

**Linear complementarity (LC) systems** In discrete time these systems are given by the equations

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}_1\boldsymbol{u}(k) + \boldsymbol{B}_2\boldsymbol{w}(k), \tag{5.72}$$

$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}_1\boldsymbol{u}(k) + \boldsymbol{D}_2\boldsymbol{w}(k), \tag{5.73}$$

$$\boldsymbol{v}(k) = \boldsymbol{E}_1\boldsymbol{x}(k) + \boldsymbol{E}_2\boldsymbol{u}(k) + \boldsymbol{E}_3\boldsymbol{w}(k) + \boldsymbol{g}_4, \tag{5.74}$$

$$\boldsymbol{0} \leq \boldsymbol{v}(k) \perp \boldsymbol{w}(k) \geq \boldsymbol{0}, \tag{5.75}$$

with $\boldsymbol{v}(k), \boldsymbol{w}(k) \in \mathbb{R}^s$ and where $\perp$ denotes the orthogonality of vectors (i.e. $\boldsymbol{v}(k) \perp \boldsymbol{w}(k)$ means that $\boldsymbol{v}^{\mathrm{T}}(k)\boldsymbol{w}(k) = 0$). We call $\boldsymbol{v}(k)$ and $\boldsymbol{w}(k)$ the complementarity variables.

**Extended linear complementarity (ELC) systems** Several types of hybrid systems can be modeled as extended linear complementarity (ELC) systems:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}_1\boldsymbol{u}(k) + \boldsymbol{B}_2\boldsymbol{d}(k), \tag{5.76}$$

$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}_1\boldsymbol{u}(k) + \boldsymbol{D}_2\boldsymbol{d}(k), \tag{5.77}$$

$$\boldsymbol{E}_1\boldsymbol{x}(k) + \boldsymbol{E}_2\boldsymbol{u}(k) + \boldsymbol{E}_3\boldsymbol{d}(k) \leqslant \boldsymbol{g}_4, \tag{5.78}$$

$$\sum_{i=1}^{p} \prod_{j\in\phi_i} \big(\boldsymbol{g}_4 - \boldsymbol{E}_1\boldsymbol{x}(k) - \boldsymbol{E}_2\boldsymbol{u}(k) - \boldsymbol{E}_3\boldsymbol{d}(k)\big)_j = 0, \tag{5.79}$$

where $\boldsymbol{d}(k) \in \mathbb{R}^r$ is an auxiliary variable. Condition (5.79) is equivalent to

$$\prod_{j \in \phi_i} \left(\boldsymbol{g}_4 - \boldsymbol{E}_1 \boldsymbol{x}(k) - \boldsymbol{E}_2 \boldsymbol{u}(k) - \boldsymbol{E}_3 \boldsymbol{d}(k)\right)_j = 0 \quad \text{for each } i \in \{1, 2, \ldots, p\}, \tag{5.80}$$

due to the inequality conditions (5.78). This implies that (5.78)–(5.79) can be considered as a system of linear inequalities (i.e. (5.78)), where there are $p$ groups of linear inequalities (one group for each index set $\phi_i$) such that in each group at least one inequality should hold with equality.

**Max-min-plus-scaling (MMPS) systems** In [578] a class of discrete-event systems has been introduced that can be modeled using the operations maximization, minimization, addition, and scalar multiplication. Expressions that are built using these operations are called max-min-plus-scaling (MMPS) expressions.

**Definition 5.3 (Max-min-plus-scaling expression)** *A max-min-plus-scaling expression $f$ of the variables $x_1, \ldots, x_n$ is defined by the grammar*

$$f := x_i | \alpha | \max(f_k, f_l) | \min(f_k, f_l) | f_k + f_l | \beta f_k, \tag{5.81}$$

*with $i \in \{1, 2, \ldots, n\}$, $\alpha$, $\beta \in \mathbb{R}$, and where $f_k$, $f_l$ are again MMPS expressions. (The symbol $|$ stands for OR and the definition is recursive.)*

An MMPS expression is, for example

$$5x_1 - 3x_2 + 7 + \max(\min(2x_1, -8x_2), x_2 - 3x_3).$$

Consider now systems that can be described by

$$\boldsymbol{x}(k+1) = \boldsymbol{\mathcal{M}}_x(\boldsymbol{x}(k), \boldsymbol{u}(k), \boldsymbol{d}(k)), \tag{5.82}$$

$$\boldsymbol{y}(k) = \boldsymbol{\mathcal{M}}_y(\boldsymbol{x}(k), \boldsymbol{u}(k), \boldsymbol{d}(k)), \tag{5.83}$$

$$\boldsymbol{\mathcal{M}}_{\mathrm{c}}(\boldsymbol{x}(k), \boldsymbol{u}(k), \boldsymbol{d}(k)) \leqslant \boldsymbol{c}, \tag{5.84}$$

where $\boldsymbol{\mathcal{M}}_x$, $\boldsymbol{\mathcal{M}}_y$, and $\boldsymbol{\mathcal{M}}_{\mathrm{c}}$ are MMPS expressions in terms of the components of $\boldsymbol{x}(k)$, the input $\boldsymbol{u}(k)$, and the auxiliary variables $\boldsymbol{d}(k)$, which are all real-valued. Such systems will be called MMPS systems.

### 5.3.2 Systems equivalence

In this section we prove that MLD, LC, ELC, PWA and MMPS systems are equivalent (although in some cases additional assumptions are required). The relations between the models are depicted in Fig. 5.2.

**MLD and LC systems**

**Proposition 5.2** *Every MLD system can be written as an LC system.*

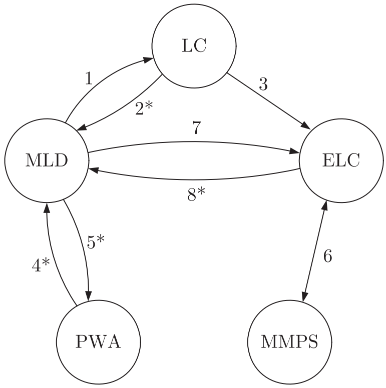


Fig. 5.2 Graphical representation of the links among the classes of hybrid systems considered in this paper. An arrow going from class A to class B means that A is a subset of B. The number next to each arrow corresponds to the proposition that states this relation. Moreover, arrows with a star ($\star$) require conditions to establish the indicated inclusion.

As mentioned before, all proofs of the equivalence results presented here can be found in [305].

**Proposition 5.3** *Every LC system can be written as an MLD system, provided that the variables* $\boldsymbol{w}(k)$ *and* $\boldsymbol{v}(k)$ *are (componentwise) bounded.*

Proposition 5.3 assumes that upper bounds on $\boldsymbol{w}$ and $\boldsymbol{v}$ are known. This hypothesis is not restrictive in practice, as these quantities are related to continuous inputs and states of the system, which are usually bounded for physical reasons.

### LC and ELC systems

**Proposition 5.4** *Every LC system can be written as an ELC system.*

### PWA and MLD systems

A PWA system of the form (5.68) is called well-posed, if (5.68) is uniquely solvable in $\boldsymbol{x}(k+1)$ and $\boldsymbol{y}(k)$, once $\boldsymbol{x}(k)$ and $\boldsymbol{u}(k)$ are specified. The following proposition has been stated in [65] and is an easy extension of the corresponding result in [62] for piecewise linear (PWL) systems (i.e. PWA systems with $\boldsymbol{f}_i = \boldsymbol{g}_i = \mathbf{0}$):

**Proposition 5.5** *Every well-posed PWA system can be rewritten as an MLD system assuming that the set of feasible states and inputs is bounded.*

*Remark 5.2* As MLD models only allow for nonstrict inequalities in (5.71), in rewriting a discontinuous PWA system as an MLD model strict inequalities like $x(k) < 0$ (where assume here for the sake of simplicity of the exposition and without loss of generality that $x(k)$ is a scalar) must be approximated by $x(k) \leq -\varepsilon$ for some $\varepsilon > 0$ (typically the machine precision), with the assumption that $-\varepsilon < x(k) < 0$ cannot occur due to the finite number of bits used for representing real numbers (no problem exists when the PWA system is continuous, where the strict inequality can be equivalently rewritten as nonstrict, or $\varepsilon = 0$). See [62] for more details and 5.8 for an example. From a strictly theoretical point of view, the inclusion stated in Proposition 5.5 is therefore not exact for discontinuous PWA systems, and the same clearly holds for an LC, ELC or MMPS reformulation of a discontinuous PWA system when the route via MLD is taken. One way to circumvent such an inexactness is to allow part of the inequalities in (5.71) to be strict. On the other hand, from a numerical point of view this issue is not relevant. The equivalence of LC and MLD systems (cf. Proposition 5.2 and 5.3) implies that all continuous PWA can be exactly written as LC systems as well. A similar result for continuous PWA systems can be derived from [217].

The MLD system (5.69) is called completely well-posed, if $\boldsymbol{x}(k+1)$, $\boldsymbol{y}(k)$, $\boldsymbol{\delta}(k)$ and $\boldsymbol{z}(k)$ are uniquely defined in their domain, once $\boldsymbol{x}(k)$ and $\boldsymbol{u}(k)$ are assigned [62]. The reverse statement of Proposition 5.5 has been established in [65] under the condition that the MLD system is completely well-posed:

**Proposition 5.6** *A completely well-posed MLD system can be rewritten as a PWA system.*

Constructive procedures for converting MLD systems into PWA form were provided in [55, 56] (and implemented in the Hybrid Toolbox [57], see Chapter 10) and in [265]. Equivalences between PWA systems and other hybrid model classes have been also investigated in [136], where the authors examine a relationship existing among linear hybrid automata (LHA) and piecewise affine (PWA) systems, showing in a constructive way that a LHA can be equivalently represented as a continuous-time PWA system.

### MMPS and ELC systems

**Proposition 5.7** *The classes of MMPS and ELC systems coincide.*

### MLD and ELC systems

**Proposition 5.8** *Every MLD system can be rewritten as an ELC system.*

*Remark 5.3* Note that the condition $\delta_i(k) \in \{0, 1\}$ is also equivalent to the MMPS constraint $\max(-\delta_i(k), \delta_i(k) - 1) = 0$ or $\min(\delta_i(k), 1 - \delta_i(k)) = 0$.

**Proposition 5.9** *Every ELC system can be written as an MLD system, provided that the quantity $\boldsymbol{g}_4 - \boldsymbol{E}_1\boldsymbol{x}(k) - \boldsymbol{E}_2\boldsymbol{u}(k) - \boldsymbol{E}_3\boldsymbol{d}(k)$ is (componentwise) bounded.*

Note that (just as for Proposition 5.3) the boundedness hypothesis in Proposition 5.9 is not restrictive in practice, since the inputs and states of the system are usually bounded for physical reasons.

**Example 5.8** *Equivalent hybrid systems*

To demonstrate the equivalences proven above, we consider the example [62]

$$x(k+1) = \begin{cases} 0.8x(k) + u(k), & \text{if } x(k) \geq 0, \\ -0.8x(k) + u(k), & \text{if } x(k) < 0, \end{cases} \tag{5.85}$$

with $m \leq x(k) \leq M$. In [62] it is shown that (5.85) can be written as

$$\begin{aligned} &x(k+1) = -0.8x(k) + u(k) + 1.6z(k), && \\ &-m\delta(k) \leq x(k) - m, && x(k) \leq (M+\varepsilon)\delta(k) - \varepsilon, \\ &z(k) \leq M\delta(k), && z(k) \geq m\delta(k), \\ &z(k) \leq x(k) - m(1-\delta(k)), && z(k) \geq x(k) - M(1-\delta(k)), \end{aligned} \tag{5.86}$$

and the condition $\delta(k) \in \{0,1\}$. Note that the strict inequality $x(k) < 0$ has been replaced by $x(k) \leq -\varepsilon$, where $\varepsilon > 0$ is a small number (typically the machine precision). In view of Remark 5.2 observe that $\varepsilon = 0$ results in a mathematically exact MLD model. In this case the model is well-posed, but not completely well-posed as $x(k) = 0$ allows both $\delta(k) = 0$ and $\delta(k) = 1$. (An MLD model is called well-posed, if $\boldsymbol{x}(k+1)$ and $\boldsymbol{y}(k)$ are uniquely determined, once $\boldsymbol{x}(k)$ and $\boldsymbol{u}(k)$ are given. Note that there are no requirements on $\boldsymbol{\delta}(k)$ and $\boldsymbol{z}(k)$.)

One can verify that (5.85) can be rewritten as the MMPS model

$$x(k+1) = -0.8x(k) + 1.6\max(0, x(k)) + u(k), \tag{5.87}$$

as the LC formulation

$$x(k+1) = -0.8x(k) + u(k) + 1.6z(k), \tag{5.88}$$

$$0 \leq w(k) = -x(k) + z(k) \perp z(k) \geq 0, \tag{5.89}$$

and as the ELC representation

$$\begin{aligned} &x(k+1) = -0.8x(k) + u(k) + 1.6d(k), \\ &-d(k) \leq 0, \\ &x(k) - d(k) \leq 0, \\ &0 = \big(x(k) - d(k)\big)\big(-d(k)\big). \end{aligned}$$

While the MLD representation (5.86) requires bounds on $x(k)$, $u(k)$ to be specified (although such bounds can be arbitrarily large), the PWA, MMPS, LC, and ELC expressions do not require such a specification.

Note that we only need one max-operator in (5.87) and one complementarity pair in (5.88)–(5.89). If we would transform the MLD system (5.86) into, e.g., the LC model as indicated by the equivalence proof, this would require nine complementarity pairs. Hence, it is clear that the proofs only show the conceptual equivalence, but do not result in the most compact models. □

**Outlook** In this section we have discussed the equivalence of five classes of discrete-time hybrid systems: MLD, LC, ELC, PWA, and MMPS systems. For some of the transformations additional conditions like boundedness of the state and input variables or well-posedness had to be made. These results allow one to transfer properties and tools from one class to another. So for the study of a particular hybrid system that belongs to any of these classes, one can choose the most convenient modeling framework.

In the continuous-time framework, which is the natural habitat for most of the applications for LC systems, such broad equivalence relations are out of the question. There are relations though of LC systems to other specific classes of nonsmooth systems such as specific differential inclusions based on the normal cones of convex analysis and so-called projected dynamical systems. The reader may consult [124, 303] for these relationships.

## 5.4 Solution concepts and well-posedness

This section considers the fundamental system-theoretic property of well-posedness for hybrid dynamical systems. We intend to provide an overview on the available results on existence and uniqueness of solutions for given initial conditions in the context of various description formats for hybrid systems and their corresponding solution concepts.

### 5.4.1 Problem statement

On an abstract level, scientific modeling may be defined as the process of finding common descriptions for groups of observed phenomena. Often, several description forms are possible.

**Example 5.9** *Flying ball*

To take an example from not very recent technology, suppose we want to describe the flight of iron balls fired from a cannon. One description can be obtained by noting that such balls approximately follow parabolas, which may be parametrized in terms of firing angle, cannon ball weight, and amount of gun powder used. Another possible description characterizes the trajectories of the cannon balls as solutions of certain differential equations. The latter description may be viewed as being fairly *indirect*; after all it represents trajectories only as solutions to some problem, rather than expressing directly what the trajectories are, as the first description form does. On the other hand, the description by means of differential equations is applicable to a wider range of phenomena, and one may, therefore, feel that it represents a deeper insight. Besides, interconnection (composition) becomes much easier since it is in general much easier to write down equations than to determine the solutions of the interconnected system. □

There are many examples in science where, as above, an implicit description (that is, a description in terms of a mathematical problem to be solved) is useful and possibly more powerful than explicit descriptions. Whenever an implicit description is used, however, one has to show that the description is a "good" one in the sense that the stated problem has a well-defined solution. This is essentially the issue of well-posedness.

Many different description formats have been proposed in recent years for hybrid systems. Some proposed forms are quite direct, others lead to rather indirect descriptions. The direct forms have advantages from the point of view of *analysis*, but the indirect forms are often preferable from the perspective of *modeling* (specification); examples will be seen below. The more indirect a description form is, the harder it becomes to show that solutions are well-defined. This section intends to provide a survey on the available results on existence and uniqueness of solutions for given initial conditions in the context of the description formats for hybrid systems as considered in this handbook.

### 5.4.2 Model classes

This section summarizes the models of hybrid systems that will be investigated later with respect to the existence and well-posedness of a solution.

**Hybrid automata** Hybrid automata were already defined in Section 1.2 and Section 2.1 and we refer to the formal definition of this model class based on the 8-tuple $H = (\mathcal{Q}, \mathcal{X}, \boldsymbol{f}, \mathit{Init}, \mathit{Inv}, \mathcal{E}, \mathcal{G}, \mathcal{R})$ given there.

**Differential equations with discontinuous right-hand sides** During the past decades, extensive studies have been made of *differential equations with discontinuous right-hand sides* (cf. in particular [237] and [639, 640]). For a typical example, consider the following specification:

$$\dot{x} = \begin{cases} f_1(x), & \text{when } h(x) > 0, \\ f_2(x), & \text{when } h(x) < 0, \end{cases} \tag{5.90}$$

where $h$ is a real-valued function. A system of this form can be looked at either as a discontinuous dynamical system or as a hybrid system of a particular form. The specification above is obviously incomplete since no statement is made about the situation in which $h(x) = 0$. One way to arrive at a solution concept is to adopt a suitable *relaxation*. Specifically, Filippov [237] proposed rewriting the equations in a *convex* relaxation (5.90) as

$$\dot{x} \in \mathcal{F}(x), \tag{5.91}$$

where the set-valued function $\mathcal{F}(x)$ is defined by

$$F(x) = \begin{cases} \{f_1(x)\}, & \text{when } h(x) > 0, \\ \{f_2(x)\}, & \text{when } h(x) < 0, \\ \{y \mid \exists a \in [0,1] \text{ s.t. } y = af_1(x) + (1-a)f_2(x)\}, & \text{when } h(x) = 0, \end{cases} \tag{5.92}$$

where it is assumed (for simplicity) that $f_1$ and $f_2$ are given as continuous functions defined on $\{x \mid h(x) \geq 0\}$ and $\{x \mid h(x) \leq 0\}$, respectively.

The discontinuous dynamical system has now been reformulated as a *differential inclusion*, and so solution concepts and well-posedness results can be applied that have been developed for systems of this type [26]. Other methods to obtain differential inclusions are proposed by Utkin ("control equivalent definition") and Aizerman and Pyatnitskii (Sect. 5.4.4). In case the vector fields $\boldsymbol{f}_i(\boldsymbol{x})$ are linear (i.e. of the form $\boldsymbol{A}_i\boldsymbol{x}$ for some matrix $\boldsymbol{A}_i$) and the switching surface is given by a linear function $h$, then the system (5.90) is called a *piecewise linear system*. These systems will receive special attention below.

**Hybrid inclusions** A conceptually simple model, but still powerful to model many classes of interest, was developed recently in [133, 271, 273]. It extends the differential inclusion (5.91) by restricting its "flow region" to a set $\mathcal{C}$ and including resets of the state variable in the "jump set" $\mathcal{D}$. As such, the model consists of the data of two subsets $\mathcal{C}$ and $\mathcal{D}$ of $\mathbb{R}^n$, and two set-valued mappings $\mathcal{F}$ and $\mathcal{G}$, from $\mathcal{C}$, respectively from $\mathcal{D}$, to $\mathbb{R}^n$. The hybrid system is written as

$$\dot{\boldsymbol{x}} \in \mathcal{F}(\boldsymbol{x}) \;\text{ if } \boldsymbol{x} \in \mathcal{C}, \tag{5.93a}$$

$$\boldsymbol{x}^+ \in \mathcal{G}(\boldsymbol{x}) \;\text{ if } \boldsymbol{x} \in \mathcal{D}, \tag{5.93b}$$

The state variable is now given by $\boldsymbol{x}(t) \in \mathbb{R}^n$ for time $t \in \mathbb{R}$, but some parts of the state vector are also allowed to take only integer values.

**Complementarity systems** Complementarity systems have been discussed already in detail in Section 5.2. The reader is referred to that section for an exposition on this class of hybrid systems.

### 5.4.3 Solution concepts

A description format for a class of dynamical systems only specifies a collection of trajectories if one provides a notion of solution. Actually the term "solution" already more or less suggests an implicit description format; in computer science terms, one may also say that a definition should be given of what is understood by a *run* (or an *execution*) of a system description. Formally speaking, description formats are a matter of syntax: they specify what is a well-formed expression. The notion of solution provides semantics: to each well-formed expression it associates a collection of functions of time. In the presentation of description formats above, the syntactic and semantic aspects have not been strictly separated, for reasons of readability. Here we review in a more formal way solution concepts for several of the description formats that were introduced.

**Solution concepts for hybrid automata** We will use the (autonomous) hybrid automata formulation as in Section 1.2 and Section 2.1 based on the 8-tuple $H = (\mathcal{Q}, \mathcal{X}, \boldsymbol{f}, Init, Inv, \mathcal{E}, \mathcal{G}, \mathcal{R})$. To formalize the solution concept based on this model syntax, we will use the following definitions.

**Definition 5.4 (Hybrid time trajectory)** [427] *A hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$ is a finite ($N < \infty$) or infinite ($N = \infty$) sequence of intervals of the real line, such that:*

- $I_i = [\tau_i, \tau_i']$ *with* $\tau_i \le \tau_i' = \tau_{i+1}$ *for* $0 \le i < N$*;*
- *if* $N < \infty$*, either* $I_N = [\tau_N, \tau_N']$ *or* $I_N = [\tau_N, \tau_N']$ *with* $\tau_N \le \tau_N' \le \infty$*.*

A hybrid time trajectory does not allow left accumulation points . Indeed, the event times set $\mathcal{E} := \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and the corresponding sequence of intervals cannot be rewritten in terms of a hybrid time trajectory. Hence, the above definition excludes implicitly specific Zeno behavior and that this concept has a "preferred direction of time." This is caused by the fact that it assumes that the set of event times is well-ordered by the usual order of the reals, but not necessarily by the reverse order; in other words, event times may accumulate to the right, but not to the left. (An ordered set $\mathcal{S}$ is said to be well-ordered if each nonempty subset of $\mathcal{S}$ has a least element.) This lack of symmetry with respect to time can be removed by allowing the set of event times $\mathcal{E}$ to be of a more general type. Similar asymmetries in time are also the case for the solutions of hybrid inclusions and the forward solutions of complementarity systems as discussed below. Interestingly, Filippov solutions for discontinuous dynamical systems do have a more symmetric notion of time, which guarantees that time-reversed solutions remain to be solutions of the time-reversed system. This property is lost for the executions of hybrid automata, solutions to hybrid inclusions and forward solutions to complementarity systems (see also [532] for a further discussion).

We say that the hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$ is a prefix of $\tau' = \{J_i\}_{i=0}^M$ and write $\tau \le \tau'$, if they are identical or $\tau$ is finite, $M \ge N$, $I_i = J_i$ for $i = 0, 1, \ldots, N-1$, and $I_N \subseteq J_N$. In case $\tau$ is a prefix of $\tau'$ and they are not identical, $\tau$ is a *strict* prefix of $\tau'$.

**Definition 5.5 (Execution)** *An execution* $\chi$ *of a hybrid automaton is a collection* $\chi = (\tau, \lambda, \xi)$ *with:*

- $\tau = \{I_i\}_{i=0}^N$ *a hybrid time trajectory;*
- $\lambda = \{\lambda_i\}_{i=0}^N$ *with* $\lambda_i : I_i \to \mathcal{Q}$*; and*
- $\xi = \{\xi_i\}_{i=0}^N$ *with* $\xi_i : I_i \to \mathcal{X}$

*satisfying*

- *initial condition* $(\lambda(\tau_0), x(\tau_0)) \in$ *Init;*
- *continuous evolution for all* $i$*:*
  - $\lambda_i$ *is constant, i.e.,* $\lambda_i(t) = \lambda_i(\tau_i)$ *for all* $t \in I_i$*;*
  - $\xi_i$ *is the solution to the differential equation* $\dot{\xi} = f(\lambda_i(t), \xi(t))$ *on the interval* $I_i$ *with initial condition* $\xi_i(\tau_i)$ *at* $\tau_i$*;*
  - *for all* $t \in [\tau_i, \tau_i')$ *it holds that* $\xi_i(t) \in Inv(\lambda_i(t))$*;*
- *discrete evolution for all* $i$*,* $e = (\lambda_i(\tau_i'), \lambda_{i+1}(\tau_{i+1})) \in \mathcal{E}$*,* $\xi(\tau_i') \in \mathcal{G}(e)$ *and* $(\xi_i(\tau_i'), \xi_{i+1}(\tau_{i+1})) \in \mathcal{R}(e)$*.*

**Solution concepts for differential equations with discontinuous right-hand side**
As we have seen above, some differential equations with discontinuous right-hand side can be considered from the perspective of differential inclusions. The standard solution concept for differential inclusions is the following. A vector function $\boldsymbol{x}(t)$ defined on an interval $[a, b]$ is said to be a *solution* of the differential inclusion

$\dot{\boldsymbol{x}} \in \mathcal{F}(\boldsymbol{x})$, where $\mathcal{F}(\cdot)$ is a set-valued function, if $\boldsymbol{x}(\cdot)$ is absolutely continuous and satisfies $\dot{\boldsymbol{x}}(t) \in \mathcal{F}(\boldsymbol{x}(t))$ for almost all $t \in [a, b]$. The requirement of absolute continuity guarantees the existence of the derivative almost everywhere. One may note that the solution concept for differential inclusions does not have a preferred direction of time, as opposed to the notion of an execution for hybrid automata.

**Solution concepts for hybrid inclusions** For the hybrid inclusions (5.93) a solution concept (cf. [133, 271, 273]) is used that shows similarities with the one adopted for the hybrid automata. It is based upon the notion of a hybrid time domain, which is tightly connected to hybrid time trajectory as in Definition 5.4, because the hybrid time trajectory includes the "event counter $j$" into the hybrid time domain.

**Definition 5.6 (Hybrid time domain)** *A compact hybrid time domain is a set $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ given by :*

$$\mathcal{D} = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\},$$

*where $J \in \mathbb{N}$ and $0 = t_0 \leq t_1 \cdots \leq t_J$. A hybrid time domain is a set $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that, for each $(T, J) \in \mathcal{D}$, $\mathcal{D} \cap ([0, T] \times \{0, \ldots, J\})$ is a compact hybrid time domain.*

Also the hybrid time domains have a "preferred direction of time" as left accumulations of the reset times $\{t_j\}$ are not allowed.

**Definition 5.7 (Hybrid trajectory)** *A hybrid trajectory is a pair (dom $\boldsymbol{x}$, $\boldsymbol{x}$) consisting of hybrid time domain dom $\boldsymbol{x}$ and a function $\boldsymbol{x}$ defined on dom $\boldsymbol{x}$ that is locally absolutely continuous in $t$ on (dom $\boldsymbol{x}$) $\cap$ $(\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}$.*

Now we are ready to formally introduce a solution to (5.93).

**Definition 5.8 (Hybrid arc)** *A hybrid trajectory $\boldsymbol{x}$ : dom $\boldsymbol{x} \rightarrow \mathbb{R}^n$ is a solution sometimes called a hybrid arc to (5.93) if:*

1. *for all $j \in \mathbb{N}$ and for almost all $t \in I_j :=$ dom $\boldsymbol{x} \cap (\mathbb{R}_{\geq 0} \times \{j\})$, we have $\boldsymbol{x}(t, j) \in \mathcal{C}$ and $\dot{\boldsymbol{x}}(t, j) \in \mathcal{F}(\boldsymbol{x}(t, j))$;*
2. *for all $(t, j) \in$ dom $\boldsymbol{x}$ such that $(t, j+1) \in$ dom $\boldsymbol{x}$, we have $\boldsymbol{x}(t, j) \in \mathcal{D}$ and $\boldsymbol{x}(t, j+1) \in \mathcal{G}(\boldsymbol{x}(t, j))$.*

**Solution concept for complementarity systems** Section 5.2 introduced the concepts of Carathéodory and forward solutions for complementarity systems. These two notions are only valid for absolute continuous solutions implying that the $\boldsymbol{x}$-part of the solutions cannot jump across events (mode switches). For (linear) complementarity systems of a higher index such as mechanical systems with unilateral constraints that induce impacts, this requirement is too strong and one has to add jump rules that connect continuous states before and after an event has taken place. Under suitable conditions (specifically, in the case of linear complementarity systems and in the case of Hamiltonian complementarity systems), a general jump rule may be given [302, 304, 572].

### 5.4.4 Well-posedness notions

In the context of systems of differential equations, the term well-posedness roughly means that there is a nice relation between trajectories and initial conditions (or, more generally, boundary conditions). There are various ways in which this idea can be made more precise, so the meaning of the term may in fact be adapted to the particular problem class at hand. Typically it is required that solutions exist and are unique for any given initial condition. Both for the existence and for the uniqueness statement, one has to specify a function class in which solutions are considered. The function class used for existence may be the same as the one used for uniqueness, or they may be different; for instance, one might prove that solutions exist in some function class and that uniqueness holds in a larger function class. In the latter situation one is able to show specific properties (the ones satisfied by the smaller function class) of solution trajectories in the larger class. In case one is dealing with a system description that includes equality and/or inequality constraints, it may be reasonable to limit the set of initial conditions to a suitably chosen set of "feasible" or "consistent" initial conditions.

If solutions exist and are unique, a given system description defines a mapping from the set of initial conditions to trajectory set. In the theory of smooth dynamical systems, it is usually taken as part of the definition of well-posedness that this mapping is continuous with respect to suitably chosen topologies. In the case of nonsmooth and hybrid dynamical systems, it frequently happens that there are certain boundaries in the continuous state space separating regions of initial conditions that generate widely different trajectories. Therefore, continuous dependence of solutions on initial conditions (at least in the sense of the topologies that are commonly used for smooth dynamical systems) may be a requirement too strong for hybrid systems. See, for instance, the mechanical example in [304] consisting of two carts connected by a hook and a spring, where the motion of the first cart is constrained by a block. This simple example illustrates the discontinuous dependence on initial conditions nicely.

One may also distinguish between various notions of well-posedness on the basis of the time interval that is involved. For instance, in the context of hybrid automata, one may say that a given automaton is *nonblocking* [427] if for each initial condition either at least one transition is enabled or an a smooth evolution according to the dynamics of one of the modes is possible on an interval of positive length. If the continuation is unique (the automaton is *deterministic* [427]), one may then say that the automaton is *initially well-posed*. This definition allows a situation in which a transition from location 1 to location 2 is immediately followed by a transition back to location 1 and so on in an infinite loop, so that $\tau_i' = \tau_i$ for all $i$ in the hybrid time trajectory corresponding to this execution indicating that this solution does not make progress in the continuous time direction $t$ (live-lock). A stronger notion is obtained by requiring that a solution exists at least on an interval $[0, \varepsilon)$ with $\varepsilon > 0$; system descriptions for which such solutions exist and are unique are called *locally well-posed*. In computer science terminology, such systems "allow time to progress." Finally, if solutions exist and are unique on the whole half-line $[0, \infty)$, then one speaks of *global well-posedness*. Local and global well-posedness can be seen to be asymmetric in their consideration of time in the sense that it considers "continuous" time $t$ to be dominant over the "discrete time" $j$ (in the terminology of hybrid time domains). For "physical" hybrid systems this asymmetry is useful as we are interested in the actual progress of real time $t$ and less interested in the number of events. Initial well-posedness is from this point of view more symmetric.

**Well-posedness of hybrid automata** Necessary and sufficient conditions for well-posedness of hybrid automata have been stated in [427]. Basically these conditions mean that transitions with non-trivial reset relations are enabled whenever continuous evolution is impossible (this property is called *nonblocking*) and that discrete transitions must be forced by the continuous flow exiting the invariant set, no two discrete transitions can be enabled simultaneously, and no point $\boldsymbol{x}$ can be mapped onto two different points $\boldsymbol{x}' \neq \boldsymbol{x}''$ by the reset relation $R(q, q')$ - this property is called *determinism*. We will formally state the results of [427] after introducing some necessary concepts and definitions.

An execution $\chi = (\tau, \lambda, \xi)$ as defined in Definition 5.5 is called *finite*, if $\tau$ is a finite sequence ending with a closed interval; *infinite*, if $\tau$ is an infinite sequence or if $\sum_i (\tau_i' - \tau_i) = \infty$; and *maximal* if it is not a strict prefix of any other execution of the hybrid automaton. We denote the set of all maximal and infinite executions of the automaton with initial state $(q_0, \boldsymbol{x}_0) \in$ *Init* by $\mathcal{H}^M_{(q_0, \boldsymbol{x}_0)}$ and $\mathcal{H}^\infty_{(q_0, \boldsymbol{x}_0)}$, respectively.

**Definition 5.9 (Nonblocking automaton)** *A hybrid automaton is called* nonblocking *if* $\mathcal{H}^\infty_{(q_0, \boldsymbol{x}_0)}$ *is nonempty for all* $(q_0, \boldsymbol{x}_0) \in$ *Init. It is called* deterministic *if* $\mathcal{H}^M_{(q_0, \boldsymbol{x}_0)}$ *contains at most one element for all* $(q_0, \boldsymbol{x}_0) \in$ *Init.*

These well-posedness concepts are similar to what we called *initial* well-posedness as they do not say anything about live-lock or the continuation beyond accumulation points of event times.

To simplify the characterization of nonblocking and deterministic automata, the following assumption has been introduced in [427]:

**Assumption 5.1** *The vector field* $\boldsymbol{f}(q, \cdot)$ *is globally Lipschitz continuous for all* $q \in \mathcal{Q}$. *The edge* $(q, q')$ *is contained in* $\mathcal{E}$ *if and only if* $\mathcal{G}(q, q') \neq \emptyset$ *and* $\boldsymbol{x} \in \mathcal{G}(q, q')$ *if and only if there is an* $\boldsymbol{x}' \in \mathcal{X}$ *such that* $(\boldsymbol{x}, \boldsymbol{x}') \in \mathcal{R}(q, q')$.

The first part of the assumption is standard to guarantee global existence and uniqueness of solutions within each location given a continuous initial state. The latter part is without loss of generality as can easily be seen [427].

A state $(\hat{q}, \hat{\boldsymbol{x}})$ is called *reachable* if there exists a finite execution $(\tau, \lambda, \xi)$ with $\tau = \{[\tau_i, \tau_i']\}_{i=0}^N$ and $(\lambda_N(\tau_N'), \xi_N(\tau_N')) = (\hat{q}, \hat{\boldsymbol{x}})$. The set *Reach* $\subseteq \mathcal{Q} \times \mathcal{X}$ denotes the collection of reachable states of the automaton.

The set of states from which continuous evolution is impossible is defined as

$$Out = \{(q_0, \boldsymbol{x}_0) \in \mathcal{Q} \times \mathcal{X} \mid \forall \varepsilon > 0 \exists t \in [0, \varepsilon)\ \boldsymbol{x}_{q_0,\boldsymbol{x}_0}(t) \notin Inv(q_0)\},$$

in which $\boldsymbol{x}_{q_0,\boldsymbol{x}_0}(\cdot)$ denotes the unique solution to $\dot{\boldsymbol{x}} = \boldsymbol{f}(q_0, \boldsymbol{x})$ with $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

**Theorem 5.14** [427] *Let Assumption 5.1 be satisfied.*

1. *A hybrid automaton is nonblocking if, for all $(q, \boldsymbol{x}) \in Reach \cup Out$, there exists $(q, q') \in \mathcal{E}$ with $\boldsymbol{x} \in \mathcal{G}(q, q')$. In case the automaton is deterministic, this condition is also necessary.*
2. *A hybrid automaton is deterministic if and only if for all $(q, \boldsymbol{x}) \in Reach$*
   - *if $\boldsymbol{x} \in \mathcal{G}(q, q')$ for some $(q, q') \in \mathcal{E}$, then $(q, \boldsymbol{x}) \in Out$;*
   - *if $(q, q') \in \mathcal{E}$ and $(q, q'') \in \mathcal{E}$ with $q' \neq q''$, then $\boldsymbol{x} \notin \mathcal{G}(q, q') \cap \mathcal{G}(q, q'')$; and*
   - *if $(q, q') \in \mathcal{E}$ and $\boldsymbol{x} \in \mathcal{G}(q, q')$, then there is at most one $\boldsymbol{x}' \in \mathcal{X}$ with $(\boldsymbol{x}, \boldsymbol{x}') \in \mathcal{R}(q, q')$.*

As a consequence of the broad class of systems covered by the results in this section, the conditions are rather implicit in the sense that for a particular example the conditions cannot be verified by direct calculations (i.e. are not in an algorithmic form). Especially, if the model description itself is implicit (e.g. piecewise affine systems or complementarity models) these results are only a start of the well-posedness analysis as the hybrid automaton model and the corresponding sets *Reach* and *Out* have to be determined first. In the next sections, we will present results that can be checked by direct computations.

The extension of the initial well-posedness results for hybrid automata to local or global existence of executions are awkward as Zeno behavior is hard to characterize or exclude, and continuation beyond Zeno times is not easy to show. This is one of the motivation to derive conditions that guarantee the existence or absence of Zeno behavior (see, e.g., [19, 159, 272, 341, 532, 583, 619, 680]) To guarantee continuation beyond Zeno times the hybrid model is sometimes extended or modified by using, e.g., relaxations [341]. As another example of an extension, consider the the bouncing ball model (Section 2.3.3) in which "global solutions" defined for all $t$ in $[0, \infty)$ can be obtained by adding the "constrained mode" $\dot{x}_1 = \dot{x}_2 = 0$. Note that in case of complementarity modelling of the bouncing ball by $\ddot{x}_1 = -g + w$, $0 \leq w \perp x_1 \geq 0$ (completed with the elastic reset map), where $w$ represents the constraint force exerted by the ground on the ball, this constrained mode with $x_1 = 0$ follows naturally. For complementarity systems, but also for differential equations with discontinuous right-hand sides such as piecewise affine systems or other switched systems, one has the advantage that the location or mode can be described as a function of the continuous state. Of course, in this case one is able to define an evolution beyond the Zeno time by proving that the (left-)limit of the continuous state exists at the Zeno point (e.g. show for the bouncing ball as in Section 2.3.3 that $\lim_{t \uparrow \tau^*} x_1(t) = 0$

and $\lim_{t \uparrow \tau^*} x_2(t) = 0$, and that from $(0, 0)^\top$ continuation in the constrained mode is clearly possible). Continuation from this limit follows then from initial or local existence.

**Well-posedness of piecewise linear systems** A problem of considerable importance is to find necessary and sufficient conditions for well-posedness of piecewise linear systems

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{A}_1 \boldsymbol{x}, & \text{when } \boldsymbol{x} \in \mathcal{C}_1, \\ \boldsymbol{A}_2 \boldsymbol{x}, & \text{when } \boldsymbol{x} \in \mathcal{C}_2, \\ & \vdots \\ \boldsymbol{A}_r \boldsymbol{x}, & \text{when } \boldsymbol{x} \in \mathcal{C}_r, \end{cases} \tag{5.94}$$

where $\mathcal{C}_i$ are certain subsets of $\mathbb{R}^n$ having the property that

$$\begin{aligned} &\mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_r = \mathbb{R}^n \\ &\operatorname{int} \mathcal{C}_i \cap \operatorname{int} \mathcal{C}_j = \emptyset, \quad i \neq j. \end{aligned} \tag{5.95}$$

This situation may naturally arise from modeling, as well as from the application of a switching linear feedback scheme (with different feedback laws corresponding to the subsets $\mathcal{C}_i$). Of course, even more general cases may be considered, or, instead, extra conditions may be imposed on the subsets $\mathcal{C}_i$. Note that the first condition in (5.95) is a necessary (but not sufficient) condition for existence of solutions for all initial conditions and the second one is necessary (but again not sufficient) for uniqueness (unless the vector fields are equal on the overlapping parts of the regions $\mathcal{C}_i$).

A particular case of the above problem, which has been investigated in depth, is the *bimodal* linear case

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{A}_1 \boldsymbol{x}, & \text{when } \boldsymbol{C}\boldsymbol{x} \geq 0, \\ \boldsymbol{A}_2 \boldsymbol{x}, & \text{when } \boldsymbol{C}\boldsymbol{x} \leq 0, \end{cases} \tag{5.96}$$

under the additional assumption that both pairs $(\boldsymbol{C}, \boldsymbol{A}_1)$ and $(\boldsymbol{C}, \boldsymbol{A}_2)$ are *observable*.

The solution concept that will be employed is the *extended Carathéodory solution*, which is a function $\boldsymbol{x} : [t_0, t_1] \to \mathbb{R}^n$, which is absolutely continuous on $[t_0, t_1]$, satisfies

$$\boldsymbol{x}(t) = \boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{x}(\tau)) \mathrm{d}\tau, \tag{5.97}$$

where $\boldsymbol{f}(\boldsymbol{x})$ is the (discontinuous) vector field given by the right-hand side of (5.96), and there are no left-accumulation points of event times on $[t_0, t_1]$.

Note that Filippov solutions involving sliding modes are not extended Carathéodory solutions. Moreover, note that if $\boldsymbol{f}(\boldsymbol{x})$ is *continuous* then necessarily there exists a $\boldsymbol{K}$ such that $\boldsymbol{A}_1 = \boldsymbol{A}_2 + \boldsymbol{K}\boldsymbol{C}$, and $\boldsymbol{f}$ is automatically Lipschitz continuous, implying local uniqueness of solutions by classical results on ordinary differential equations.

Before stating the main result we introduce some notation. First we define the $n \times n$ observability matrices corresponding to $(\boldsymbol{C}, \boldsymbol{A}_1)$, respectively $(\boldsymbol{C}, \boldsymbol{A}_2)$:

$$\boldsymbol{W}_1 := \begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A}_1 \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}_1^{n-1} \end{pmatrix}, \quad \boldsymbol{W}_2 := \begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A}_2 \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}_2^{n-1} \end{pmatrix} \tag{5.98}$$

(by assumption they both have rank $n$). Furthermore we define the following subsets of the state space $\mathbb{R}^n$:

$$\begin{aligned} \mathcal{S}_i^+ &= \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{W}_i \boldsymbol{x} \succeq 0\} \\ \mathcal{S}_i^- &= \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{W}_i \boldsymbol{x} \preceq 0\} \end{aligned} \qquad i = 1, 2, \tag{5.99}$$

where $\succeq$ denotes *lexicographic* ordering, that is $\boldsymbol{x} = 0$ or $\boldsymbol{x} \succeq 0$ if the first component of $\boldsymbol{x}$ that is nonzero is positive. Furthermore, $\boldsymbol{x} \preceq 0$ iff $-\boldsymbol{x} \succeq 0$. Then the following result from [335] can be stated:

**Theorem 5.15** *The bimodal linear system (5.96) is well-posed if and only if one of the following equivalent conditions are satisfied:*
*(a)* $\mathcal{S}_1^+ \cup \mathcal{S}_2^- = \mathbb{R}^n$;
*(b)* $\mathcal{S}_1^+ \cap \mathcal{S}_2^- = \{0\}$;
*(c)* $\boldsymbol{W}_2 \boldsymbol{W}_1^{-1}$ *is a lower-triangular matrix with positive diagonal elements.*

Possible extensions to noninvertible observability matrices, the situation of more than two modes, as well as to modification of the sets $\boldsymbol{C}\boldsymbol{x} \geq 0, \quad \boldsymbol{C}\boldsymbol{x} \leq 0$, are discussed in [335, 336].

**Complementarity systems** Several well-posedness results were already presented in Section 5.2. These results focussed on Carathéodory and forward solutions that applied to absolutely continuous trajectories only. However, in various application domains of complementarity systems the restriction to continuous trajectory is too stringent. This is the case in the context of unilaterally constrained mechanical systems (cf. [122, 123, 302, 413, 462]) in which impacts cause discontinuities in the velocities of the impacting bodies. In this section we will provide a result that applies to linear complementarity systems of the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \tag{5.100a}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t), \tag{5.100b}$$

$$0 \leq \boldsymbol{u}(t) \perp \boldsymbol{y}(t) \geq 0, \tag{5.100c}$$

in which impacts are allowed. Before doing so, we will present a result for a class of nonsmooth dynamical systems consisting of linear saturation systems and linear relay systems, which are based on “complementarity reasoning,” see [149, 151].

**Linear saturation and linear relay systems** As is well-known [217], piecewise linear relations may be described in terms of the linear complementarity problem. In the circuits and systems community (cf. [395, 641]) the complementarity formulation has already been used for *static* piecewise linear systems; this subsection may be viewed as an extension of the cited work in the sense that we consider *dynamic* systems. For the sake of simplicity, we will focus on a specific type of piecewise linear systems, namely linear saturation systems, i.e. linear systems coupled to saturation characteristics. They are of the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \tag{5.101a}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t), \tag{5.101b}$$

$$(\boldsymbol{u}(t), \boldsymbol{y}(t)) \in \mathrm{saturation}_i, \tag{5.101c}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^m$, $\boldsymbol{y}(t) \in \mathbb{R}^m$, $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are matrices of appropriate sizes, and $\mathrm{saturation}_i$ is the curve depicted in Fig. 5.3 with $e_2^i - e_1^i > 0$ and $f_1^i \geq f_2^i$. We denote the overall system (5.101) by $\mathrm{SAT}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$. Note that relay characteristics can be obtained from saturation characteristics by setting $f_1^i = f_2^i$.

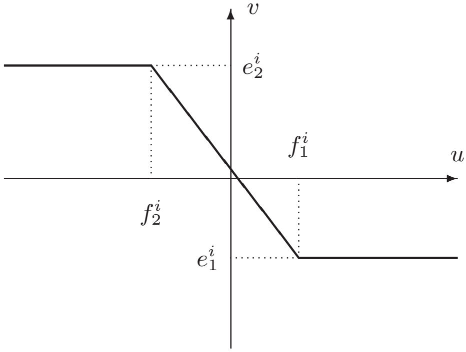


Fig. 5.3 Saturation characteristic.

One may argue that the saturation characteristic is a Lipschitz continuous function (provided that $f_1^i - f_2^i > 0$) and hence the existence and uniqueness of the solutions follow from the theory of ordinary differential equations. The following example shows that this is not correct in general if the feedthrough term $D$ is nonzero:

**Example 5.10** *Linear saturation system*

Consider the single-input single-output system

$$\dot{x} = u, \tag{5.102}$$

$$y = x - 2u, \tag{5.103}$$

where $u$ and $y$ restricted by a saturation characteristic with $e_1 = -f_1 = -e_2 = f_2 = 1/2$ as shown in Fig. 5.3. Let the periodic function $\tilde{u} : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$\tilde{u}(t) = \begin{cases} 1/2, & \text{if } 0 \le t < 1, \\ -1/2, & \text{if } 1 \le t < 3, \\ 1/2, & \text{if } 3 \le t < 4, \end{cases}$$

and $\tilde{u}(t-4) = \tilde{u}(t)$ whenever $t \ge 4$. By using this function define $\tilde{x} : \mathbb{R}_+ \to \mathbb{R}$ as

$$\tilde{x}(t) = \int_0^t \tilde{u}(s)\,\mathrm{d}s,$$

and $\tilde{y} : \mathbb{R}_+ \to \mathbb{R}$ as

$$\tilde{y} = \tilde{x} - 2\tilde{u}.$$

It can be verified that $(-\tilde{u}, -\tilde{x}, -\tilde{y})$, $(0, 0, 0)$, and $(\tilde{u}, \tilde{x}, \tilde{y})$ are all solutions of SAT(0, 1, 1, −2) with the zero initial state. □

As illustrated in the example, the Lipschitz continuity argument does not work in general when $f_1^i > f_2^i$. Also in the case, where $f_1^i = f_2^i$ this reasoning does not apply. The following theorem gives a sufficient condition for the well-posedness of linear systems with saturation characteristics. Recall that a *$P$-matrix* is a matrix with all its principal minors being positive.

**Theorem 5.16** [149, 151] *Consider SAT($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$). Let* $\boldsymbol{R} = \mathrm{diag}(e_2^i - e_1^i)$ *and* $\boldsymbol{S} = \mathrm{diag}(f_2^i - f_1^i)$. *Suppose that* $\boldsymbol{G}(\sigma)\boldsymbol{R} - \boldsymbol{S}$ *is a $P$-matrix for all sufficiently large $\sigma \in \mathbb{R}$, where*

$$\boldsymbol{G}(\sigma) = \boldsymbol{C}(\sigma I - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}.$$

*Then, there exists a unique forward solution of SAT($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$) for all initial states.*

**Linear complementarity systems with jumps** Up to this point, the results on well-posedness for complementarity systems concerned solutions of which the $\boldsymbol{x}$-part is continuous. As mentioned before, for applications such as constrained mechanical systems (e.g. the bouncing ball) discontinuities in the state variables are required. For linear complementarity systems as in (5.100) a distributional framework was used to obtain an *extension of the forward solution concept* (see [304] for details). The work [304] presented also sufficient conditions for local well-posedness. In case of one complementarity pair, these conditions are also sufficient for global well-posedness.

Consider the LCS$(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ as in (5.100) with Markov parameters $\boldsymbol{H}^0 = \boldsymbol{D}$ and $\boldsymbol{H}^i = \boldsymbol{C}\boldsymbol{A}^{i-1}\boldsymbol{B}$, $i = 1, 2, \ldots$ and define the leading row and column indices by

$$\rho_j := \inf\{i \in \mathbb{N} \mid \boldsymbol{H}^i_{j\bullet} \neq 0\}, \quad \eta_j := \inf\{i \in \mathbb{N} \mid \boldsymbol{H}^i_{\bullet j} \neq 0\},$$

where $j \in \{1, \ldots, k\}$ and $\inf \emptyset := \infty$. The *leading row coefficient matrix* $\mathcal{M}$ and *leading column coefficient matrix* $\mathcal{N}$ are then given for *finite* leading row and column indices by

$$\mathcal{M} := \begin{pmatrix} \boldsymbol{H}_{1\bullet}^{\rho_1} \\ \vdots \\ \boldsymbol{H}_{k\bullet}^{\rho_k} \end{pmatrix} \text{ and } \mathcal{N} := (\boldsymbol{H}_{\bullet 1}^{\eta_1} \ldots \boldsymbol{H}_{\bullet k}^{\eta_k}).$$

**Theorem 5.17** [304] *If the leading column coefficient matrix $\mathcal{N}$ and the leading row coefficient matrix $\mathcal{M}$ are both defined and P-matrices, then LCS($\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$) has a unique* local *forward solution (with jumps) on an interval of the form $[0, \varepsilon)$ for some $\varepsilon > 0$. Moreover, live-lock (an infinite number of events at one time instant) does not occur.*

**Differential equations with discontinuous right-hand sides** Differential equations of the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t)) \tag{5.104}$$

with $\boldsymbol{f}$ being piecewise continuous in a domain $\mathcal{G}$ and with the set $\mathcal{M}$ of discontinuity points having measure zero, received quite some attention in the literature. Major roles have been played in this context by Filippov [237] and Utkin [640]. An example of such a system with two "modes" was given in (5.90). As mentioned in Subsection 5.4.2, solution concepts have been defined by replacing the basic differential equation (5.104) by a differential inclusion of the form

$$\dot{\boldsymbol{x}}(t) \in \mathcal{F}(t, \boldsymbol{x}(t)), \tag{5.105}$$

where $\mathcal{F}$ is constructed from $\boldsymbol{f}$. The solution concept is then inherited from the realm of differential inclusions [26].

**Definition 5.10 (Solution of differential inclusion)** *The function $\boldsymbol{x} : \Omega \rightarrow \mathbb{R}^n$ is called a solution of the differential inclusion (5.105) if $\boldsymbol{x}$ is absolutely continuous on the time-interval $\Omega$ and satisfies $\dot{\boldsymbol{x}}(t) \in \mathcal{F}(t, \boldsymbol{x}(t))$ for almost all $t \in \Omega$.*

There are several ways to transform $f$ into $\mathcal{F}$ and we will restrict ourselves to the two most famous ones and briefly discuss an alternative transformation proposed by Aizerman and Pyatnitskii [6]. For further details see [237].

In the *convex definition* [237], as already briefly mentioned in Section 5.4.2 the set $\mathcal{F}_a(t, x)$ is taken to be the smallest convex closed set containing all the limit values of the function $\boldsymbol{f}(\bar{t}, \bar{x})$ for $\bar{x} \longrightarrow x$, $\bar{t} = t$ and $(\bar{t}, \bar{x}) \notin M$.

The *control equivalent definition* proposed by Utkin [640] (see also page 54 in [237]) applies to equations of the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), u_1(t, \boldsymbol{x}), \ldots, u_r(t, \boldsymbol{x})), \tag{5.106}$$

where $\boldsymbol{f}$ is continuous in its arguments, but $u_i(t,\boldsymbol{x})$ is a scalar-valued function being discontinuous only on a smooth surface $\mathcal{S}_i$ given by $\phi_i(\boldsymbol{x}) = 0$. We define the sets $U_i(t,\boldsymbol{x})$ as $\{u_i(t,\boldsymbol{x})\}$ when $\boldsymbol{x} \notin \mathcal{S}_i$ and in case $\boldsymbol{x} \in \mathcal{S}_i$ by the closed interval with end-points $u_i^-(t,\boldsymbol{x})$ and $u_i^+(t,\boldsymbol{x})$. The values $u_i^-(t,\boldsymbol{x})$ and $u_i^+(t,\boldsymbol{x})$ are the limiting values of the function $u_i$ on both sides of the surface $S_i$ which we assume to exist. The differential equation (5.106) is replaced by (5.105) with $\mathcal{F}_b(t,\boldsymbol{x}) = \boldsymbol{f}(t,\boldsymbol{x},U_1(t,\boldsymbol{x}),\ldots,U_r(t,\boldsymbol{x}))$.

*Remark 5.4* In case $\mathcal{F}_c(t,\boldsymbol{x})$ is chosen as the smallest convex closed set containing $\mathcal{F}_b(t,\boldsymbol{x})$, then the general definition of Aizerman and Pyatnitskii [6] is obtained. In case $f$ is affine in $u_1,\ldots,u_r$ and the surfaces $S_1,\ldots,S_r$ are all different and at the point of intersection the normal vectors are linearly independent, all the before mentioned definitions coincide, i.e. $\mathcal{F}_a = \mathcal{F}_b = \mathcal{F}_c$.

The well-posedness results of the differential equation (5.104) or (5.106) can now be based on the theory available for differential inclusions (cf. [26, 237] and the references therein). A set-valued function $\mathcal{F}$ is called *upper semicontinuous* at $p_0$, if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $\mathcal{F}(p+\delta\mathbb{B}) \subseteq \mathcal{F}(p_0)+\varepsilon\mathbb{B}$, where $\mathbb{B}$ denotes the unit ball. $\mathcal{F}$ is called upper semicontinuous on a set $\mathcal{D}$, if $\mathcal{F}$ is upper semicontinuous in each point of the set $\mathcal{D}$.

**Definition 5.11 (Basic condition)** *We say that the set-valued map $\mathcal{F}(t,\boldsymbol{x})$ satisfies the* basic conditions*, if:*
- *for all $(t,\boldsymbol{x}) \in \mathcal{G}$ the set $\mathcal{F}(t,\boldsymbol{x})$ is nonempty, bounded, closed, and convex*
- *$\mathcal{F}$ is upper semicontinuous in $t$, $\boldsymbol{x}$.*

The following result is described on page 77 of the monograph [237].

**Theorem 5.18** (Theorems 2.7.1 and 2.7.2 in [237]) *If $\mathcal{F}(t,\boldsymbol{x})$ satisfies the basic conditions in the domain $\mathcal{G}$, then for any point $(t_0,\boldsymbol{x}_0) \in \mathcal{G}$ there exists a solution of the problem*

$$\dot{\boldsymbol{x}}(t) \in \mathcal{F}(t,\boldsymbol{x}(t)), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{5.107}$$

*If the basic conditions are satisfied in a closed and bounded domain $\mathcal{G}$, then each solution can be continued on both sides up to the boundary of the domain $\mathcal{G}$.*

In combination with the following result Theorem 5.18 proves the existence of solutions for the differential inclusions related to $\mathcal{F}_a$, $\mathcal{F}_b$, and $\mathcal{F}_c$:

**Theorem 5.19** (Page 67 in [237]) *The sets $\mathcal{F}_a(t,\boldsymbol{x})$, $\mathcal{F}_b(t,\boldsymbol{x})$ and $\mathcal{F}_c(t,\boldsymbol{x})$ are nonempty, bounded, and closed. $\mathcal{F}_a(t,\boldsymbol{x})$ and $\mathcal{F}_c(t,\boldsymbol{x})$ are also convex. $\mathcal{F}_a$ is upper semicontinuous in $\boldsymbol{x}$, and $\mathcal{F}_b$ and $\mathcal{F}_c$ are upper semicontinuous in $t$, $\boldsymbol{x}$.*

Theorem 5.18 and 5.19 together show the existence of solutions when Filippov's convex definition is used under the condition that $f$ is time-invariant. In case $f$ is not time-invariant, additional assumptions are needed to arrive at $\mathcal{F}$ being upper semicontinuous in $t$ as well (cf. page 68 in [237]). For the definition of Aizerman and Pyatnitskii (i.e. using $\mathcal{F}_c$) existence of solutions is guaranteed. In case $\mathcal{F}_b(t, \boldsymbol{x})$ is convex for all relevant $(t, \boldsymbol{x})$ (e.g. if the conditions mentioned in Remark 5.4 are satisfied), then existence follows as well. If the convexity assumption is not satisfied, the existence result still holds if upper semicontinuity is replaced by continuity (cf. page 79 in [237]). In fact, the two major cases studied in Chapter 3 of [26] are related to these two situations: (i) the values of $\mathcal{F}$ are compact and convex and $\mathcal{F}$ is upper semicontinuous; and (ii) the values of $\mathcal{F}$ are compact, but not necessarily convex and $\mathcal{F}$ is continuous.

Now we will discuss the issue of uniqueness. Right uniqueness (in the Filippov sense) holds for the differential equation (5.104) at the point $(t_0, \boldsymbol{x}_0)$, if there exists $t_1 > t_0$ such that each two solutions of this equation satisfying the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ coincide on the interval $[t_0, t_1]$ or on the interval on which they are both defined. Right uniqueness holds for a domain $\mathcal{D}$ if from each point $(t_0, \boldsymbol{x}_0) \in \mathcal{D}$ right uniqueness holds.

Not too many uniqueness results are available in the literature. The most useful result given in [237] is related to the following situation. Let the domain $G \subset \mathbb{R}^n$ be separated by a smooth surface $\mathcal{S}$ into domains $G^-$ and $G^+$. Let $f$ and $\partial f / \partial x_i$ be continuous in the domains $G^-$ and $G^+$ up to the boundary such that $f^-(t, \boldsymbol{x})$ and $f^+(t, \boldsymbol{x})$ denote the limit values of the function $f$ at $(t, \boldsymbol{x})$, $\boldsymbol{x} \in \mathcal{S}$ from the regions $G^-$ and $G^+$, respectively. We define $h(t, \boldsymbol{x}) = f^+(t, \boldsymbol{x}) - f^-(t, \boldsymbol{x})$ as the discontinuity vector over the surface $\mathcal{S}$. Moreover, let $n(\boldsymbol{x})$ be the normal vector to $\mathcal{S}$ at point $\boldsymbol{x}$ directed from $G^-$ to $G^+$.

**Theorem 5.20** *Consider the differential equation (5.104) with $f$ as above. Let $\mathcal{S}$ be a twice continuously differentiable surface and suppose that the function $h$ is continuously differentiable. If for each $t \in (a, b)$ and each point $\boldsymbol{x} \in \mathcal{S}$ at least one of the inequalities $n(\boldsymbol{x})^{\mathrm{T}} f^-(t, \boldsymbol{x}) > 0$ or $n(\boldsymbol{x})^{\mathrm{T}} f^+(t, \boldsymbol{x}) < 0$ (possibly different inequalities for different $\boldsymbol{x}$ and $t$) is fulfilled, then right uniqueness holds for (5.104) in the domain $\mathcal{G}$ for $t \in (a, b)$ in the sense of Filippov.*

The criterion above clearly holds for general nonlinear systems, but needs to be verified on a point-by-point basis. Alternatively, the results on complementarity systems, piecewise affine systems or linear saturation systems are more straightforward to check as it requires, for instance, the computation of the determinants of all principal minors of the transfer function of the underlying linear system, or determine the signs of the leading Markov parameters. However, the latter theory applies to specific classes of hybrid systems and uses a *different* solution concept. Hence, uniqueness is not proved in the Filippov sense, but in a forward sense.

**Hybrid inclusions** Adding reset maps and restricting the "flow region" for the above differential inclusions (5.105) leads to the hybrid inclusions (5.93). In [271] the following basic conditions are adopted:

- $\mathcal{C}$ and $\mathcal{D}$ are closed sets;
- $\mathcal{F}$ is outer semicontinuous in the sense that for all $\boldsymbol{x} \in \mathbb{R}^n$ and all sequences $\{\boldsymbol{x}_i\}$ with $\boldsymbol{x}_i \to \boldsymbol{x}$, $y_i \in \mathcal{F}(\boldsymbol{x}_i)$ such that $y_i \to y$, it holds that $y \in \mathcal{F}(\boldsymbol{x})$;
- $\mathcal{F}$ is locally bounded (i.e. for any compact set $K \subseteq \mathbb{R}^n$ there exists $m > 0$ such that for all $\boldsymbol{x} \in K$ it holds that $\mathcal{F}(\boldsymbol{x}) \subseteq m\mathbb{B}$, where $\mathbb{B}$ is the unit ball) and $\mathcal{F}(\boldsymbol{x})$ is nonempty and convex for all $\boldsymbol{x} \in \mathcal{C}$;
- $\mathcal{G}$ is outer semicontinuous and and $\mathcal{G}(\boldsymbol{x})$ is nonempty for all $\boldsymbol{x} \in \mathcal{D}$.

Note that in the case of locally bounded set-valued mappings with closed values, outer semicontinuity agrees with upper semicontinuity. In general this is not true [271].

Based on these basic conditions, Proposition 2.4 in [271] states existence results for these hybrid inclusions. Actually, [271] follows here closely the lines of [27], where a similar result was obtained for so-called *impulse differential inclusions*. To formulate the existence result we will use the following concepts. The tangent cone $T_C(x)$ to a set $\mathcal{C}$ at the point $\boldsymbol{x} \in \mathcal{C}$ consists of all $v \in \mathbb{R}^n$ for which there exist real numbers $\alpha_i > 0$ with $\alpha_i \to 0$ and vectors $v_i \to v$ such that for $i = 1, 2, \ldots$ $x + \alpha_i v_i \in \mathcal{C}$. A solution (in the form of a hybrid arc as defined in Definition 5.8) to (5.93) is called maximal if it cannot be extended and is called complete if its domain is unbounded (either in the "$j$" and/or "$t$" directions). The notions of maximal and complete solutions are similar in nature as maximal and infinite executions of hybrid automata in Section 5.4.4.

**Theorem 5.21** *Consider the system (5.93) with the above basic conditions are fulfilled. If $\boldsymbol{x}_0 \in \mathcal{D}$ or $\boldsymbol{x}_0 \in \mathcal{C}$ and for some neighborhood $U$ of $\boldsymbol{x}_0$ it holds that*

$$\boldsymbol{x}' \in \mathcal{U} \cap C \text{ implies that } T_C(\boldsymbol{x}') \cap \mathcal{F}(\boldsymbol{x}') \neq \emptyset, \tag{5.108}$$

*then there exists a hybrid arc $\boldsymbol{x}$ of the hybrid inclusion with $\boldsymbol{x}(0,0) = \boldsymbol{x}_0$ and dom $\boldsymbol{x} \neq (0,0)$. If (5.108) holds for any $\boldsymbol{x}_0 \in \mathcal{C} \setminus \mathcal{D}$, then for any maximal solution $\boldsymbol{x}$ at least one of the following is true:*
*(i) $\boldsymbol{x}$ is complete;*
*(ii) $\boldsymbol{x}$ eventually leaves every compact subset of $\mathbb{R}^n$;*
*(iii) for some $(T,J) \in$ dom $x$ with $(T,J) \neq (0,0)$ we have $\boldsymbol{x}(T,J) \notin \mathcal{C} \cup D$.*
*Case (iii) does not occur if additionally we have for all $\boldsymbol{x}_0 \in \mathcal{D}$ that $\mathcal{G}(\boldsymbol{x}_0) \subseteq C \cup D$.*

This result can be considered as an initial well-posedness result. In particular, it does not give any guarantees that a solution can be defined on a domain containing some $(t, J)$ with $t > 0$ (as live-lock is not excluded) nor for $t \to \infty$ (due to finite escape times or right-accumulations of reset times). Uniqueness of trajectories is not considered in the context of hybrid inclusions. Note that statement (ii) above expresses a kind of "finite escape time" condition, which is similar as in Theorem 5.18 for differential inclusions only.

### 5.4.5 Comparison of some solution concepts

The difference between Filippov and forward and extended Carathéodory solutions will be discussed in the context of the class of systems for which all these concepts apply. In particular, we will study the plant

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t);\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t), \tag{5.109}$$

in a closed loop with the relay feedback

$$\boldsymbol{u}(t) = -\mathrm{sgn}(y(t)). \tag{5.110}$$

Note that, in the context of Theorem 5.16, we are dealing with the situation in which $R = 2I$ and $S = 0$. Note also that $\mathcal{F}_a = \mathcal{F}_b = \mathcal{F}_c$ for such linear relay systems and that the corresponding solution concepts coincide and will therefore be referred to as "Filippov solutions" from now on.

**Example 5.11** *Difference between Filippov and forward concepts*

The difference between the forward solutions and Filippov solution is related to Zeno behavior and is nicely demonstrated by an example constructed by Filippov (page 116 in [237]), which is given by

$$\dot{x}_1 = -u_1 + 2u_2, \tag{5.111a}$$
$$\dot{x}_2 = -2u_1 - u_2, \tag{5.111b}$$
$$y_1 = x_1, \tag{5.111c}$$
$$y_2 = x_2, \tag{5.111d}$$
$$u_1 = -\mathrm{sgn}(y_1), \tag{5.111e}$$
$$u_2 = -\mathrm{sgn}(y_2). \tag{5.111f}$$

This system has, besides the zero solution (which is both a Filippov and a forward solution), an infinite number of other trajectories (being Filippov, but not forward solutions) starting from the origin. The nonzero solutions (Fig. 5.4) leave the origin due to left-accumulations of the relay switching times and are Filippov solutions, but are not forward solutions. However, this example does not satisfy the conditions for uniqueness given in Theorem 5.16 in Section 5.4.4. Hence, it is not clear if the conditions in Section 5.4.4 are sufficient for Filippov uniqueness as well. □

The latter problem mentioned in the example is studied in [532] for the case where (5.109) is a single-input-single-output (SISO) system. Theorem 5.16 states that the positivity of the leading Markov parameter $\boldsymbol{H}^\rho$ with $\boldsymbol{H}^i = \boldsymbol{C}\boldsymbol{A}^{i-1}\boldsymbol{B}$, $i = 1, 2, \ldots$ and $\rho = \min\{i \mid \boldsymbol{H}^i \neq 0\}$ implies existence uniqueness in forward sense. What about uniqueness in Filippov's sense?

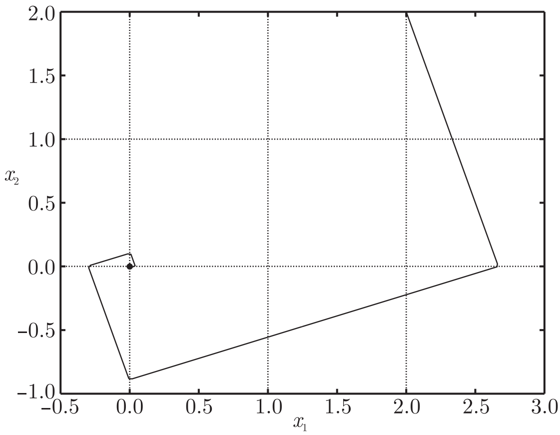


Fig. 5.4 Trajectory in the phase plane of (5.111).

**Theorem 5.22** [532] *Consider the system (5.109)–(5.110). The following statements hold for the relative degree $\rho$ being 1 or 2:*

$\rho = 1$*: The system (5.109)–(5.110) has a unique Filippov solution for all initial conditions if and only if the leading Markov parameter $\boldsymbol{H}^{\rho}$ is positive.*

$\rho = 2$*: The system (5.109)–(5.110) has a unique Filippov solution for initial condition $\boldsymbol{x}(0) = 0$ if and only if the leading Markov parameter $\boldsymbol{H}^{\rho}$ is positive.*

*Moreover, in the case $\boldsymbol{H}^1 = \boldsymbol{CB} > 0$, Filippov solutions do not have left-accumulations of relay switching times.*

Interestingly, the above theorem presents conditions that exclude particular types of Zeno behavior.

Up to this point, one might hope that the positivity of the leading Markov parameter is also sufficient for Filippov uniqueness for higher relative degrees. However, in [532] a counter-example is presented of the form (5.109)–(5.110), with (5.109) being a triple integrator.

**Example 5.12** *Linear relay example*

This relay system has one forward solution (being identically zero) starting in the origin (as expected, as the leading Markov parameter is positive), but has infinitely many Filippov solutions of which one is the zero solution and the other starts with a left-accumulation point of relay switching times [532]. This example can also be considered in the light of the piecewise linear systems (5.96) considered in Section 5.4.4, which are of the form

$$\begin{cases} \text{mode 1}: & \dot{\boldsymbol{x}} = \boldsymbol{A}_1 \boldsymbol{x}, \quad \text{if} \quad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} \geq 0, \\ \text{mode 2}: & \dot{\boldsymbol{x}} = \boldsymbol{A}_2 \boldsymbol{x}, \quad \text{if} \quad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} \leq 0, \end{cases} \tag{5.112}$$

with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = (1\,0\,0\,0). \tag{5.113}$$

An extended Carathéodory solution concept (5.97) for this type of systems and necessary and sufficient conditions for existence and uniqueness are presented in [335] (see Theorem 5.15). As this solution concept does not allow for sliding modes and left-accumulation points of event times, the above system does not have any extended Carathéodory solution starting from the initial state $(0, 0, 0, 1)^{\mathrm{T}}$ as can easily be seen (cf. also Theorem 5.15).

In summary, the triple integrator connected to a (negative) relay forms a nice comparison between the three mentioned solution concepts; for the system (5.112) with (5.113) and $x_0 = (0, 0, 0, 1)^{\mathrm{T}}$, there exist [532]

- *no* extended Carathéodory solution;
- *one* forward solution; and
- *infinitely many* Filippov solutions. □

For specific applications in discontinuous feedback control the Filippov solution concept allows trajectories, which are not practically relevant for the stabilization problem at hand. So-called Euler (or sampling) solutions seem to be more appropriate in this context [176, 178]. Also in this case the discontinuous dynamical system is replaced by a differential inclusion with the difference that a particular choice of the controller is made at the switching surface. This choice determines which trajectories are actually Euler solutions by forming the limits of certain numerical integration routines (cf. [176, 178] for more details).

In Section 2.4.2 of [237] some further results can be found on uniqueness. The most general result in [237] for uniqueness in the setting of Filippov's convex definition uses the exclusion of left-accumulation points as one of the conditions to prove uniqueness. Unfortunately, it is not clear how such assumptions should be verified. As a consequence, Theorem 5.22 is quite useful. In Section 2.4.2 of [237] one can also find some results on continuous dependence of solutions on initial data. See also the recent survey of [188] on discontinuous dynamical systems.

### 5.4.6 Zenoness

The above examples, and also the discussion in Chapter 2, indicate that the Zeno phenomenon in all its forms complicates simulation and many analysis and design problems, including the well-posedness issue. Depending on which type of Zenoness is allowed in the solution concept, the answer to the well-posedness problem differs. So, conditions stating the existence or absence of certain variants of Zenoness are of interest. Such conditions are generally hard to come by, but some rather recent works provide some interesting insights in this difficult problem. The reader might want to consult [19, 159, 272, 341, 532, 583, 619, 680] and the references therein. Some of these works also indicate possibilities on how to proceed (define solutions) beyond Zeno points.

## Bibliographical notes

Introductions to model-predictive control are given in [58, 140, 432, 470, 555].

A tutorial overview on the mathematical aspects of discontinuous differential equations is given in [188]. Impulsive systems are considered in the monograph [292].