

# Decentralized Static Output-Feedback Control via Networked Communication

N.W. Bauer, M.C.F. Donkers, N. van de Wouw, W.P.M.H. Heemels

**Abstract**—This paper provides an approach to the design of decentralized switched output-feedback controllers for large-scale linear plants where the controllers, sensors and actuators are connected via a shared communication network subject to time-varying transmission intervals and delays. Due to the communication medium being shared, it is impossible to transmit all control commands and measurement data simultaneously over the communication network. As a consequence, a protocol is needed to orchestrate what data is sent over the network at each transmission instant. To effectively deal with the shared communication medium using static controllers, we adopt a switched controller structure that switches based on available control inputs at each transmission time. By taking a discrete-time switched linear system perspective, we are able to derive a general model that captures all these networked and decentralized control aspects. The proposed synthesis method is based on decomposing the closed-loop model into a multi-gain switched static output-feedback form. This decomposition allows for the formulation of linear matrix inequality based synthesis conditions which, if satisfied, provide stabilizing switched controllers, which are both decentralized and robust to network effects. A numerical example illustrates the aforementioned developed theory.

## I. INTRODUCTION

Recently, there has been an enormous interest in the control of large-scale networked systems that are physically distributed over a wide area [16]. Examples of such distributed systems are electrical power distribution networks, water transportation networks, industrial factories and energy collection networks (such as wind farms). This problem setting has a number of features that seriously challenge the controller design.

The first feature which challenges controller design is that the controller is decentralized, in the sense that it consists of a number of local controllers that do not share information. The difficulty of decentralized control synthesis lies in the fact that each local controller has only local information to utilize for control, which implies that the other local control actions are perceived as (unknown) disturbances. This synthesis problem is, in general, non-convex. In [19], it was shown that for linear time-invariant systems, only decentralized (block diagonal) controllers interconnected with plants

of an *identical* block diagonal structure satisfy a property called ‘quadratic invariance,’ and, hence, allow for convex synthesis of optimal static feedback controllers. For the specific case of block diagonal static *state-feedback* control design, [8] discovered that through a change of variable, sufficient linear matrix inequality (LMI) synthesis conditions could be formulated which guarantee robust stability with respect to convex bounded uncertainties.

The second feature which challenges controller design comes from the fact that when considering control of a large-scale system, it would be unreasonable to assume that all states are measured. Therefore an output-based controller is needed. In particular, we consider a static feedback setup. In [23], an algorithm based on sufficient LMI conditions for synthesis of robust decentralized static controllers with respect to unknown non-linear subsystem coupling, which is sector bounded and state dependent, was presented. However, [23], as well as [8], considered the communication channels between sensors, actuators and controllers to be ideal.

The third feature which challenges controller design arises from the fact that the implementation of a decentralized control strategy may not be financially possible without a way to inexpensively connect the sensors, actuators and controllers. Indeed, the advantages of using a wired/wireless network compared to dedicated point-to-point (wired) communication are inexpensive and easily modifiable communication links. However, the drawback is that the control system is susceptible to undesirable (possibly destabilizing) side-effects see e.g. [12]. So, the decentralized controller needs to have certain robustness properties when using a communication network. There are roughly five recognized networked control system (NCS) side-effects: time-varying transmission intervals, time-varying delays, packet dropouts, quantization and a shared communication medium (the latter implying that not all information can be sent over the network at once). For modeling simplicity, we only consider time-varying transmission intervals and the communication medium to be shared in this work, although extensions including the other side effects can be envisioned within the presented framework.

In the NCS literature, there are many existing results on stability analysis which consider linear static controllers [3], [7], [17], [21], linear dynamic controllers [6], [22], nonlinear dynamic controllers [1], [10], [18] and model-based controllers [15]. However, results in controller synthesis for NCSs are rare [12]. LMI conditions for synthesis of state-feedback controllers [2] and static output-feedback controllers [9] only became available recently and both considered the *centralized* controller synthesis problem setting.

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Resuming, we note that although a decentralized output-based switched control structure is reasonable to use in practice, its design is extremely complex due to the fact that we simultaneously face the issues of (i) a decentralized control structure (ii) limited measurement information and (iii) communication side-effects. The contribution of this paper is twofold: firstly, a model describing the controller decentralization and the communication side-effects is derived and, secondly, our most significant contribution is LMI-based synthesis conditions for decentralized switched static output-feedback controllers that are robust to communication imperfections.

### A. Nomenclature

We denote  $A^\top \in \mathbb{R}^{m \times n}$  as the transpose of the matrix  $A \in \mathbb{R}^{n \times m}$  and  $\|A\| := \sqrt{\lambda_{\max}(A^\top A)}$  the spectral norm of a matrix  $A$ , which is the square-root of the maximum eigenvalue of the matrix  $A^\top A$ . For a matrix  $A \in \mathbb{R}^{n \times m}$  and two subsets  $\mathbf{I} \subseteq \{1, \dots, n\}$  and  $\mathbf{J} \subseteq \{1, \dots, m\}$ , the  $(\mathbf{I}, \mathbf{J})$ -submatrix of  $A$  is defined as  $(A)_{\mathbf{I}, \mathbf{J}} := (a_{ij})_{i \in \mathbf{I}, j \in \mathbf{J}}$ . In case  $\mathbf{I} = \{1, \dots, n\}$ , we also write  $(A)_{\bullet, \mathbf{J}}$ .

## II. THE MODEL & PROBLEM DEFINITION

We consider a collection of coupled continuous-time linear subsystems  $\mathcal{P}_1, \dots, \mathcal{P}_N$ , see Fig. 1, given by

$$\mathcal{P}_i : \begin{cases} \dot{x}_i(t) = A_i x_i(t) + B_i \hat{u}_i(t) \\ \quad + \sum_{\substack{j=1 \\ j \neq i}}^N (A_{i,j} x_j(t) + B_{i,j} \hat{u}_j(t)) \\ y_i(t) = C_i x_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^N C_{i,j} x_j(t), \end{cases} \quad (1)$$

for  $i \in \{1, \dots, N\}$ , where  $x_i \in \mathbb{R}^{n_{x_i}}$ ,  $\hat{u}_i \in \mathbb{R}^{n_{u_i}}$ , and  $y_i \in \mathbb{R}^{n_{y_i}}$  are the subsystem state, input and output vectors, respectively. The subsystem interaction matrices,  $A_{i,j}, B_{i,j}, C_{i,j}$ ,  $i \neq j$ , represent how subsystem  $j$  affects subsystem  $i$  via state, input and output coupling, respectively. We consider this subsystem collection to be disjoint, i.e. the entire collection of subsystems can be compactly written as

$$\mathcal{P} : \begin{cases} \dot{x}(t) = Ax(t) + B\hat{u}(t) \\ y(t) = Cx(t), \end{cases} \quad (2)$$

with state  $x = [x_1^\top, x_2^\top, \dots, x_N^\top]^\top \in \mathbb{R}^{n_x}$ , control input  $\hat{u} = [\hat{u}_1^\top, \hat{u}_2^\top, \dots, \hat{u}_N^\top]^\top \in \mathbb{R}^{n_u}$  and measured output  $y = [y_1^\top, y_2^\top, \dots, y_N^\top]^\top \in \mathbb{R}^{n_y}$ . The matrix  $A$  is defined as

$$A := \begin{bmatrix} A_1 & A_{1,2} & \cdots & A_{1,N} \\ A_{2,1} & A_2 & & \vdots \\ \vdots & & \ddots & \\ A_{N,1} & \cdots & & A_N \end{bmatrix} \quad (3)$$

and the matrices  $B$  and  $C$  in (2) are defined similarly. The objective of this paper is to present an approach for analysis and synthesis of a controller for system (2) that has the following features: (i) decentralized; (ii) output-based; (iii) robustly stabilizes  $x = 0$  with respect to uncertain time-varying transmission intervals  $h_k \in [\underline{h}, \bar{h}]$ ; (iv) stabilizes  $x = 0$  in the presence of a shared communication medium:

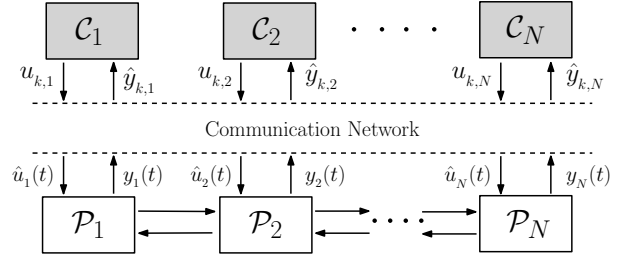


Fig. 1. Decentralized NCS.

not all measured outputs and control inputs can be communicated simultaneously and a network protocol schedules which information is sent at the transmission instants.

Due to these design features, we consider a decentralized control structure consisting of  $N$  local controllers  $\mathcal{C}_i$ ,  $i \in \{1, \dots, N\}$ , which communicate with the sensors and actuators of the plant via a shared network. The decentralized control structure we consider, ‘parallels’ the chosen plant decomposition as in (1). This is depicted in Fig. 1, where the  $i^{\text{th}}$  controller receives measurements from and sends control commands to the  $i^{\text{th}}$  subsystem only. Specifically, we consider the control law to be of the form

$$\mathcal{C}_i : \quad u_{k,i} = K_{\sigma_k, i} \hat{y}_{k,i},$$

where  $u_{k,i} \in \mathbb{R}^{n_{u_i}}$ ,  $\hat{y}_{k,i} \in \mathbb{R}^{n_{y_i}}$  with  $t_k$ , the  $k^{\text{th}}$  transmission instant,  $k \in \mathbb{N}$ , and  $\sigma_k \in \{1, \dots, \bar{N}\}$  is a switching variable related to the shared communication medium, which will be explained in the next section. This control law can be compactly written as

$$\mathcal{C} : \quad u_k = K_{\sigma_k} \hat{y}_k, \quad (4)$$

where

$$K_l = \text{diag}(K_{l,1}, K_{l,2}, \dots, K_{l,N}), \quad l \in \{1, \dots, \bar{N}\}, \quad (5)$$

with  $K_{l,i} \in \mathbb{R}^{n_{u_i} \times n_{y_i}}$ .

In Section II-A, a description of the network imperfections is provided. In Section II-B, a closed-loop model suitable for controller synthesis is derived incorporating all the aforementioned aspects.

### A. Network Description

Communication between sensors, actuators and controllers will take place via a shared network, see Fig. 1. Here, we will consider two network effects: namely, time-varying transmission intervals and a shared communication medium, where the latter imposes the need for a scheduling protocol to determine what measurement and control command data is transmitted at each transmission instant.

Assuming that the transmission intervals  $h_k := t_{k+1} - t_k$  satisfy  $h_k \in [\underline{h}, \bar{h}]$  for some  $0 < \underline{h} \leq \bar{h}$  and all  $k \in \mathbb{N}$ , and a zero-order-hold assumption on the inputs  $\hat{u}$ , i.e.

$$\hat{u}(t) = \hat{u}_k \text{ for all } t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}, \quad (6)$$

the exact discrete-time equivalent of (2) is

$$\mathcal{P}_{h_k} : \begin{cases} x_{k+1} = \bar{A}_{h_k} x_k + \bar{B}_{h_k} \hat{u}_k \\ y_k = C x_k, \end{cases} \quad (7)$$

where  $\bar{A}_{h_k} := e^{A h_k}$  and  $\bar{B}_{h_k} := \int_0^{h_k} e^{As} ds B$ . In (7),  $x_k := x(t_k)$ ,  $y_k := y(t_k)$ , with  $t_k$  the transmission instants, and  $\hat{u}_k$  is the discrete-time control action available at the plant at  $t = t_k$ .

Since the plant and controller are communicating through a network with a shared communication medium, the actual input of the plant  $\hat{u}_k \in \mathbb{R}^{n_u}$  is not equal to the controller output  $u_k$  and the actual input of the controller  $\hat{y}_k \in \mathbb{R}^{n_y}$  is not equal to the sampled plant output  $y_k$ . Instead,  $\hat{u}_k$  and  $\hat{y}_k$  are ‘networked’ versions of  $u_k$  and  $y_k$ , respectively.

To explain the effect of the shared communication medium and thus the difference between  $\hat{y}_k$  and  $y_k$ , and  $\hat{u}_k$  and  $u_k$ ,  $k \in \mathbb{N}$ , one has to realize that the plant has  $n_y$  sensors and  $n_u$  actuators. The actuators and sensors are grouped into  $\bar{N}$  nodes, where, in principle it is allowed that a node can contain both sensors and actuators. The sets of actuator and sensor indices corresponding to node  $l$  are denoted by

$$\bar{\mathbf{J}}_l^u \subseteq \{1, \dots, n_u\}, \quad \bar{\mathbf{J}}_l^y \subseteq \{1, \dots, n_y\},$$

respectively.

At each transmission instant, only one node obtains access to the network and transmits its corresponding  $u$  and/or  $y$  values. Only the transmitted values will be updated, while all other values remain unchanged. This constrained data exchange can be expressed as

$$\hat{u}_k = \Gamma_{\sigma_k}^u u_k + (I - \Gamma_{\sigma_k}^u) \hat{u}_{k-1} \quad (8a)$$

$$\hat{y}_k = \Gamma_{\sigma_k}^y y_k + (I - \Gamma_{\sigma_k}^y) \hat{y}_{k-1}, \quad (8b)$$

where the value of  $\sigma_k \in \{1, \dots, \bar{N}\}$  indicates which node is given access to the network at the transmission instant  $k \in \mathbb{N}$  and  $\Gamma_l^u \in \mathbb{R}^{n_u \times n_u}$  and  $\Gamma_l^y \in \mathbb{R}^{n_y \times n_y}$ , for  $l \in \{1, \dots, \bar{N}\}$ , are diagonal matrices where

$$(\Gamma_l^u)_{i,i} := \begin{cases} 1, & \text{if } i \in \bar{\mathbf{J}}_l^u, \\ 0, & \text{otherwise,} \end{cases}$$

$$(\Gamma_l^y)_{i,i} := \begin{cases} 1, & \text{if } i \in \bar{\mathbf{J}}_l^y, \\ 0, & \text{otherwise.} \end{cases}$$

The mechanisms determining  $\sigma_k$  at transmission instant  $t_k$  are known as protocols. In this paper, we focus on the general class of periodic protocols [6], which are characterized by

$$\sigma_{k+\tilde{N}} = \sigma_k, \text{ for all } k \in \mathbb{N}, \quad (9a)$$

$$\{\sigma_k \mid 1 \leq k \leq \tilde{N}\} \supseteq \{1, \dots, \bar{N}\}, \quad (9b)$$

where  $\tilde{N} \geq \bar{N}$  and  $\tilde{N} \in \mathbb{N}$  is the period of the protocol. Note that (9b) means that every node is addressed at least once in every period of the protocol. This condition is very natural as nodes that are never used do not need to be defined.

**Remark 1** *Note that in the case when a node, that contains both sensors and actuators, gains access to the network, the control law (4) requires that, first, the measurement data is received, then, the control commands are computed and, finally, the control commands are transmitted to the actuators.*

To characterize the decentralized NCS, we need to determine the sets of actuators and sensors that are associated with

node  $l \in \{1, \dots, \bar{N}\}$  and belong to subsystem  $i \in \{1, \dots, N\}$ . To achieve this we can use the structure present in the disjoint system decomposition. Due to the fact that we consider the decomposition of (2) to be disjoint, as given in (1), we have that the input vector  $\hat{u}_k$  and output vector  $y_k$  are ordered such that the set of indices corresponding to actuators  $\hat{u}_k$  and sensors  $y_k$  belonging to subsystem  $i$  are defined as

$$\mathbf{J}_i^u := \{\sum_{j=0}^{i-1} n_{u_j} + 1, \dots, \sum_{j=0}^i n_{u_j}\},$$

$$\mathbf{J}_i^y := \{\sum_{j=0}^{i-1} n_{y_j} + 1, \dots, \sum_{j=0}^i n_{y_j}\},$$

respectively, for  $i \in \{1, \dots, N\}$ , where  $n_{u_0} = n_{y_0} := 0$  and  $n_{u_i}$  and  $n_{y_i}$  denote the number of actuators and sensors, respectively, belonging to subsystem  $i \in \{1, \dots, N\}$ . With these sets defined, we have that the set  $\bar{\mathbf{J}}_l^u \cap \mathbf{J}_i^u$  consists exactly of the indices of the actuators that are associated with node  $l$  and belong to subsystem  $i$ . A similar interpretation holds for  $\bar{\mathbf{J}}_l^y \cap \mathbf{J}_i^y$ , regarding the indices of the sensors.

### B. Closed-Loop Model

To derive an expression for the closed-loop dynamics, we will define the state vector  $\bar{x}_k := [x_k^\top \ \hat{y}_{k-1}^\top \ \hat{u}_{k-1}^\top]^\top \in \mathbb{R}^n$ , where  $n = n_x + n_u + n_y$ . Combining (4), (7) and (8) results in the overall closed-loop system

$$\bar{x}_{k+1} = \tilde{A}_{\sigma_k, h_k} \bar{x}_k, \quad (10)$$

where

$$\tilde{A}_{l,h} = \begin{bmatrix} \bar{A}_{h_k} + \bar{B}_{h_k} \Gamma_{\sigma_k}^u K_{\sigma_k} \Gamma_{\sigma_k}^y C & \bar{B}_{h_k} \Gamma_{\sigma_k}^u K_{\sigma_k} (I - \Gamma_{\sigma_k}^y) & \bar{B}_{h_k} (I - \Gamma_{\sigma_k}^u) \\ \Gamma_{\sigma_k}^y C & (I - \Gamma_{\sigma_k}^y) & 0 \\ \Gamma_{\sigma_k}^u K_{\sigma_k} \Gamma_{\sigma_k}^y C & \Gamma_{\sigma_k}^u K_{\sigma_k} (I - \Gamma_{\sigma_k}^y) & (I - \Gamma_{\sigma_k}^u) \end{bmatrix}, \quad (11)$$

$l \in \{1, \dots, \bar{N}\}$ , and  $h \in \mathbb{R}$ . The closed-loop system (10) is a discrete-time switched linear parameter-varying (SLPV) system where the switching, as given by  $\sigma_k$ , is due to the communication medium being shared and the parameter uncertainty is caused by the uncertainty in the transmission interval  $h_k \in [\underline{h}, \bar{h}]$ .

### III. POLYTOPIC OVERAPPROXIMATION

In the previous section, we obtained a decentralized NCS model in the form of a switched uncertain system. However, the form of the model in (10), (11) is not convenient to develop efficient techniques for controller synthesis due to the nonlinear dependence of  $\tilde{A}_{\sigma_k, h_k}$  in (11) on the uncertain parameter  $h_k$ . To make the system amenable for synthesis, a procedure is employed to overapproximate system (10), (11) by a polytopic system with norm-bounded additive uncertainty, i.e.

$$\bar{x}_{k+1} = \sum_{m=1}^M \alpha_k^m (\mathcal{F}_{\sigma_k, m} + \mathcal{G}_m \Delta_k \mathcal{H}_{\sigma_k}) \bar{x}_k, \quad (12)$$

where  $\mathcal{F}_{l,m} \in \mathbb{R}^{n \times n}$ ,  $\mathcal{G}_m \in \mathbb{R}^{n \times 2n_x}$ ,  $\mathcal{H}_l \in \mathbb{R}^{2n_x \times n}$ , for  $l \in \{1, \dots, \bar{N}\}$  and  $m \in \{1, \dots, M\}$ , with  $M$  the number of vertices of the polytope. The vector

$\alpha_k = [\alpha_k^1 \dots \alpha_k^M]^\top \in \Omega$ , for all  $k \in \mathbb{N}$ , is time-varying with

$$\Omega = \left\{ \alpha \in \mathbb{R}^M \mid \sum_{m=1}^M \alpha^m = 1 \text{ and } \alpha^m \geq 0 \right. \\ \left. \text{for } m \in \{1, \dots, M\} \right\} \quad (13)$$

and  $\Delta_k \in \mathbf{\Delta}$ , for all  $k \in \mathbb{N}$ , with the additive uncertainty set  $\mathbf{\Delta} \subseteq \mathbb{R}^{2n_x \times 2n_x}$  given by

$$\mathbf{\Delta} = \left\{ \text{diag}(\Delta^1, \dots, \Delta^{2Q}) \mid \Delta^{q+jQ} \in \mathbb{R}^{n_{\lambda_q} \times n_{\lambda_q}}, \right. \\ \left. \|\Delta^{q+jQ}\| \leq 1, q \in \{1, \dots, Q\}, j \in \{0, 1\} \right\}, \quad (14)$$

where  $n_{\lambda_q} \times n_{\lambda_q}$ ,  $q \in \{1, \dots, Q\}$ , are the dimensions of the  $q^{\text{th}}$  real Jordan block [13] of  $A$  and  $Q$  is the number of real Jordan blocks of  $A$ . System (12) is an overapproximation of system (10) in the sense that

$$\left\{ \tilde{A}_{l,h} \mid h \in [\underline{h}, \bar{h}] \right\} \\ \subseteq \left\{ \sum_{m=1}^M \alpha^m (\mathcal{F}_{l,m} + \mathcal{G}_m \Delta \mathcal{H}_l) \mid \alpha \in \Omega, \Delta \in \mathbf{\Delta} \right\}, \quad (15)$$

for all  $l \in \{1, \dots, \bar{N}\}$ . Due to this inclusion, stability of (12) for all  $\alpha_k \in \Omega$  and  $\Delta_k \in \mathbf{\Delta}$ ,  $k \in \mathbb{N}$ , implies stability of (10) for all  $h_k \in [\underline{h}, \bar{h}]$ . Although many overapproximation techniques are available, see e.g. the survey [11], here we employ a gridding-based procedure based on [6] to overapproximate system (10), such that (15) holds. This choice is motivated by the favorable properties that this method possesses [11].

To obtain an overapproximation of (10) in the form (12), a set of grid points  $\{\tilde{h}_1, \dots, \tilde{h}_M\}$ , where  $\tilde{h}_m \in [\underline{h}, \bar{h}]$ ,  $m \in \{1, \dots, M\}$ , must be chosen. The choice of grid points directly influences the tightness of the overapproximation. There are procedures in the literature which determine the set of grid points  $\{\tilde{h}_1, \dots, \tilde{h}_M\}$  by iteratively placing each grid point at the location of the worst-case approximation error, thus, iteratively tightening the overapproximation. Due to lack of space we cannot provide the procedure here and instead refer the reader to [6] for details. Following a procedure similar to that of [6] leads to an overapproximation (12) of (10) satisfying (15), with

$$\mathcal{F}_{l,m} := \tilde{A}_{l, \tilde{h}_m}$$

for  $l \in \{1, \dots, \bar{N}\}$  and  $m \in \{1, \dots, M\}$  and, with  $B$  given in (2), we define

$$\mathcal{H}_l := \begin{bmatrix} T^{-1} & 0 & 0 \\ T^{-1}BK_l\Gamma_l^u C & T^{-1}BK_l(I - \Gamma_l^y) & T^{-1}B(I - \Gamma_l^y) \end{bmatrix}, \quad (16)$$

for  $l \in \{1, \dots, \bar{N}\}$  and

$$\mathcal{G}_m := \begin{bmatrix} T & T \\ 0 & 0 \\ 0 & 0 \end{bmatrix} U_m, \quad (17)$$

for  $m \in \{1, \dots, M\}$ , in which  $T$  is given by the real Jordan form decomposition [13] of the matrix  $A$ , as in (2), i.e.  $A := T\Lambda T^{-1}$ , where  $T$  is an invertible matrix and  $\Lambda = \text{diag}(\Lambda_1, \dots, \Lambda_Q)$  with  $\Lambda_q \in \mathbb{R}^{n_{\lambda_q} \times n_{\lambda_q}}$ ,  $q \in \{1, \dots, Q\}$ , the  $q^{\text{th}}$  real Jordan block of  $A$ . Additionally,

$$U_m := \text{diag}(\delta_{1,m}^A I_1, \dots, \delta_{Q,m}^A I_Q, \delta_{1,m}^E I_1, \dots, \delta_{Q,m}^E I_Q)$$

where  $I_q$  is the  $n_{\lambda_q} \times n_{\lambda_q}$  identity matrix and  $\delta_{q,m}$  is the worst case approximation error for each real Jordan block,  $\Lambda_q$ ,  $q \in \{1, \dots, Q\}$  and for each grid point  $m \in \{1, \dots, M\}$ . See [6] for details on how to compute  $\delta_{q,m}$ .

We care to stress that the most appealing aspect of this particular overapproximation technique is the fact that it introduces arbitrarily little conservatism when employed in (quadratic-type) Lyapunov-based stability analysis [6].

#### IV. DECENTRALIZED CONTROLLER SYNTHESIS

In this section, we will present the main contribution of this paper consisting of LMI-based conditions for *designing* the decentralized controller gains  $K_l$  in (4), by using the overapproximated model (12). Before we can use the model (12) for synthesis, an essential step must be taken so that (12) can be rewritten in a form suitable for controller synthesis.

The essential step in achieving LMI-based synthesis conditions is reformulating (11) such that the design variables are non-structured matrices, instead of the structured (block diagonal) matrices  $K_{\sigma_k}$ , as in (5), that are currently present. To achieve this, we define

$$\Upsilon_{l,i}^u := \begin{cases} (\Gamma_l^u)_{\bullet, \bar{\mathbf{J}}_l^u \cap \mathbf{J}_i^u}, & \text{if } l \in \mathbf{L}_{u,i}, \\ 0, & \text{otherwise,} \end{cases} \quad (18a)$$

$$\bar{\Upsilon}_i^y := (I^y)_{\bullet, \mathbf{J}_i^y}, \quad (18b)$$

for  $l \in \{1, \dots, \bar{N}\}$ , where  $I^y \in \mathbb{R}^{n_y \times n_y}$  is the identity matrix and

$$\mathbf{L}_{u,i} := \{l \in \{1, \dots, \bar{N}\} \mid \bar{\mathbf{J}}_l^u \cap \mathbf{J}_i^u \neq \emptyset\}, \quad (19)$$

are the sets of node indices which contain at least one actuator from subsystem  $i$ . Finally, we define

$$\bar{K}_{l,i} := \begin{cases} (K_l)_{\bar{\mathbf{J}}_l^u \cap \mathbf{J}_i^u, \mathbf{J}_i^y}, & \text{if } l \in \mathbf{L}_{u,i}, \\ 0, & \text{otherwise,} \end{cases} \quad (20)$$

for  $l \in \{1, \dots, \bar{N}\}$ . With these matrices defined, we have that the following equation holds for  $l \in \{1, \dots, \bar{N}\}$

$$\Gamma_l^u K_l = \sum_{i=1}^N \Upsilon_{l,i}^u \bar{K}_{l,i} \bar{\Upsilon}_i^{y\top}. \quad (21)$$

Now, (21) allows the closed-loop matrix  $\tilde{A}_{\sigma_k, h_k}$  in (11) to be expressed in terms of the non-structured matrices  $\bar{K}_{\sigma_k, i}$  instead of the structured (block diagonal) matrices  $K_{\sigma_k}$ . The benefit of this is that the synthesis problem for decentralized control, which naturally imposes ‘structural’ constraints, can now be formulated as a ‘non-structured’ synthesis problem.

Recall that we now have a system of the form (12), where the matrices  $\mathcal{F}_{l,m} = \tilde{A}_{l, \tilde{h}_m}$  are given by (11) with  $h \in \{\tilde{h}_1, \dots, \tilde{h}_M\}$ , and the matrices  $\mathcal{H}_l$  and  $\mathcal{G}_m$  are given in (16) and (17), respectively. Using (21), we can decompose  $\mathcal{F}_{l,m}$  and  $\mathcal{H}_l$  in the following way

$$\mathcal{F}_{l,m} = \mathcal{A}_{l,m} + \sum_{i=1}^N \mathcal{B}_{l,m,i} \bar{K}_{l,i} \mathcal{C}_{l,i}, \quad (22a)$$

$$\mathcal{H}_l = \mathcal{D}_l + \sum_{i=1}^N \mathcal{E}_{l,i} \bar{K}_{l,i} \mathcal{C}_{l,i}, \quad (22b)$$

where

$$\mathcal{A}_{l,m} := \begin{bmatrix} \bar{A}_{\bar{h}_m} & 0 & \bar{B}_{\bar{h}_m} (I - \Gamma_l^u) \\ \Gamma_l^y C & (I - \Gamma_l^y) & 0 \\ 0 & 0 & (I - \Gamma_l^u) \end{bmatrix},$$

$$\mathcal{B}_{l,m,i} := \begin{bmatrix} \bar{B}_{\bar{h}_m} \Upsilon_{l,i}^u \\ 0 \\ \Upsilon_{l,i}^u \end{bmatrix},$$

$$\mathcal{C}_{l,i} := [\bar{\Upsilon}_i^{y\top} \Gamma_l^y C \quad \bar{\Upsilon}_i^{y\top} (I - \Gamma_l^y) \quad 0],$$

$$\mathcal{D}_l := \begin{bmatrix} T^{-1} & 0 & 0 \\ 0 & 0 & T^{-1}B(I - \Gamma_l^u) \end{bmatrix}, \quad \mathcal{E}_{l,i} := \begin{bmatrix} 0 \\ T^{-1}B\Upsilon_{l,i}^u \end{bmatrix}.$$

Notice that (12) with (22) describes a discrete-time SLPV system with norm-bounded uncertainty. No results are available in the literature to synthesize the controller gains  $\bar{K}_l$ , at present. However, LMI-based synthesis conditions can be obtained by generalizing the results in [4], [5] in three directions. In particular, the first extension is the accommodation of norm-bounded uncertainty  $\mathcal{G}_m \Delta \mathcal{H}_{\sigma_k}$  in (12), where  $\Delta \in \mathbf{\Delta}$ , and the second extension is that the switching sequence (9) that we consider is ordered (periodic in this case), whereas [5] considered the case of arbitrary switching. Finally, the third extension is generalizing the set of LMI-based conditions in [5] so that solutions for the *multi-gain* switched static output-feedback problem can be included. Although the required extensions of the ideas in [5] contribute toward our main result, we would like to emphasize that using (21) for the formulation of (22) is the foundation upon which our main result is built. Stabilizing controller gains  $K_l$  for the NCS given by (10) with  $h_k \in [\underline{h}, \bar{h}]$  and a protocol satisfying (9) can be synthesized according to the following theorem. To state our main contribution below, the matrix set

$$\mathbf{R} := \{\text{diag}(r_1 I_1, \dots, r_Q I_Q, r_{Q+1} I_1, \dots, r_{2Q} I_Q) \in \mathbb{R}^{2n_x \times 2n_x} \mid r_{\bar{q}} > 0\} \quad (23)$$

is used, where  $I_q$  is the  $n_{\lambda_q} \times n_{\lambda_q}$  identity matrix complying with the  $q^{\text{th}}$  real Jordan Block of  $A$ .

**Theorem 1** Consider the system (10), (11) with  $h_k \in [\underline{h}, \bar{h}]$ ,  $k \in \mathbb{N}$ , and its overapproximation given by (12), (17), (22). Furthermore, assume that the protocol satisfies (9)

and any node  $l \in \{1, \dots, \bar{N}\}$ , which contains at least one sensor from subsystem  $i$ , i.e.  $\bar{\mathbf{J}}_l^y \cap \mathbf{J}_i^y \neq \emptyset$ , consists of linearly independent subsystem sensors, i.e.  $(C)_{\bar{\mathbf{J}}_l^y \cap \mathbf{J}_i^y, \bullet}$  has full row rank. Now, suppose there exist symmetric matrices  $P_j$ , matrices  $R_{j,m} \in \mathbf{R}$ , with  $\mathbf{R}$  as in (23), and matrices  $G_l$ ,  $Z_{l,i}$  and  $X_{l,i}$  where  $j \in \{1, \dots, \bar{N}\}$ ,  $m \in \{1, \dots, M\}$ ,  $l \in \{1, \dots, \bar{N}\}$ ,  $i \in \{1, \dots, N\}$  such that

$$\left[ \begin{array}{cc|cc} G_{\sigma_j} + G_{\sigma_j}^\top - P_j & \Xi_1(j, m)^\top & 0 & \Xi_2(j)^\top \\ \star & P_{j+1} & \mathcal{G}_m R_{j,m} & 0 \\ \star & \star & R_{j,m} & 0 \\ \star & \star & \star & R_{j,m} \end{array} \right] \succ 0, \quad (24)$$

for  $j \in \{1, \dots, \bar{N}\}$ ,  $m \in \{1, \dots, M\}$ , and

$$X_{l,i} \mathcal{C}_{l,i} = \mathcal{C}_{l,i} G_l, \quad \text{for } l \in \mathbf{L}_{u,i}, \quad i \in \{1, \dots, N\} \quad (25)$$

for which we define

$$\begin{aligned} \Xi_1(j, m) &:= \mathcal{A}_{\sigma_j, m} G_{\sigma_j} + \sum_{i=1}^N \mathcal{B}_{\sigma_j, m, i} Z_{\sigma_j, i} \mathcal{C}_{\sigma_j, i}, \\ \Xi_2(j) &:= \mathcal{D}_{\sigma_j} G_{\sigma_j} + \sum_{i=1}^N \mathcal{E}_{\sigma_j, i} Z_{\sigma_j, i} \mathcal{C}_{\sigma_j, i}, \end{aligned}$$

for  $j \in \{1, \dots, \bar{N}\}$ ,  $m \in \{1, \dots, M\}$ , with  $P_{\bar{N}+1} := P_1$  and the set  $\mathbf{L}_{u,i}$  is given in (19). Then the controller gains  $K_l$ , defined by (21), (20) and  $\bar{K}_{l,i} = Z_{l,i} X_{l,i}^{-1}$ ,  $l \in \mathbf{L}_{u,i}$ ,  $i \in \{1, \dots, N\}$ , render the system (10), with  $h_k \in [\underline{h}, \bar{h}]$ ,  $k \in \mathbb{N}$ , and the mentioned periodic protocol, UGES.

## V. EXAMPLE

In this section, we will illustrate the presented theory with a numerical example. The unstable plant that we aim to stabilize is in the form (2) where the matrices  $A$ ,  $B$ , and  $C$  are given by (30). The three disjoint systems are indicated by dashed lines. We will compare three different controller structures, denoted C1, C2 and C3.

**C1** - The first controller (C1) is a centralized controller ( $N = 1$ ) of the form (4) where the communication medium is not shared, meaning all sensors and actuators are in one node ( $\bar{N} = 1$ ) and  $\Gamma_1^u = \Gamma_1^y = I$ . This is the simplest setting for which Theorem 1 applies.

**C2** - The second controller (C2) is a decentralized controller ( $N = 3$ ) of the form (4) where the decentralized structure is indicated in (30) with the dashed lines and the communication medium is not shared, meaning all sensors and actuators are in one node ( $\bar{N} = 1$ ) and  $\Gamma_1^u = \Gamma_1^y = I$ .

$$\left[ \begin{array}{c|c} A & B \\ \hline C & \end{array} \right] = \begin{bmatrix} -3.7 & 0 & 0 & -1.0 & 0.1 & -2.0 & 0.4 & -0.4 & 0.4 & 0.7 & 0.9 & -3.6 & -3.9 & 2.7 \\ 0 & -5.1 & 0.7 & -0.2 & -0.9 & 1.2 & 0.5 & 0.9 & 0.3 & 1.9 & 2.4 & -1.2 & 2.0 & -3.9 \\ 0 & 0.7 & -2.6 & -0.1 & -0.5 & -0.7 & -0.6 & 0.6 & -0.8 & -1.5 & 3.4 & 2.5 & -0.4 & 1.5 \\ -1.0 & -0.2 & -0.1 & -3.7 & -1.3 & -0.6 & 2.1 & -1.8 & -1.4 & -0.3 & -1.9 & -4.0 & -3.5 & -1.0 \\ 0.1 & -0.9 & -0.5 & -1.3 & -4.5 & -0.3 & 1.8 & -0.9 & 0.8 & -0.1 & -2.6 & -2.9 & -0.3 & 2.7 \\ -2.0 & 1.2 & -0.7 & -0.6 & -0.3 & -6.0 & 0.5 & -1.6 & 0.4 & -0.1 & -0.8 & -2.4 & -0.7 & 0 \\ 0.4 & 0.5 & -0.6 & 2.1 & 1.8 & 0.5 & -3.1 & 1.8 & -0.2 & 0 & 3.5 & -2.4 & 2.8 & -1.7 \\ -0.4 & -0.9 & -0.6 & -1.8 & -0.9 & -1.6 & 1.8 & -3.9 & 0.8 & -0.6 & -3.4 & 0.8 & 0.2 & -0.6 \\ 0.4 & 0.3 & -0.8 & -1.4 & 0.8 & 0.4 & -0.2 & 0.8 & -3.7 & 0.7 & -0.7 & -1.8 & -2.4 & -1.6 \\ 0.7 & 1.9 & -1.5 & 0.3 & -0.1 & -0.1 & 0 & -0.6 & 0.7 & -1.3 & 3.2 & -2.4 & 1.4 & -2.5 \\ \hline -2.5 & -1.0 & 2.6 & -1.3 & -1.0 & 2.4 & -1.9 & 3.2 & -3.5 & -0.5 & & & & \\ 1.5 & 2.9 & 1.2 & 0.3 & 1.6 & 3.7 & -2.0 & -2.4 & 4.0 & -2.2 & & & & \\ -1.6 & -2.9 & 2.6 & 1.8 & 0.4 & -0.2 & 3.0 & -1.6 & 0.7 & -0.6 & & & & \\ 0.3 & 0.8 & 1.3 & -1.5 & -0.4 & 3.1 & 1.9 & 1.3 & -0.6 & 2.1 & & & & \\ -2.8 & 0 & -1.3 & 2.7 & 1.6 & -2.6 & -2.9 & -1.7 & 0.1 & 0.2 & & & & \\ 1.6 & 3.2 & -1.7 & 0.6 & 1.0 & 3.9 & -4.0 & -0.2 & -1.3 & 1.1 & & & & \end{bmatrix} \quad (30)$$

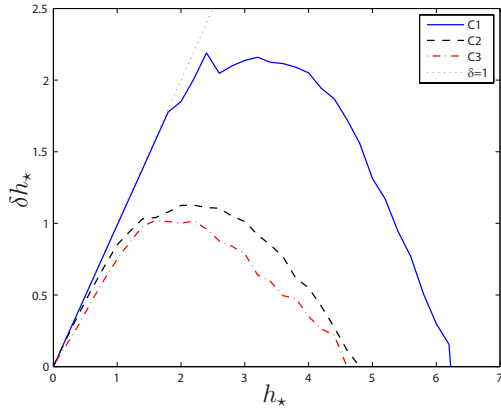


Fig. 2. Maximum  $\frac{1}{2}(\bar{h} - \underline{h}) = \delta h_*$  for which stabilizing controller gains for the system (2), (30) could be synthesized for C1, C2 and C3 using Theorem 1.

**C3** - The third controller (C3) is a decentralized controller ( $N = 3$ ) of the form (4) where the decentralized structure is indicated in (30) with the dashed lines and the communication medium is shared. We specify that there are  $\tilde{N} = 2$  nodes, where  $\Gamma_1^u = \text{diag}(1, 1, 0, 0)$ ,  $\Gamma_2^u = \text{diag}(0, 0, 1, 1)$ ,  $\Gamma_1^y = \text{diag}(1, 1, 1, 0, 0, 0)$  and  $\Gamma_2^y = \text{diag}(0, 0, 0, 1, 1, 1)$ . We specify the protocol to be the well-known Round-Robin protocol given by (9) with  $\sigma_l = l$ ,  $l \in \{1, 2\}$  and  $\tilde{N} = 2$ .

For each of the controllers C1, C2 and C3 we took different values of a nominal transmission interval, denoted  $h_*$ , and used the YALMIP interface [14] with the SeDuMi solver [20] to verify the conditions of Theorem 1 in order to find stabilizing gains  $K_l$  which maximize  $\delta \in [0, 1)$  such that the NCS (10) is stable for  $[\underline{h}, \bar{h}] = [(1 - \delta)h_*, (1 + \delta)h_*]$ ,  $k \in \mathbb{N}$ , i.e. we consider a symmetric uncertainty interval around  $h_*$ . For C1, C2 and C3 we considered an overapproximation (12) of the closed-loop dynamics (10) with  $M = 10$  grid points.

The result of applying Theorem 1 to C1, C2 and C3 is plotted in Fig. 2. The regions for which closed-loop stability can be guaranteed for C1, C2 and C3 lie below the lines corresponding to C1, C2 and C3. One can see that, as expected, the case with synthesizing a centralized controller and a communication medium which is not shared (C1) yields the largest stable region. Synthesizing a controller which is stabilizing in the presence of a shared communication medium and decentralized structural constraints (C3) results the smallest region.

## VI. CONCLUSION

In this paper, we have presented LMI-based synthesis conditions for designing decentralized static controllers in the presence of a shared communication medium, which are robust with respect to time-varying transmission intervals and time-varying delays. This paper presented direct LMI-based synthesis conditions using an (arbitrarily tight) overapproximated model. This achievement was realized by expressing the controller design problem as a multi-gain switched static output-feedback problem (with additive uncertainty), for which the gains can be efficiently solved by LMI-based feasibility conditions. These LMI-based synthesis conditions, if satisfied, provide stabilizing gains for both the decentralized problem setting and the NCS problem setting

in isolation, as well as the unification of these two problem settings.

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