An ℓ_2-Consistent Event-triggered Control Policy for Linear Systems

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Abstract

In this article, we consider the design of an event-triggered ℓ_2-control policy, for a setting where a scheduler is arbitrating state transmissions from the sensors to the controller of a discrete-time linear system. We start by introducing a periodic time-triggered ℓ_2-controller for different transmission time-periods with a given ℓ_2-gain bound using the minimax game-theoretical approach. After that, we propose an ℓ_2-consistent event-triggered controller in the sense that it guarantees at least the same ℓ_2-gain bound as the designed periodic time-triggered ℓ_2-controller, however with a larger, or at most equal, average inter-transmission time. In practice, for typical disturbances, the proposed event-triggered scheme can lead to significant gains, both in terms of communication savings and disturbance attenuation, compared to periodic time-triggered policies, which is illustrated through a numerical example.

Key words: Networked control systems, ℓ_2-consistent event-triggered controller, ℓ_2-gain, Dynamic game theory.

1 Introduction

The advent of new communication technologies, such as 5G, will further facilitate the rapid expansion of networked control systems (NCS) in many (industrial) branches of our society in the years to come. In NCSs, sensors, controllers and actuators communicate through shared communication networks. Applications include vehicle platooning, cloud-based control, smart grids, and robot swarms. In configurations where communication between agents happens periodically, the well-developed theory of sampled-data control (Chen and Francis, 2012) can be used to guarantee stability and performance of these systems. However, periodic communication for control applications can be rather resource-inefficient. In fact, control applications require large bandwidth for high communication frequencies and, when relying on wireless technologies, can lead to a large power consumption, which can be prohibitive when using battery powered communication devices. Therefore, managing and reducing the communication between sensors, controllers and actuators is crucial in many networked control applications.

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Event-triggered controllers (ETCs) have been proposed in the literature as an alternative to periodic time-triggered controllers in order to decrease the communication load in NCSs, while at the same time preserving stability and performance requirements see, e.g., Åström and Bernhardsson (2002); Tabuada (2007); Heemels et al. (2008); Lunze and Lehmann (2010); Heemels et al. (2012); Molin and Hirche (2014); Behera et al. (2018); Nowzari et al. (2019) and the references therein. In a loop with an ETC, data transmissions between agents (sensors, controllers, actuators) are triggered based on well-defined events such as abrupt changes in the value of data or when estimation errors exceed certain thresholds. A large number of studies has been carried out so far in this research area with promising results in reducing the communication burden of the control loops, see, e.g., Mazo and Tabuada (2008); Postoyan et al. (2011); Wu et al. (2013); Araujo et al. (2014); Antunes and Heemels (2014); Weerakkody et al. (2016) and Mastrangelo et al. (2019). In some studies, ETCs are designed in order to guarantee stability of the system (Mazo and Tabuada, 2008; Postoyan et al., 2011; Mamduhi et al., 2017). Others also provide guarantees on an average quadratic cost of the event-triggered control-loops (Araujo et al., 2014; Antunes and Heemels, 2014; Goldenshluger and Mirkin, 2017; Balaghi I. and Antunes,

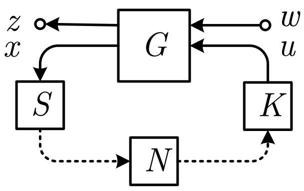

Fig. 1. The state feedback ℓ_2-controller with a resource-constraint communication network. G, K, S and N refer to the plant, the controller, the scheduler and the network, respectively.

2017; Brunner et al., 2018; Asadi Khashooei et al., 2018; Balaghi I. et al., 2018).

Another important performance criterion for control-loops is the ℓ_2- or $\mathcal{L}_2$-gain, which captures the worst-case disturbance attenuation level from an exogenous input to a performance output of the control loop for discrete-time or continuous-time systems, respectively. In a networked control configuration with communication limitations, the setup as depicted in Fig. 1 is of interest, where a feedback controller K attenuates the effect of the disturbance input w on the performance output z of the plant G. Here, a scheduler S determines the time instances when the measured state should be communicated to the controller through a communication network N. The event-triggered scheduler should be designed together with an appropriate controller to guarantee a certain ℓ_2- or $\mathcal{L}_2$-gain bound for the closed-loop system, while the available communication network should be able to handle the required data transmissions.

In recent years, researchers took different approaches in order to design event-triggered ℓ_2- or $\mathcal{L}_2$-controllers. In particular, conditions for the $\mathcal{L}_2$-stability of the proposed event-triggered transmission policies in a sampled-data control system configuration are given in Yan et al. (2015); Peng and Han (2013), by constructing Lyapunov-Krasovskii functionals. In Kishida et al. (2017), finite-gain $\mathcal{L}_2$-stability is guaranteed for an uncertain linear system by jointly designing an event-triggered mechanism in updating the control inputs and a self-triggered mechanism in determining the next sampling time of the sensors. The exponential stability and $\mathcal{L}_2$-gain analysis of a NCS, where the sensor to controller and the controller to actuator communications are both based on event-triggered mechanisms, is studied by using the delay system approach in Hu and Yue (2013). Moreover, there are some other studies establishing the $\mathcal{L}_2$-stability of the systems with ETCs (Wang and Lemmon, 2009; Yu and Antsaklis, 2013) or providing guaranteed values for the ℓ_2-gain of discrete-time linear systems with an ETC (Heemels et al., 2013). In addition, an ETC is designed for output-feedback linear systems by considering the $\mathcal{L}_\infty$-gain of the closed-loops in Donkers and Heemels (2010). For nonlinear systems, ETCs are proposed in Dolk et al. (2017) and Abdelrahim et al. (2017) that guarantee a finite $\mathcal{L}_p$-gain for closed-loop systems and prevent the Zeno behaviour in data transmissions.

In principle, employing an ETC in NCSs is beneficial only if it results in a better performance in comparison to time-triggered periodic control when both transmit with the same average transmission rate. This concept was first introduced in Antunes and Asadi Khashooei (2016) and referred to as consistency. In recent years, consistent ETCs in the sense of average quadratic cost have been proposed in both centralized and decentralized NCS configurations, see, e.g., Goldenshluger and Mirkin (2017); Brunner et al. (2018); Asadi Khashooei et al. (2018); Balaghi I. et al. (2018), see also an early result for scalar systems in Åström and Bernhardsson (2002).

We can also extend the notion of consistency to event-triggered ℓ_2- or $\mathcal{L}_2$-control loops. Accordingly, an ETC is called ℓ_2- or $\mathcal{L}_2$-consistent if it guarantees the same ℓ_2- or $\mathcal{L}_2$-gain bound as any periodic time-triggered ℓ_2- or $\mathcal{L}_2$-controller, however, with a smaller or at most the same average transmission rate (Balaghi I. et al., 2019). In spite of all works previously mentioned in the context of event-triggered ℓ_2- or $\mathcal{L}_2$-control, the design of an ℓ_2- or $\mathcal{L}_2$-consistent ETC has not received much attention so far. In fact, we are only aware of two very recent references related to our work, see Balaghi I. et al. (2019) and Mi and Mirkin (2019). Our previous work (Balaghi I. et al., 2019) differs from the present paper as it focusses on designing a fixed, a priori given, transmission sequence, and not a policy, while Mi and Mirkin (2019) derive an ETC with similar $\mathcal{L}_2$-consistent properties as the one we present in this paper. However, they are given for continuous-time systems (and thus $\mathcal{L}_2$-gain), and, most importantly, follow a very different approach based on the Youla parametrization, whereas we consider discrete-time systems and follow a game-theoretical approach. As both results are developed independently and follow different approaches for different settings, they are of independent interest.

To be precise, in this work, for a given fixed transmission time period, we design a periodic time-triggered ℓ_2-controller for any feasible ℓ_2-gain bound, following a game-theoretical approach. Then, we design an ETC guaranteeing an equal ℓ_2-gain bound as that of the designed periodic time-triggered ℓ_2-controller, however, with a larger (or at least equal) average inter-transmission time. In fact, based on our proposed ETC, when the realization of the disturbance input follows the worst-case scenario, then the proposed ETC triggers data transmissions periodically. However, when the disturbance input deviates from the worst-case scenario, then our proposed ETC is able to skip data transmissions thereby guaranteeing a larger average inter-transmission time than the time period of the periodic controller, while they both guarantee the same ℓ_2-gain bound for the system.

Implicit in the NCS of interest in the current work it that (possibly large) packets of information are sent to the controller (and there is no error in the transmitted values) and it is the objective of the scheduler to keep the number of transmissions (average communication rate) as small as possible, while guaranteeing certain performance objectives. An alternative perspective, also considered in the literature (see, e.g., Ishii and Francis (2002)), is to keep the communication frequency constant, but reduce the size of the packets to be transmitted (and thus there is a discrepancy between the actual measurements and the transmitted quantized value) and thereby also realize a small bit rate. The problem of interest in this line of research is to determine the accuracy (or the number of bits) of each communicated data packet on the relation of a given control objective or to find the minimal number of bits needed in order to realize a certain objective. Although not considered in this paper, there are also some recent works, where this idea is jointly employed with event-triggered transmission mechanisms, which can at the same time reduce the communication frequency as is investigated, for instance, in Tallapragada and Cortés (2016); Abdelrahim et al. (2019); Ling (2020), all also the references therein, to achieve exponential and input-to-state stability for linear systems, respectively.

The remainder of this paper is organized as follows. The problem of interest is introduced in Section 2 and an ℓ_2-consistent ETC is proposed in Section 3. The effectiveness of the novel ETC in decreasing the communication load is demonstrated through a numerical example in Section 4. Finally, Section 5 presents concluding remarks. The proofs of lemmas and theorems can be found in the appendix.

Notation: For $r,s\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, we define $\mathbb{N}_r^s=\{t\in\mathbb{N}_0|r\leqslant t\leqslant s\}$ and ℓ_2^d as the Hilbert space of square summable sequences $w:=\{w_k\}_{k\in\mathbb{N}_0}$, where $w_k\in\mathbb{R}^d$ for all $k\in\mathbb{N}_0$, and $\sum_{k=0}^{\infty} w_k^\top w_k<\infty$. The ℓ_2-norm of $w\in\ell_2^d$ is given by $\|w\|_{\ell_2}:=\sqrt{\sum_{k=0}^{\infty}\|w_k\|^2}$, where $\|w_k\|^2=w_k^\top w_k$. Moreover, $\lfloor x\rfloor$ indicates the floor of an $x\in\mathbb{R}$, and for matrices A, and B, we define $\mathrm{diag}(A,B)$ for the corresponding block diagonal matrix.

2 Problem setting

We introduce the NCS with periodic communication in Section 2.1 and the NCS with event-triggered communication in Section 2.2. The problem of interest is stated in Section 2.3.

2.1 Networked control system with periodic communication

Consider the system architecture in Fig. 1 in which the plant G is given by a discrete-time linear time-invariant (LTI) system

$$x_{k+1}=Ax_k+Bu_k+Dw_k, \tag{1}$$

where $x_k\in\mathbb{R}^n$, $u_k\in\mathbb{R}^m$, $w_k\in\mathbb{R}^d$ are the state, the control input and the disturbance, respectively, at discrete time $k\in\mathbb{N}_0$. Let $w\in\ell_2^d$ and assume that the disturbance generator at every time step has access to all the state vectors from the initial up to the current time-step. Therefore, $w_k=\mathcal{T}_k(\mathcal{E}_k)$, where

$$\mathcal{E}_k:=\{x_i|i\in\mathbb{N}_0^k\}, \tag{2}$$

for some mapping $\mathcal{T}_k:\mathcal{E}_k\to\mathbb{R}^d$, $k\in\mathbb{N}_0$. Moreover, let $\delta_k=1$ if x_k is transmitted to the controller at time $k\in\mathbb{N}_0$ and let $\delta_k=0$, otherwise. For the periodic transmission policy with a given time period $\tau\in\mathbb{N}$, we set $\delta_k=\pi_k^\tau$, where

$$\pi_k^\tau:=\begin{cases}1, & \text{if } k \text{ is zero or an integer multiple of } \tau\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

Then any periodic control policy can be formulated as

$$u_k:=\mathcal{R}_k^{\pi_\tau}(\mathcal{F}_k^{\pi_\tau}), \tag{4}$$

where at every $k\in\mathbb{N}_0$,

$$\mathcal{F}_k^{\pi_\tau}:=\{x_i|i\in\mathbb{N}_0^k\wedge\pi_i^\tau=1\} \tag{5}$$

is the information set available for the controller and $\mathcal{R}_k^{\pi_\tau}:\mathcal{F}_k^{\pi_\tau}\to\mathbb{R}^m$ is an appropriate mapping. Although we use here this general definition, in practice, the periodic control policies of interest (see Lemmas 1 and 2 below) will only depend on the last transmitted state. Therefore, the controller does not need to store all the received state vectors in memory (which can possibly require a large memory). The goal of an ℓ_2-controller is to attenuate the effect of the disturbance input w_k on the performance output

$$z_k:=\left[(Ex_k)^\top \quad (Fu_k)^\top\right]^\top \tag{6}$$

of the system, where we assume that F has full column rank. Let $E^\top E=Q$ and without loss of generality we can now assume $F^\top F=I$. Therefore, $\|z_k\|^2=x_k^\top Qx_k+u_k^\top u_k$ at every $k\in\mathbb{N}_0$. We need the following assumptions and the definition of global asymptotic stability in the sequel.

Assumption 1 *It holds that*

(i) (A,B) is stabilizable and $(Q^{\frac{1}{2}},A)$ is observable,
(ii) D is full column rank. □

Note that these assumptions are rather standard in the ℓ_2-control context (see also Point 3. after Theorem 1 below).

Definition 1 (Global asymptotic stability (Aliyu, 2017)) *The system (1) with $w=(0,0,\dots)$ and a given control input policy is said to be globally asymptotically stable (at equilibrium point $x_e=0$), if*

(i) the control loop is Lyapunov stable, i.e., for every $\zeta>0$, there exists a $\delta>0$ such that for all initial states $x_0\in\mathbb{R}^n$ with $\|x_0\|\leqslant\delta$, it holds that $\|x_k\|\leqslant\zeta$ for every $k\in\mathbb{N}_0$,

(ii) the corresponding state trajectory x_k converges to $x_e=0$ as time goes to infinity, i.e., $\lim_{k\to\infty} x_k=0$. □

Next, we formally define the concept of τ-periodic ℓ_2-controller for the system (1).

Definition 2 (τ-Periodic ℓ_2-controller (Aliyu, 2017)) *Given $\gamma\in\mathbb{R}_{>0}$ and $\tau\in\mathbb{N}$, a periodic control policy $\mathcal{R}_k^{\pi_\tau}:\mathcal{F}_k^{\pi_\tau}\to\mathbb{R}^m$, $k\in\mathbb{N}_0$, for the system (1) and (6), where $\mathcal{F}_k^{\pi_\tau}$ follows (5), such that*

(i) the closed-loop control system given by (1) and (4) is globally asymptotically stable when $w=(0,0,\dots)$,

(ii) when $x_0=0$,[1] then for all $w\in\ell_2^d$,

$$\|z\|_{\ell_2}^2-\gamma^2\|w\|_{\ell_2}^2\leqslant-\epsilon\|w\|_{\ell_2}^2 \tag{7}$$

holds for some positive ϵ (independent of w),

is referred to as a τ-periodic ℓ_2-controller with ℓ_2-gain bound γ. Moreover, the infimum value of $\gamma\in\mathbb{R}_{>0}$ for which a τ-periodic ℓ_2-controller exists with ℓ_2-gain bound γ is called the infimal ℓ_2-gain of (1) and (6), and is denoted by γ_τ^*. □

Let us define

$$J:=\|z\|_{\ell_2}^2-\gamma^2\|w\|_{\ell_2}^2, \tag{8}$$

for all $w\in\ell_2^d$. Based on (7), when $w=(0,0,\dots)$, the designed τ-periodic ℓ_2-controller should result in J to be equal or less than zero for $x_0=0$. However, when $w\neq(0,0,\dots)$, then J should always be strictly less than zero. Before designing a τ-periodic ℓ_2-controller with ℓ_2-gain bound γ, one should evaluate the existence of such a controller for the given value of $\gamma\in\mathbb{R}_{>0}$, i.e., decide if $\gamma>\gamma_\tau^*$. However, for a given $\gamma\in\mathbb{R}_{>0}$, a τ-periodic ℓ_2-controller with ℓ_2-gain bound γ exists if and only if, for $x_0=0$, the following minimax optimization problem results in a non-positive value, i.e. $J^*\leqslant 0$, where

$$J^*=\min_{\{u_k=\mathcal{R}_k^{\pi_\tau}(\mathcal{F}_k^{\pi_\tau})\}_{k\in\mathbb{N}_0}}\max_{\{w_k=\mathcal{T}_k(\mathcal{E}_k)\}_{k\in\mathbb{N}_0}} J. \tag{9}$$

This can be concluded from the arguments in Başar and Bernhard (2008). Therefore, the infimal ℓ_2-gain of the closed-loop system with τ-periodic transmission is the infimum value of the set of $\gamma\in\mathbb{R}_{>0}$ for which the minimax problem (9) has a non-positive value. Moreover, if for a given $\gamma\in\mathbb{R}_{>0}$, J^* is non-positive, then the optimal control policy determined based on (9) is a τ-periodic ℓ_2-controller in the sense of Definition 2. In the following two lemmas, we provide a τ-periodic ℓ_2-controller with ℓ_2-gain bound γ by solving the minimax problem (9). Lemma 1 considers the special case $\tau=1$ and Lemma 2 provides the results for general $\tau\in\mathbb{N}$.

Lemma 1 (1-Periodic ℓ_2-controller) *Let Assumption 1 hold. Then*

(i) there exists a $\hat{\gamma}\in\mathbb{R}_{>0}$ such that for every $\gamma>\hat{\gamma}$, the Ricatti equation

$$M=A^\top MH^{-1}A+Q, \tag{10}$$

where $H=I+(BB^\top-\gamma^{-2}DD^\top)M$, has a positive definite solution M and $\gamma^2I-D^\top MD>0$. Moreover, the infimum value of $\hat{\gamma}$ for which the above holds coincides with the infimal ℓ_2-gain of 1-periodic ℓ_2-controllers, i.e. γ_1^*.

(ii) for any $\gamma>\gamma_1^*$, the control policy

$$u_k^*=Kx_k, \tag{11}$$

where

$$K=-B^\top MH^{-1}A, \tag{12}$$

is a 1-periodic ℓ_2-controller with ℓ_2-gain bound γ.

(iii) for any $\gamma>\gamma_1^*$ and $\tau=1$, the performance index (8) is upper bounded as

$$\begin{aligned}J\leqslant\textstyle\sum_{k=0}^{\infty}\big[&(u_k-u_k^*)^\top\Phi_1(u_k-u_k^*)\\&-(Dw_k-Dw_k^*)^\top\Psi_1(Dw_k-Dw_k^*)\big],\end{aligned} \tag{13}$$

for $\Phi_1=(I-B^\top MH^{-1}B)^{-1}$, where $\Phi_1\geqslant I$, $\Psi_1=\gamma^2D(D^\top D)^{-1}(D^\top D)^{-1}D^\top-M$, where $\Psi_1>0$, and

$$w_k^*=\gamma^{-2}D^\top M(I-\gamma^{-2}DD^\top M)^{-1}(Ax_k+Bu_k). \tag{14}$$

□

Before considering the general condition in Lemma 2, where $\tau\in\mathbb{N}$, let us introduce another time variable $\iota\in\mathbb{N}_0$, where $\iota=\lfloor\frac{k}{\tau}\rfloor$ and define the following augmented control and disturbance inputs at every $\iota\in\mathbb{N}_0$,

$$\begin{aligned}U_\iota&:=[u_{\iota\tau}^\top,\dots,u_{(\iota+1)\tau-1}^\top]^\top,\\W_\iota&:=\big[(Dw_{\iota\tau})^\top,\dots,(Dw_{(\iota+1)\tau-1})^\top\big]^\top.\end{aligned} \tag{15}$$

Lemma 2 (τ-Periodic ℓ_2-controller for $\tau\in\mathbb{N}$) *Let Assumption 1 hold. Then*

[1] We can easily investigate the condition with an unknown initial condition by adding one extra time to the time-horizon and considering the initial condition as the disturbance of the previous time (Başar and Bernhard, 2008).

(i) *there exists a $\hat{\gamma}\in\mathbb{R}_{>0}$ such that for every $\gamma>\hat{\gamma}$,* (10) *has a positive definite solution M and $\gamma^2 I-\bar{D}_\tau^\top \bar{M}_\tau \bar{D}_\tau>0$ for which $\bar{M}_\tau=diag(I_{\tau-1}\otimes Q, M)$ and*

$$\bar{D}_\tau=\begin{bmatrix} D & 0 & 0 \\ AD & D & 0 \\ & & \cdots \\ A^{\tau-1}D & A^{\tau-2}D & D \end{bmatrix}_{\tau\times\tau}.$$

Moreover, the infimum value of $\hat{\gamma}$ for which the above holds coincides with the infimal ℓ_2-gain of τ-periodic ℓ_2-controllers for $\tau\in\mathbb{N}$, i.e. γ_τ^*.

(ii) *for any $\gamma>\gamma_\tau^*$, the control policy*

$$u_k^*=K\hat{x}_{k|k}, \tag{16}$$

where K follows (12) *and*

$$\hat{x}_{k+1|k}=H^{-1}A\hat{x}_{k|k}, \quad \hat{x}_{k|k}=\begin{cases} x_k, & \text{if } \pi_k^\tau=1 \\ \hat{x}_{k|k-1}, & \text{otherwise,} \end{cases} \tag{17}$$

is a τ-periodic ℓ_2-controller with ℓ_2-gain bound γ.

(iii) *consider*

$$\begin{aligned} U_\iota^* &= [u_{\iota\tau}^{*\top},\dots,u_{(\iota+1)\tau-1}^{*\top}]^\top, \\ W_\iota^* &= \big[(Dw_{\iota\tau})^{*\top},\dots,(Dw_{(\iota+1)\tau-1})^{*\top}\big]^\top, \end{aligned} \tag{18}$$

where u_k^* follows (16) *for all $k\in\mathbb{N}_0$, and*

$$w_k^*=\begin{cases} \bar{S}_h(Ax_k+Bu_k), & \text{if } h=\tau-1 \\ \bar{S}_h(Ax_k+Bu_k)+\Pi_{h+1}\tilde{U}_{k+1}, & \text{otherwise,} \end{cases} \tag{19}$$

where $h=k-\iota\tau$, $h\in\mathbb{N}_0^{\tau-1}$,

$$\begin{aligned} &\bar{S}_h=\gamma^{-2}D^\top\Theta_{h+1}V_h^{-1}, \quad V_h=I-\gamma^{-2}DD^\top\Theta_{h+1}, \\ &\Pi_h=\gamma^{-2}D^\top(I-\gamma^{-2}DD^\top\Theta_h)^{-\top}Z_h, \\ &\Theta_{h+1}=\begin{cases} Q+A^\top\Theta_{h+2}V_{h+1}^{-1}A, & \text{if } h\in\mathbb{N}_0^{\tau-2} \\ M, & \text{if } h=\tau-1, \end{cases} \\ &Z_h=\begin{cases} \big[A^\top\Theta_{h+1}V_h^{-1}B \quad A^\top V_h^{-\top}Z_{h+1}\big], & \text{if } h\in\mathbb{N}_0^{\tau-2} \\ A^\top M(I-\gamma^{-2}DD^\top M)^{-1}B, & \text{if } h=\tau-1, \end{cases} \end{aligned} \tag{20}$$

and $\tilde{U}_{k+1}=[u_{k+1}^\top\dots u_{(\iota+1)\tau-1}^\top]^\top$. Then, for any $\gamma>\gamma_\tau^*$, the performance index (8) *is upper bounded as*

$$\begin{aligned} J\leqslant\textstyle\sum_{\iota=0}^{\infty}&[(U_\iota-U_\iota^*)^\top\Phi_\tau(U_\iota-U_\iota^*) \\ &-(W_\iota-W_\iota^*)^\top\Psi_\tau(W_\iota-W_\iota^*)], \end{aligned} \tag{21}$$

where $\Phi_\tau:=Y_0$, for all $\tau\in\mathbb{N}$, and Y_0 is determined based on the following backward iteration

$$Y_h^{-1}=\begin{bmatrix} I & 0 \\ 0 & Y_{h+1}^{-1} \end{bmatrix}-\begin{bmatrix} B^\top & 0 \\ 0 & \bar{B}_{h+1} \end{bmatrix}X\begin{bmatrix} B & 0 \\ 0 & \bar{B}_{h+1}^\top \end{bmatrix}, \tag{22}$$

for all $h\in\mathbb{N}_0^{\tau-2}$, where $Y_{\tau-1}^{-1}=I-B^\top MH^{-1}B$,

$$X=\begin{bmatrix} MH^{-1} & H^{-1} \\ H^{-1} & M^{-1}(H^{-1}-I) \end{bmatrix},$$

and for all $h\in\mathbb{N}_0^{\tau-1}$,

$$\bar{B}_h=-\begin{bmatrix} K \\ K(H^{-1}A) \\ \vdots \\ K(H^{-1}A)^{\tau-1-h} \end{bmatrix}. \tag{23}$$

Moreover,

$$\begin{aligned} \Psi_\tau:=diag\big(&\gamma^2 D(D^\top D)^{-1}(D^\top D)^{-1}D^\top-\Theta_1 \\ &,\dots,\gamma^2 D(D^\top D)^{-1}(D^\top D)^{-1}D^\top-M\big). \end{aligned}$$

□

Lemmas 1 and 2 do not only provide a τ-periodic ℓ_2-controller for (1), (6), and a given $\tau\in\mathbb{N}$ but also introduce upper bounds for J in (13) and (21) that will be useful in the design of a NCS with event-triggered communication in Section 3, as we will see.

Remark 1 *It is important to mention that Lemmas 1 and 2 still hold without any change if at every time-step $k\in\mathbb{N}_{\iota\tau}^{(\iota+1)\tau-1}$, the disturbance generator has access also to the control inputs from k up to $(\iota+1)\tau-1$, i.e., the information set available for the disturbance generator follows*

$$\mathcal{E}_k=\{x_i|i\in\mathbb{N}_0^k\}\cup\{u_k,\dots,u_{(\iota+1)\tau-1}\}. \tag{24}$$

Moreover, the disturbance input policies w_k^* given in (14) *and* (19) *are the worst-case disturbance scenarios, when the disturbance generator has access to the information set* (24) *at all times.*

2.2 Networked control system with event-triggered communication

The NCS we are interested in has the same plant G as in (1) and the information set of the disturbance generator also follows (2) (or (24)). However, data transmission to the controller follows a state-dependent mechanism, which is called an event-triggered transmission policy, and we can formulate it as

$$\delta_k = \mu_k(\mathcal{H}_k) \in \{0, 1\}, \tag{25}$$

where

$$\mathcal{H}_k := \{x_i | i \in \mathbb{N}_0^k\} \cup \{\delta_i | i \in \mathbb{N}_0^{k-1}\} \tag{26}$$

is the information set available for the scheduler at $k \in \mathbb{N}_0$. Then, any appropriate control policy is defined as

$$u_k = \mathcal{R}_k^\mu(\mathcal{F}_k^\mu), \tag{27}$$

where

$$\mathcal{F}_k^\mu := \{x_i | i \in \mathbb{N}_0^k \wedge \mu_i(\mathcal{H}_i) = 1\} \tag{28}$$

is the information set available for the controller at $k \in \mathbb{N}_0$ based on an event-triggered scheduling policy defined in (25) and $\mathcal{R}_k^\mu : \mathcal{F}_k^\mu \to \mathcal{R}^m$ is a suitable mapping. Similarly to the periodic control case, in practice, the event-triggered scheduling and control policies of interest (see, e.g., the proposed one in Section 3) will only depend on a few members $\mathcal{H}_k$ and $\mathcal{F}_k^\mu$, respectively. Therefore, the controller does not need to store all the received state vectors (and thus does not need a large memory). We call an event-triggered scheduler and its related controller an ETC, which is denoted by $\eta = (\mu, \mathcal{R}^\mu)$. Furthermore, we introduce the average transmission rate associated with an event-triggered scheduling policy μ and a disturbance sequence $w \in \ell_2^d$ as $\bar{f}_\eta(w) = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mu_t(\mathcal{H}_t)$ and the average inter-transmission time as $\bar{\Omega}_\eta(w) = 1/\bar{f}_\eta(w)$. Next, we define the concept of an event-triggered ℓ_2-controller.

Definition 3 (Event-triggered ℓ_2-controller (Yan et al., 2015)) *Given $\gamma \in \mathbb{R}_{>0}$, an ETC $\eta = (\mu, \mathcal{R}^\mu)$ for the system* (1) *and* (6) *satisfying that*

(i) the closed-loop control system (1) *and* (27) *is globally asymptotically stable when $w = (0, 0, \dots)$,*
(ii) under the assumption of zero initial condition $J \leqslant 0$ for all $w \in \ell_2^d$, where J follows (8),

is referred to as an event-triggered ℓ_2-controller with ℓ_2-gain bound γ. Moreover, the infimum value of $\gamma \in \mathbb{R}_{>0}$, where (i) and (ii) hold for an ETC η designed for (1) *and* (6) *is called the infimal ℓ_2-gain of the event-triggered control loop and is denoted by γ_η^*.* □

Remark 2 *One could define condition (ii) in Definition 2 exactly in the same way as in Definition 3. However, in this case, $\gamma^2 I - \bar{D}_\tau^\top \bar{M}_\tau \bar{D}_\tau \geqslant 0$ would be the necessary condition for the existence of a τ-periodic ℓ_2-controller for a given ℓ_2-gain bound γ and $\tau \in \mathbb{N}$, while $\gamma^2 I - \bar{D}_\tau^\top \bar{M}_\tau \bar{D}_\tau > 0$ is the sufficient condition for the existence of the proposed τ-periodic ℓ_2-control policies in* (11) *and* (16)*. The current condition (ii) of Definition 2 is important to determine $\gamma^2 I - \bar{D}_\tau^\top \bar{M}_\tau \bar{D}_\tau > 0$ as both the necessary and sufficient conditions for the existence of a τ-periodic ℓ_2-controller for a given ℓ_2-gain bound γ and $\tau \in \mathbb{N}$. It is also important to mention that based on Definitions 2 and 3, event-triggered ℓ_2-controllers have to satisfy a weaker condition than τ-periodic ℓ_2-controllers. However, since ϵ in Definition 2 is allowed to be arbitrarily small, this difference in definitions is negligible.*

2.3 Problem statement

The τ-periodic ℓ_2-controllers with ℓ_2-gain bound γ determined in Lemmas 1 and 2 periodically update their state estimates based on the full-state measurements of the sensors. In this way, these controllers can guarantee a desired disturbance attenuation level γ for all disturbance inputs. For every τ-periodic ℓ_2-controller given in Lemmas 1 and 2 we can propose an event-triggered ℓ_2-controller counterpart η, which guarantees the same disturbance attenuation level γ for the system. However, based on the realization of the disturbance inputs, its scheduler can skip some of these periodic data transmissions needed by the τ-periodic ℓ_2-controller, thereby requiring fewer transmissions and thus resulting in larger (or equal) values of $\bar{\Omega}_\eta(w)$ in comparison to τ. This ETC is called ℓ_2-consistent according to the following definition.

Definition 4 (ℓ_2-consistent event-triggered controller) *For any given $\tau \in \mathbb{N}$ and any ℓ_2-gain bound $\gamma > \gamma_\tau^*$ of the system* (1) *and* (6)*, an event-triggered ℓ_2-controller $\eta = (\mu, \Psi^\mu)$ is said to be ℓ_2-consistent with ℓ_2-gain bound γ if*

(i) η has ℓ_2-gain bound γ,
(ii) in comparison to the τ-periodic ℓ_2-controller (16) *(or equivalently* (11)*, in case $\tau = 1$) with ℓ_2-gain bound γ, the average inter-transmission time of η is larger than, or at least equal to, τ, i.e. $\bar{\Omega}_\eta(w) \geqslant \tau$ for all $w \in \ell_2^d$.* □

The goal of this work is to propose an ℓ_2-consistent ETC for the NCS depicted in Fig. 1.

3 ℓ_2-consistent event-triggered controller

We propose an ℓ_2-consistent ETC in this section. For simplicity we start, in Section 3.1, with the case in which $\tau = 1$, since the main ideas can already be conveyed for this case. In Section 3.2, we consider the general case in which $\tau \in \mathbb{N}$.

3.1 Special case $\tau = 1$

Based on Definition 4, in comparison to a 1-periodic ℓ_2-controller (11) with ℓ_2-gain bound $\gamma > \gamma_1^*$, the scheduler of an ℓ_2-consistent ETC should skip data transmissions at some time-steps, while still guaranteeing the same ℓ_2-gain bound γ. We know that the control policy (11) requires the state information at every time-step. However, in our desired ETC setting, the controller does not have the state information at all times and can, therefore, only use an estimation $\bar{x}_{k|k}$ of the state x_k at time $k\in\mathbb{N}_0$. In particular, we select the controller associated with our desired ℓ_2-consistent ETC policy as

$$u_k = K\bar{x}_{k|k}, \tag{29}$$

where K is given as in (12) and $\bar{x}_{k|k}$ is the state estimate in the controller. We propose three state estimators. Two are described as

$$\bar{x}_{k+1|k} = N\bar{x}_{k|k},\quad \bar{x}_{k|k} = \begin{cases} x_k, & \text{if } \mu_k(\mathcal{H}_k)=1 \\ \bar{x}_{k|k-1}, & \text{otherwise,} \end{cases} \tag{30}$$

at all $k\in\mathbb{N}$ for $N\in\{I,A\}$ and $\bar{x}_{0|0}=x_0$. The choice $N=I$ boils down to keeping the estimated state constant if data is not transmitted to the controller and $N=A$ boils down to updating the estimated state based on the system dynamics by ignoring the effects of both the control input and the disturbance when $\mu_k(\mathcal{H}_k)=0$. Additionally,

$$\bar{x}_{k+1|k} = A\bar{x}_{k|k} + Bu_k,\quad \bar{x}_{k|k} = \begin{cases} x_k, & \text{if } \mu_k(\mathcal{H}_k)=1 \\ \bar{x}_{k|k-1}, & \text{otherwise,} \end{cases} \tag{31}$$

at all $k\in\mathbb{N}$ and $\bar{x}_{0|0}=x_0$, is another (possibly more reasonable for some special disturbance inputs) state estimator in the controller. In the following theorem, we propose an event-triggered scheduling policy, which together with (29) and (30) (or (31)), results in an ℓ_2-consistent ETC in the sense of Definition 4.

Theorem 1 *Consider system (1) and (6) and let Assumption 1 hold. For a given $\gamma>\gamma_1^*$, consider the event-triggered scheduling policy*

$$\mu_k(\mathcal{H}_k) := \begin{cases} 1, & \text{if } k=0 \text{ or } G_k(\hat{U}_k,\hat{W}_k)>0 \\ 0, & \text{otherwise,} \end{cases} \tag{32}$$

where $G_0(\hat{U}_0,\hat{W}_0):=0$ and at every $k\in\mathbb{N}$,

$$G_k(\hat{U}_k,\hat{W}_k) := \textstyle\sum_{i=l_k+1}^{k}\big[(\hat{u}_i-u_i^*)^\top\Phi_1(\hat{u}_i-u_i^*) - (Dw_{i-1}-D\hat{w}_{i-1}^*)^\top\Psi_1(Dw_{i-1}-D\hat{w}_{i-1}^*)\big],$$

in which $l_k=\max\{r\in\mathbb{N}_0^{k-1}|\mu_r(\mathcal{H}_r)=1\}$ is the last triggering time before k, u_i^* is given as in (11) and for all $i\in\mathbb{N}_{\ell_k}^{k-1}$,

$$\hat{w}_i^* := SAx_i + (L-S)A\bar{x}_{i|i},$$

where $S=\gamma^{-2}D^\top M(I-\gamma^{-2}DD^\top M)^{-1}$, $L=\gamma^{-2}D^\top MH^{-1}$. Moreover, $\hat{W}_k=\{w_i|i\in\mathbb{N}_{l_k}^{k-1}\}$ and $\hat{U}_k=\{\hat{u}_i|i\in\mathbb{N}_{l_k+1}^{k}\}$, are the actual values of disturbances and control inputs, respectively, where for all $i\in\mathbb{N}_{l_k+1}^{k}$,

$$\hat{u}_i := \begin{cases} K\bar{x}_{i|i-1}, & \text{if } i=k, \\ u_i, & \text{otherwise,} \end{cases}$$

u_i is determined based on (29) and $\bar{x}_{i|i-1}$ follows either (30) for $N\in\{I,A\}$ or (31). Then, the ETC (32) and (29) is ℓ_2-consistent with ℓ_2-gain bound γ. □

We highlight next some features of the ETC proposed in Theorem 1.

1: Based on the event-triggered scheduling policy (32), a deviation of the actual disturbance inputs from the worst-case disturbance scenario given by (14) acts as a "reward" in order to skip data transmissions and let the control inputs deviate from the one determined for 1-periodic ℓ_2-controller in (11). This reward can counteract the penalty incurred by skipping data transmissions (as then $u_k\neq u_k^*$). This is the main intuition behind the proposed ETC in Theorem 1. Moreover, as it can be easily concluded, if the disturbance inputs follow $w_k=w_k^*$ for all $k\in\mathbb{N}_0$, where $\{w_k^*|k\in\mathbb{N}_0\}$ can be seen as a worst case disturbance input, then the proposed event-triggered scheduling policy (32) always triggers data transmissions, i.e., $\mu_k(\mathcal{H}_k)=1$ at all $k\in\mathbb{N}_0$, unless $\hat{u}_k=u_k^*$, which is typically not the case.

2: We can show that for the 1-periodic ℓ_2-controller determined in Lemma 1, and for all $\gamma>\gamma_1^*$, $J^*=x_0^\top Mx_0$, where M is given in Lemma 1. As we select a smaller value for γ, M will become larger (in the sense that $M_{\gamma_1}\succ M_{\gamma_2}$ if $\gamma_2>\gamma_1$). Furthermore, based on (10), we can conclude that MH^{-1} will also become larger. Now, since $\Phi_1=(I-B^\top MH^{-1}B)^{-1}$ and $\Psi_1=\gamma^2(DD^\top)^{-1}-M$, then Φ_1 becomes larger and Ψ_1 becomes smaller. Therefore, the scheduling law (32) is expected to trigger more transmissions for smaller values of γ and the same disturbance input sequence w.

3: In order to evaluate the event-triggered condition (32) at every time $k\in\mathbb{N}_0$, the scheduler needs the values of $\{Dw_t|t\in\mathbb{N}_0^{k-1}\}$ and $\{Dw_t^*|t\in\mathbb{N}_0^{k-1}\}$, which can be calculated by using $Dw_{k-1}=x_k-Ax_{k-1}-Bu_{k-1}$ and $w_{k-1}^*=S(Ax_{k-1}+Bu_{k-1})$ given the condition that the scheduler receives x_k at every $k\in\mathbb{N}_0$ and knows the control policy, from which the control inputs u_{k-1} can be replicated. Therefore, (ii) in Assumption 1 helps to calculate the values of the disturbance inputs, needed in the event-triggered scheduling policy, based on the state measurements, and there is no need for measuring them independently.

3.2 General case $\tau\in\mathbb{N}$

For any $\gamma>\gamma_\tau^*$, the τ-periodic ℓ_2-controller (16) requires periodic state transmission after every $\tau\in\mathbb{N}$ time-steps. However, in this section, we propose an event-triggered ℓ_2-controller with ℓ_2-gain bound $\gamma>\gamma_\tau^*$, which can skip data transmissions at some of these time-steps. Let us introduce the augmented control policy associated with our desired ETC as

$$U_\iota = \bar{K}_\tau\bar{x}_{\iota\tau|\iota\tau}, \tag{33}$$

where $\bar{K}_\tau = \bar{B}_0$ in which $\bar{B}_0$ is determined based on (23) and similar to the previous section, we can either have

$$\bar{x}_{(\iota+1)\tau|\iota\tau} = \bar{N}\bar{x}_{\iota\tau|\iota\tau}, \quad \bar{x}_{\iota\tau|\iota\tau} = \begin{cases} x_{\iota\tau}, & \text{if } \mu_{\iota\tau}(\mathcal{H}_{\iota\tau}) = 1 \\ \bar{x}_{\iota\tau|(\iota-1)\tau}, & \text{otherwise,} \end{cases} \tag{34}$$

for $\bar{N} \in \{I, A^\tau\}$, or

$$\begin{aligned} \bar{x}_{(\iota+1)\tau|\iota\tau} &= A^\tau \bar{x}_{\iota\tau|\iota\tau} + [A^{\tau-1}B, \ldots, B]U_\iota, \\ \bar{x}_{\iota\tau|\iota\tau} &= \begin{cases} x_{\iota\tau}, & \text{if } \mu_{\iota\tau}(\mathcal{H}_{\iota\tau}) = 1 \\ \bar{x}_{\iota\tau|(\iota-1)\tau}, & \text{otherwise,} \end{cases} \end{aligned} \tag{35}$$

as the state estimator in the controller for all $\iota \in \mathbb{N}$ and $\bar{x}_{0|0} = x_0$, depending on the characteristic of the disturbance input. In the following theorem, we propose an event-triggered scheduler which together with (33) and (34) (or (35)) result in an ℓ_2-consistent ETC based on Definition 4.

Theorem 2 *Consider system* (1) *and* (6) *and let Assumption 1 hold. For a given* $\tau \in \mathbb{N}$ *and* $\gamma > \gamma_\tau^*$*, consider the event-triggered scheduling policy*

$$\mu_k(\mathcal{H}_k) := \begin{cases} 1, & \textit{if } k = 0 \ \vee \\ & \Big(k = \iota\tau \ \wedge \ \bar{G}_\iota(\hat{\mathcal{U}}_\iota, \hat{\mathcal{W}}_\iota) > 0, \ \textit{for some } \iota \in \mathbb{N}\Big), \\ 0, & \textit{otherwise,} \end{cases} \tag{36}$$

where $\bar{G}_0(\hat{\mathcal{U}}_0, \hat{\mathcal{W}}_0) := 0$ *and at every* $\iota \in \mathbb{N}$,

$$\begin{aligned} \bar{G}_\iota(\hat{\mathcal{U}}_\iota, \hat{\mathcal{W}}_\iota) := \textstyle\sum_{i=l_\iota+1}^{\iota} \big[&(\hat{U}_i - U_i^*)^\top \Phi_\tau (\hat{U}_i - U_i^*) \\ &- (W_{i-1} - \hat{W}_{i-1}^*)^\top \Psi_\tau (W_{i-1} - \hat{W}_{i-1}^*) \big] \end{aligned}$$

in which $l_\iota = \sup\{r \in \mathbb{N}_0^{\iota-1} | \mu_{r\tau}(\mathcal{H}_{r\tau}) = 1\}$ *is the last triggering time before* ι, $\hat{W}_i^* = D[\hat{w}_{i\tau}^{*\top}, \ldots, \hat{w}_{(i+1)\tau-1}^{*\top}]^\top$ *and* U_i^* *follows* (18) *for all* $i \in \mathbb{N}_{\ell_\iota}^{\iota-1}$, *where for* $L = \gamma^{-2}D^\top M H^{-1}$,

$$\hat{w}_{i\tau+h}^* := \bar{S}_h A x_{i\tau+h} + (L - \bar{S}_h) A (H^{-1}A)^h \bar{x}_{i\tau|i\tau},$$

for every $h \in \mathbb{N}_0^{\tau-1}$ *in which* $\bar{S}_h$ *follows* (20). *Moreover,* $\hat{\mathcal{W}}_\iota = \{W_i | i \in \mathbb{N}_{l_\iota}^{\iota-1}\}$ *and* $\hat{\mathcal{U}}_\iota = \{\hat{U}_i | i \in \mathbb{N}_{l_\iota+1}^{\iota}\}$, *are the actual values of disturbances and control inputs, respectively, where*

$$\hat{U}_i := \begin{cases} \bar{K}_\tau \bar{x}_{i\tau|(i-1)\tau}, & \textit{if } i = \iota \\ U_i, & \textit{otherwise,} \end{cases}$$

for all $i \in \mathbb{N}_{l_\iota+1}^{\iota}$, U_i *is determined based on* (33), $\bar{x}_{i\tau|(i-1)\tau}$ *follows either* (34) *for* $N \in \{I, A^\tau\}$ *or* (35) *and* W_i *follows* (15). *Then, the ETC policy* (36) *and* (33) *is* ℓ_2*-consistent with* ℓ_2*-gain bound* γ. □

Note that the ℓ_2-consistent ETC proposed in Theorem 2 also has the features discussed after Theorem 1.

The ℓ_2-consistency of the proposed ETC policies in Theorems 1 and 2 indicates that for the same $\gamma \in \mathbb{R}_{>0}$ as the disturbance attenuation level where $\gamma > \gamma_\tau^*$, in case the disturbance input does not follow the worst-case scenario given by (19), the event-triggered scheduler can skip data transmissions at some times required by the τ-periodic controller (16) and results in a larger average inter-transmission time than τ while guaranteeing the same ℓ_2-gain bound γ. Moreover, for the proposed ETC, the behaviour of the average inter-transmission time with respect to γ when $\gamma > \gamma_\tau^*$ is not necessarily increasing and it highly depends on the actual disturbance input of the system. This can be clearly seen in Fig. 3 below corresponding to a numerical example.

4 Numerical example

Consider a scalar system where $A = 1.1$, $B = 1$, $D = 1$ are the parameters of the linear model (1), and $Q = 1$. Moreover, we take $w_k = e^{-\frac{k}{200}} \sin(\frac{k}{25})$, $k \in \mathbb{N}_0$, as the unknown disturbance input of the system. The infimal ℓ_2-gain of the system for periodic control with the inter-transmission time-steps $\tau \in \{1, 2, 3, 4\}$ are $\gamma_1^* = 1.487$, $\gamma_2^* = 2.202$, $\gamma_3^* = 2.999$ and $\gamma_4^* = 3.871$. According to Lemmas 1 and 2, for any $\tau \in \mathbb{N}$ and $\gamma > \gamma_\tau^*$, we can design a τ-periodic ℓ_2-controller with ℓ_2-gain bound γ. Then based on Theorems 1 or 2, we can design its ℓ_2-consistent ETC counterpart for this system. Based on Definition (4), the proposed ℓ_2-consistent ETC can result in the same attenuation level (ℓ_2-gain bound) as the PTC (11) or (16), however, with a smaller (or at most an equal) average transmission rate. However, we will show that for the given system and the disturbance input, it is even possible to achieve smaller disturbance attenuation levels by following the proposed ETC in comparison to the PTC (11) or (16), while they both have the same average transmission rate.

Firstly, we consider $\tau = 1$ and design an ETC based on Theorem 1. The controller follows (29), where the state estimation in the controller is determined based on (30) for $N = A$. Fig. 2(a) shows the state trajectory related to the ETC when $\tau = 1$ and $\gamma = 1.630 > \gamma_1^*$ is its corresponding ℓ_2-gain bound. This ETC results in $\bar{\Omega}_\eta(w) = 2.033$, where $\bar{\Omega}_\eta(w)$ denotes the average inter-transmission time of scheduler. However, if the scheduler triggers transmissions periodically with $\tau = 2$, then we know that the infimal ℓ_2-gain of the system with periodic control is $\gamma_2^* = 2.202$. The state trajectory of the periodic controller (16) for $\gamma = \gamma_2^* + \epsilon$, where $\epsilon > 0$ is a small real number, and $\tau = 2$ is shown in Fig. 2(a), which indicates the better disturbance attenuation of the ETC while they both have almost the same average transmission rate for the given disturbance input w.

Fig. 2(b) compares similar situations when the ETC is designed for $\tau = 2$ and $\gamma = 2.924$ based on Theorem 2. The controller follows (33) where the state estimation in the

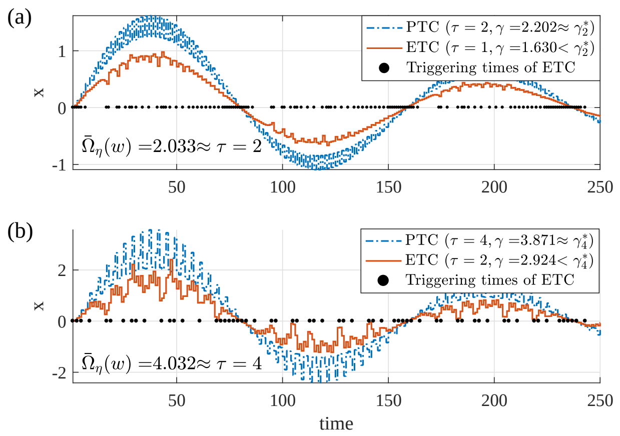

Fig. 2. Illustration of the improved performance of the proposed ETC in comparison to periodic time-triggered controller (PTC) in disturbance attenuation with the same average transmission rate (for the given disturbance input w) when (a) ETC designed for $\tau=1$ and PTC designed for $\tau=2$ (b) ETC designed for $\tau=2$ and PTC designed for $\tau=4$.

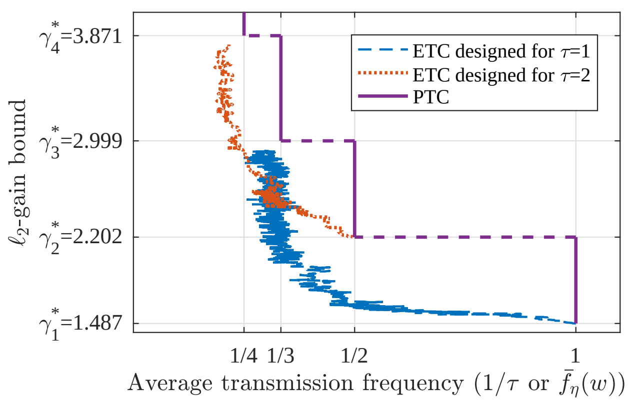

Fig. 3. Trade-off curves resulted by the ℓ_2-consistent ETC designed for $\tau=1$ and $\tau=2$ and the given disturbance input w in comparison to the one which can be achieved by PTC.

controller is determined based on (34) for $N=A^2$. Again, for this system, the ℓ_2-gain bound of the system with the ETC is significantly smaller than the minimum value that results from periodic control while they both have almost the same average inter-transmission time-steps to the controller $\big(\bar{\Omega}_\eta(w)\approx\tau=4\big)$ for the given disturbance input w. Fig. 3 is more generic and illustrates the ℓ_2-consistency of the proposed ETC when $\tau=1$ and $\tau=2$. For the given disturbance input w, we find the average inter-transmission time of the system with the ℓ_2-consistent ETC designed for different values of $\gamma>\gamma^*_\tau$, a time horizon of 250 and zero initial condition. The solid line shows the trade-offs one can achieve by following a periodic time-triggered control strategy. We easily see the better trade-offs for the proposed ℓ_2-consistent ETC in comparison to PTCs. In principle, based on the theory (Theorems 1 and 2), for every $\tau\in\mathbb{N}$ and $\gamma\in[\gamma^*_\tau,\gamma^*_{\tau+1})$ the trade-offs for the proposed ETC (36) and (33) (or (32) and (29) when $\tau=1$) are guaranteed to be bellow (or at most on) the stairwise curve of PTCs for any linear system (1) and disturbance input w.

5 Conclusions

In this work, we investigated the design of event-triggered controllers (ETCs) for discrete-time linear systems by considering the ℓ_2-gain as a performance criterion of the closed-loop system. Firstly, for every transmission time period, we determined a periodic ℓ_2-controller for a given ℓ_2-gain bound, following a game-theoretical approach. Then, we introduced the notion of ℓ_2-consistency, which refers to any ETC that guarantees the same ℓ_2-gain bound as that of the designed periodic ℓ_2-controller, however, with a larger or an equal average inter-transmission time. Next, we proposed the design of an ℓ_2-consistent ETC with some interesting features. When the disturbance input follows the worst-case scenario at every time, the scheduler triggers transmissions periodically in order to guarantee an ℓ_2-gain bound for the system. However, when the disturbance input is not equal to the worst-case scenario, the ℓ_2-gain bound of our designed ETC is still guaranteed and equal to that of the designed periodic ℓ_2-controller, however, with a (significantly) larger average inter-transmission time. Possible directions for future work include considering linear plants with partial state information, non-linear plants and date bit rate constraints.

Appendix

1. *Proof of Lemma 1*

Parts i and ii can be proved by the arguments in Theorem 3.8 of Başar and Bernhard (2008), in which the stabilizability of (A,B) and the observability of $(Q^{\frac{1}{2}},A)$ are used to guarantee the existence of $\hat{\gamma}\in\mathbb{R}_{>0}$, where for all $\gamma>\hat{\gamma}$ the Ricatti equation (10) has a positive definite solution M. They are also proved (by making $\tau=1$) in a more general setting in Lemma 2. The only point that is not proved in Başar and Bernhard (2008) is the Lyapunov stability of the control loop when $w=(0,0,\dots)$, which we postpone it to the end of the present proof.

Part iii) Let us define $\tilde{J}(x_k,x_{k+1})=x_{k+1}^\top Mx_{k+1}-x_k^\top Mx_k$ for all $k\in\mathbb{N}_0$, where M is the positive definite solution of $M=A^\top MH^{-1}A+Q$ for a given $\gamma\in\mathbb{R}_{>0}$ such that $\gamma^2I-D^\top MD>0$. Then, by using (1)

$$\begin{aligned}\tilde{J}(x_k,x_{k+1})=&(Ax_k+Bu_k)^\top M(Ax_k+Bu_k)\\&-w_k^\top(\gamma^2I-D^\top MD)w_k+2(Ax_k+Bu_k)^\top MDw_k\\&-x_k^\top Mx_k+(x_k^\top Qx_k+u_k^\top u_k)-(z_k^\top z_k-\gamma^2w_k^\top w_k).\end{aligned}$$

Now by completing the squares for w_k, we obtain

$$\begin{aligned}\tilde{J}(x_k,x_{k+1})=&-(w_k-w_k^*)^\top D^\top\Psi_1 D(w_k-w_k^*)+u_k^\top u_k\\&+x_k^\top(Q-M)x_k+(Ax_k+Bu_k)^\top M\mathbb{G}(Ax_k+Bu_k)\\&-(z_k^\top z_k-\gamma^2 w_k^\top w_k),\end{aligned}$$

for $w_k^*=(D^\top\Psi_1 D)^{-1}D^\top M(Ax_k+Bu_k)$, where $\Psi_1:=\gamma^2 D(D^\top D)^{-1}(D^\top D)^{-1}D^\top-M$ and $D^\top\Psi_1 D\succ 0$. Moreover, $\mathbb{G}:=(I-\gamma^{-2}DD^\top M)^{-1}$. Now, we complete the squares for u_k.

$$\begin{aligned}&\tilde{J}(x_k,x_{k+1})=-(w_k-w_k^*)^\top D^\top\Psi_1 D(w_k-w_k^*)+(u_k-u_k^*)^\top\\&\Phi_1(u_k-u_k^*)+x_k^\top(A^\top MH^{-1}A-M+Q)x_k+\gamma^2 w_k^\top w_k-z_k^\top z_k,\end{aligned}\tag{A.1}$$

where $\Phi_1:=I+B^\top M\mathbb{G}B\succcurlyeq I$, $H:=I+(BB^\top-\gamma^{-2}DD^\top)M$, and $u_k^*=-B^\top MH^{-1}Ax_k$. Now by using the matrix inversion lemma (Henderson and Searle, 1981, equation (18)), we can show that $\Phi_1^{-1}=I-B^\top MH^{-1}B$. Then by summing all the values of $\tilde{J}(x_k,x_{k+1})$ over $k\in\mathbb{N}_0^\nu$ for an arbitrary $\nu\in\mathbb{N}$ and considering $M=A^\top MH^{-1}A+Q$,

$$\begin{aligned}&\textstyle\sum_{k=0}^{\nu}\tilde{J}(x_k,x_{k+1})=x_{\nu+1}^\top Mx_{\nu+1}-x_0^\top Mx_0\\&=-\textstyle\sum_{k=0}^{\nu}(z_k^\top z_k-\gamma^2 w_k^\top w_k)+\sum_{k=0}^{\nu}[(u_k-u_k^*)^\top\\&\Phi_1(u_k-u_k^*)-(w_k-w_k^*)^\top D^\top\Psi_1 D(w_k-w_k^*)].\end{aligned}\tag{A.2}$$

From this equation we conclude that

$$\begin{aligned}&\textstyle\sum_{k=0}^{\nu}[z_k^\top z_k-\gamma^2 w_k^\top w_k]=x_0^\top Mx_0-x_{\nu+1}^\top Mx_{\nu+1}+\sum_{k=0}^{\nu}\\&[-(w_k-w_k^*)^\top D^\top\Psi_1 D(w_k-w_k^*)+(u_k-u_k^*)^\top\Phi_1(u_k-u_k^*)].\end{aligned}$$

Since M is a positive definite matrix and $x_0=0$, then

$$\begin{aligned}J=\textstyle\sum_{k=0}^{\infty}[z_k^\top z_k-\gamma^2 w_k^\top w_k]\leqslant\sum_{k=0}^{\infty}\big[&(u_k-u_k^*)^\top\Phi_1(u_k-u_k^*)\\&-(w_k-w_k^*)^\top D^\top\Psi_1 D(w_k-w_k^*)\big],\end{aligned}$$

which proves part iii. Now we need to prove the global asymptotic stability of the control loop, when $w=(0,0,\dots)$ and the control input follows (11). We take $V(x_k)=x_k^\top Mx_k$ as the Lyapunov function candidate, where M is a positive definite solution of (10). Considering $\Delta V_k:=V(x_{k+1})-V(x_k)$, then based on (A.1), for $w=(0,0,\dots)$ and $u_k=u_k^*$ at every $k\in\mathbb{N}_0$, we have $\Delta V_k=-u_k^{*\top}u_k^*-x_k^\top Qx_k-w_k^{*\top}D^\top\Psi_1 Dw_k^*\leqslant 0$, for every $k\in\mathbb{N}_0$. Therefore, the control loop is Lyapunov stable. Moreover, based on the observability of $(Q^{\frac{1}{2}},A)$ it can be shown that the state of the control loop converges to zero as time goes to infinity, when $w=(0,0,\dots)$ and $u_k=u_k^*$ at every $k\in\mathbb{N}_0$, see Başar and Bernhard (2008, page 62). Thus, the system is globally asymptotically stable.

2. Proof of Lemma 2

i) Necessary and sufficient conditions for the existence of a τ-periodic ℓ_2-controller

According to Theorem 3.8 of Başar and Bernhard (2008), taking into account the stabilizability of (A,B) and the observability of $(Q^{\frac{1}{2}},A)$, there exists a $\hat{\gamma}_1\in\mathbb{R}_{>0}$, where for all $\gamma>\hat{\gamma}_1$ the Ricatti equation (10) has a positive definite solution M. Moreover, it is clear that there exists $\hat{\gamma}_2>0$ such that for all $\gamma>\hat{\gamma}_2$, $\gamma^2 I-\bar{D}_\tau^\top\bar{M}_\tau\bar{D}_\tau\succ 0$ holds. Then we can take $\hat{\gamma}:=\max\{\hat{\gamma}_1,\hat{\gamma}_2\}$, which establishes the first assertion.

Now to prove the second assertion in statement i, we resort to an argument in Başar and Bernhard (2008), which indicates that the conditions needed to find a controller satisfying part ii of Definition 2 are the same as the conditions needed to have $J^*\leqslant 0$, where J^* is given in (9). Solving the minimax optimization problem in (9) is equivalent to finding an appropriate value function $\mathcal{V}(x_\ell)$ such that the following Isaacs equation holds for every $\ell=\iota\tau\in\mathbb{N}_0$ and every $x_\ell\in\mathbb{R}^n$ (Başar and Olsder, 1999, Corollary 6.2),

$$\begin{aligned}\mathcal{V}(x_\ell)=&\min_{U_\iota=\bar{\mathcal{R}}_\iota(\mathcal{F}_\ell^{\pi_\tau})}\ \max_{w_\ell=\mathcal{T}_\ell(\mathcal{E}_\ell)}\cdots\max_{w_{\ell+\tau-1}=\mathcal{T}_{\ell+\tau-1}(\mathcal{E}_{\ell+\tau-1})}\\&\textstyle\sum_{k=\ell}^{\ell+\tau-1}[z_k^\top z_k-\gamma^2 w_k^\top w_k]+\mathcal{V}(x_{\ell+\tau}).\end{aligned}$$

In this minimax game, the information structures of the two players are periodic with given, generally not equal, time periods. As a result of Theorem 6.9 in Başar and Olsder (1999) the value function for this two-player zero-sum minimax game is $\mathcal{V}(x_\ell)=x_\ell^\top Mx_\ell$, where M is the positive definite solution of (10). Therefore, we have to find the conditions under which the following equality always holds for every $x_\ell\in\mathbb{R}^n$,

$$\begin{aligned}x_\ell^\top Mx_\ell=&\min_{U_\iota=\bar{\mathcal{R}}_\iota(\mathcal{F}_\ell^{\pi_\tau})}\ \max_{w_\ell=\mathcal{T}_\ell(\mathcal{E}_\ell)}\cdots\max_{w_{\ell+\tau-1}=\mathcal{T}_{\ell+\tau-1}(\mathcal{E}_{\ell+\tau-1})}\\&\textstyle\sum_{k=\ell}^{\ell+\tau-1}[z_k^\top z_k-\gamma^2 w_k^\top w_k]+x_{\ell+\tau}^\top Mx_{\ell+\tau}.\end{aligned}\tag{A.3}$$

In order to solve the optimization problem in (A.3) for $\tau\in\mathbb{N}$, first we need to follow τ maximization steps and determine $W_\iota^*=D[w_\ell^{*\top},\dots,w_{\ell+\tau-1}^{*\top}]^\top$ and then determine $U_\iota^*=[u_\ell^{*\top},\dots,u_{\ell+\tau-1}^{*\top}]^\top$ in one minimization step. Therefore, w_s^* when $s=\ell+\tau-1$ is determined as follows

$$\hat{J}_{\tau-1}(x_s,u_s):=\max_{w_s\in\mathbb{R}^d}\big[z_s^\top z_s-\gamma^2 w_s^\top w_s+x_{s+1}^\top Mx_{s+1}\big].$$

By substituting (1) into the above equation we get

$$\begin{aligned}w_s^*:=\arg\max_{w_s\in\mathbb{R}^d}\big(&z_s^\top z_s-\gamma^2 w_s^\top w_s\\&+(Ax_s+Bu_s+Dw_s)^\top M(Ax_s+Bu_s+Dw_s)\big),\end{aligned}$$

where a bounded w_s^* exists if $\gamma^2 I-D^\top MD\succ 0$, and

$$w_s^*=\gamma^{-2}D^\top\hat{V}_{\tau-1}^{-1}M(Ax_s+Bu_s),\tag{A.4}$$

for $\hat{V}_{\tau-1}=I-\gamma^{-2}MDD^{\top}$. Now by substituting (A.4) into $\hat{J}_{\tau-1}(x_s,u_s)$ we get $\hat{J}_{\tau-1}(x_s,u_s)=x_s^{\top}\Theta_{\tau-1}x_s+u_s^{\top}Y_{\tau-1}u_s+2x_s^{\top}Z_{\tau-1}u_s$, where

$$\begin{aligned}&\Theta_{\tau-1}:=Q+A^{\top}\hat{V}_{\tau-1}^{-1}MA,\ Z_{\tau-1}:=A^{\top}\hat{V}_{\tau-1}^{-1}MB,\\ &Y_{\tau-1}:=I+B^{\top}\hat{V}_{\tau-1}^{-1}MB.\end{aligned} \tag{A.5}$$

Therefore, the optimal game value at time s is a function of x_s and u_s. Now, by an induction argument, let us assume that at an arbitrary optimization step $h+1\in\mathbb{N}_1^{\tau-1}$ the optimal game value is

$$\begin{aligned}\hat{J}_{h+1}(x_{r+1},\hat{U}_{r+1}):=&x_{r+1}^{\top}\Theta_{h+1}x_{r+1}+2x_{r+1}^{\top}Z_{h+1}\hat{U}_{r+1}\\&+\hat{U}_{r+1}^{\top}Y_{h+1}\hat{U}_{r+1},\end{aligned}$$

where Θ_{h+1}, Y_{h+1} are known positive definite matrices, $r=\ell+h$, and $\hat{U}_{r+1}=[u_{r+1}^{\top},\dots,u_{\ell+\tau-1}^{\top}]^{\top}$ is the augmented control input. Then

$$\hat{J}_h(x_r,\hat{U}_r):=\max_{w_r\in\mathbb{R}^d}[z_r^{\top}z_r-\gamma^2w_r^{\top}w_r+\hat{J}_{h+1}(x_{r+1},\hat{U}_{r+1})].$$

By substituting (1) into the above equation, we get

$$w_r^*=\gamma^{-2}D^{\top}\hat{V}_h^{-1}\big(\Theta_{h+1}(Ax_r+Bu_r)+Z_{h+1}\hat{U}_{r+1}\big), \tag{A.6}$$

where $\hat{V}_h=I-\gamma^{-2}\Theta_{h+1}DD^{\top}$, provided that $\gamma^2I-D^{\top}\Theta_{h+1}D>0$. Then, $\hat{J}_h(x_r,\hat{U}_r)=x_r^{\top}\Theta_hx_r+2x_r^{\top}Z_h\hat{U}_r+\hat{U}_r^{\top}Y_h\hat{U}_r$, where

$$\begin{aligned}&\Theta_h:=Q+A^{\top}\hat{V}_h^{-1}\Theta_{h+1}A,\\ &Z_h:=\begin{bmatrix}A^{\top}\hat{V}_h^{-1}\Theta_{h+1}B & A^{\top}\hat{V}_h^{-1}Z_{h+1}\end{bmatrix},\\ &Y_h:=\begin{bmatrix}I+B^{\top}\hat{V}_h^{-1}\Theta_{h+1}B & B^{\top}\hat{V}_h^{-1}Z_{h+1}\\ Z_{h+1}^{\top}\hat{V}_h^{-\top}B & Y_{h+1}+Z_{h+1}^{\top}E_hZ_{h+1}\end{bmatrix},\end{aligned} \tag{A.7}$$

for $E_h=\gamma^{-2}DD^{\top}\hat{V}_h^{-1}$. Moreover, $\hat{J}_h$ takes the same form as the one assumed for the $\hat{J}_{h+1}$, and therefore, the quadratic form assumed for the $\hat{J}_h$ is correct. Finally, consider at the optimization step ℓ, after determining w_ℓ^* the optimal game value follows $\hat{J}_0(x_\ell,\hat{U}_\ell)=x_\ell^{\top}\Theta_0x_\ell+2x_\ell^{\top}Z_0\hat{U}_\ell+\hat{U}_\ell^{\top}Y_0\hat{U}_\ell$, where Θ_0, Z_0 and Y_0 are determined based on (A.7). Then, $\hat{U}_\ell^*=\arg\min_{\hat{U}_\ell\in\mathbb{R}^{m\tau}}[x_\ell^{\top}\Theta_0x_\ell+2x_\ell^{\top}Z_0\hat{U}_\ell+\hat{U}_\ell^{\top}Y_0\hat{U}_\ell]$, which results in $U_\iota^*:=\hat{U}_\ell^*=\bar{K}_\tau x_\ell$, where

$$\bar{K}_\tau=-Y_0^{-1}Z_0^{\top}, \tag{A.8}$$

and $\hat{J}_0(x_\ell):=x_\ell^{\top}(\Theta_0-Z_0Y_0^{-1}Z_0^{\top})x_\ell=x_\ell^{\top}Mx_\ell=\mathcal{V}(x_\ell)$. We can prove that $\bar{K}_\tau=\bar{B}_0$, where $\bar{B}_0$ is determined based on (23), see, Balaghiinaloo (2020, Appendix F). Therefore, we have $M=\Theta_0-Z_0Y_0^{-1}Z_0^{\top}$. However, since the value of τ is arbitrary, then for every $h\in\mathbb{N}_0^{\tau-1}$,

$$M=\Theta_h-Z_hY_h^{-1}Z_h^{\top}. \tag{A.9}$$

Based on (A.9), we can obtain the same Ricatti equation as in (10) by considering $h=\tau-1$,

$$M=\Theta_{\tau-1}-Z_{\tau-1}Y_{\tau-1}^{-1}Z_{\tau-1}^{\top}=Q+A^{\top}MH^{-1}A,$$

where $H=I+(BB^{\top}-\gamma^{-2}DD^{\top})M$. However, according to $\gamma^2I-D^{\top}\Theta_{h+1}D>0$ we have to check the following conditions for the existence of the optimal solution for (A.3),

$$\Lambda_h(\gamma):=\gamma^2I-D^{\top}\Theta_hD>0 \tag{A.10}$$

at all $h\in\mathbb{N}_1^{\tau}$, where

$$\Theta_h=\begin{cases}M, & \text{if } h=\tau\\ Q+A^{\top}\hat{V}_h^{-1}\Theta_{h+1}A & \text{otherwise.}\end{cases} \tag{A.11}$$

One can easily establish that $\gamma^2I-\bar{D}_\tau^{\top}\bar{M}_\tau\bar{D}_\tau>0$ is equivalent to the series of inequalities in (A.10), see, Balaghiinaloo (2020, Appendix F).

Proving $J\leqslant-\epsilon\|w\|_{\ell_2}^2$ for $U_\iota=U_\iota^*$ *at all* $\iota\in\mathbb{N}_0$, all $w\in\ell_2^d$ *and* $x_0=0$*:* Following the same procedure as the one given in the proof of Lemma 1, we can obtain

$$\begin{aligned}x_{\ell+\tau}^{\top}Mx_{\ell+\tau}-x_\ell^{\top}Mx_\ell=&(U_\iota-U_\iota^*)^{\top}Y_0(U_\iota-U_\iota^*)-\textstyle\sum_{i=\ell}^{\ell+\tau-1}\\&[(w_i-w_i^*)^{\top}D^{\top}\bar{\Psi}_{i-\ell}D(w_i-w_i^*)-(z_i^{\top}z_i-\gamma^2w_i^{\top}w_i)],\end{aligned} \tag{A.12}$$

for any arbitrary transmission time $\ell=\iota\tau\in\mathbb{N}_0$ and $\bar{\Psi}_h:=\gamma^2D(D^{\top}D)^{-1}(D^{\top}D)^{-1}D^{\top}-\Theta_{h+1}$ for all $h\in\mathbb{N}_0^{\tau-1}$. Then by taking the summation of both sides of the equation over the transmission times, from initial up to an arbitrary transmission time-step $\nu\in\mathbb{N}$,

$$\begin{aligned}&\textstyle\sum_{k=0}^{\nu}(z_k^{\top}z_k-\gamma^2w_k^{\top}w_k)=x_0^{\top}Mx_0-x_{\nu+\tau}^{\top}Mx_{\nu+\tau}+\sum_{\iota=0}^{\nu}\\&[(U_\iota-U_\iota^*)^{\top}\Phi_\tau(U_\iota-U_\iota^*)-(W_\iota-W_\iota^*)^{\top}\Psi_\tau(W_\iota-W_\iota^*)],\end{aligned}$$

where $\Psi_\tau=\gamma^2\hat{D}_\tau(\hat{D}_\tau^{\top}\hat{D}_\tau)^{-1}(\hat{D}_\tau^{\top}\hat{D}_\tau)^{-1}\hat{D}_\tau^{\top}-\hat{\Theta}_\tau$ in which $\hat{D}_\tau=I_\tau\otimes D$, $\hat{\Theta}_\tau=\mathrm{diag}(\Theta_1,\dots,\Theta_{\tau-1},M)$ and $\Phi_\tau=Y_0$. Thus,

$$\begin{aligned}J=&\textstyle\sum_{k=0}^{\infty}(z_k^{\top}z_k-\gamma^2w_k^{\top}w_k)\leqslant x_0^{\top}Mx_0+\sum_{\iota=0}^{\infty}\\&[(U_\iota-U_\iota^*)^{\top}\Phi_\tau(U_\iota-U_\iota^*)-(W_\iota-W_\iota^*)^{\top}\Psi_\tau(W_\iota-W_\iota^*)].\end{aligned} \tag{A.13}$$

Now following the similar arguments as in (Limebeer et al., 1992, Theorem 2.1), we can show that for $U_\iota=U_\iota^*$ at all $\iota\in\mathbb{N}_0$ and $x_0=0$, $J\leqslant-\epsilon\|w\|_{\ell_2}^2$ holds for all $w\in\ell_2^d$ and some positive ϵ. Note that in case $W_\iota=W_\iota^*$ at all $\iota\in\mathbb{N}_0$ and for $x_0=0$, we can conclude $w=(0,0,\dots)$, where still $J\leqslant-\epsilon\|w\|_{\ell_2}^2$ holds for any positive ϵ.

Proving the necessity of (A.10), *such that* $U_\iota = U_\iota^*$ *at all* $\iota \in \mathbb{N}_0$ *satisfies* $J \leqslant -\epsilon \|w\|_{\ell_2}^2$, *for* $x_0=0$ *and all* $w \in \ell_2^d$: Let us assume that $\Lambda_h(\gamma) = D^\mathsf{T} \bar{\Psi}_h D$ for a $h \in \mathbb{N}_1^\tau$ is not a positive definite matrix. In this case, we will introduce a $w \neq (0,0,\dots)$, for which it is not possible to find an $\epsilon > 0$ such that $J \leqslant -\epsilon \|w\|_{\ell_2}^2$. Suppose $d^* \neq 0$ is an eigenvector of $\Lambda_h(\gamma)$ corresponding to its zero or negative eigenvalue. Then for $x_0=0$ and a given $\iota \in \mathbb{N}_0$, we propose the following disturbance sequence

$$w_k = \begin{cases} 0, & \text{if } k < \iota\tau + h \\ w_k^* + d^*, & \text{if } k = \iota\tau + h \\ w_k^*, & \text{if } k > \iota\tau + h. \end{cases} \tag{A.14}$$

Since $x_0=0$, then $x_t=0$ and $w_t^*=0$ for all $t \in \mathbb{N}_0^k$. Now if $U_\iota = U_\iota^*$ at all $\iota \in \mathbb{N}_0$, then $(U_\iota - U_\iota^*)^\mathsf{T} \Phi_\tau (U_\iota - U_\iota^*) = 0$ and since $\Lambda_h(\gamma) d^* \leqslant 0$, then $(W_\iota - W_\iota^*)^\mathsf{T} \Psi_\tau (W_\iota - W_\iota^*) \leqslant 0$ at all $\iota \in \mathbb{N}_0$ for the given $w \neq (0,0,\dots)$. Therefore, based on (A.13), we cannot find an $\epsilon > 0$, where $J \leqslant -\epsilon \|w\|_{\ell_2}^2$ for the given nonzero disturbance input. Thus, for all $h \in \mathbb{N}_0^{\tau-1}$, $\Lambda_h(\gamma)$ should not have any zero or negative eigenvalue.

ii) τ-Periodic ℓ_2-controller

We can prove that the determined control policy $U_\iota^* = \bar{K}_\tau x_\ell$ is equivalent to the one given in (16) and (17), however it is omitted due to space limitations. In part i of the proof we showed that the control policy $U_\iota = U_\iota^*$ for all $\iota \in \mathbb{N}_0$ satisfies $J \leqslant -\epsilon \|w\|_{\ell_2}^2$ for $x_0=0$ and all $w \in \ell_2^d$. Now we just need to prove the global asymptotic stability of the control loop when $w=(0,0,\dots)$.

For this purpose, let us take $V(x_\ell) = x_\ell^\mathsf{T} M x_\ell$ as the Lyapunov function candidate at every transmission time step. Then based on (A.12), we have

$$\begin{aligned} \Delta V(x_\ell) &= x_{\ell+\tau}^\mathsf{T} M x_{\ell+\tau} - x_\ell^\mathsf{T} M x_\ell \\ &= -\textstyle\sum_{i=\ell}^{\ell+\tau-1} \left[w_i^{*\mathsf{T}} D^\mathsf{T} \bar{\Psi}_{i-\ell} D w_i^* + z_i^\mathsf{T} z_i \right] \leqslant 0, \end{aligned}$$

at every $\ell \in \mathbb{N}_0$, when $U_\iota = U_\iota^*$ and $w=(0,0,\dots)$. This indicates that the control loop is Lyapunov stable. Then, following the same arguments as in Başar and Bernhard (2008, page 62), due to the boundedness of the game upper value for $U_\iota = U_\iota^*$ at all transmission time steps $\ell = \iota\tau$, we can conclude that $Q^{\frac{1}{2}} x_k \to 0$ when $w=(0,0,\dots)$, and then based on the observability of $(Q^{\frac{1}{2}}, A)$, we can show that x_k converges to zero as time goes to infinity for any initial condition. Therefore, we can conclude the global asymptotic stability of the control loop.

iii) Performance index

We proved (21) in (A.13), where $\Phi_\tau = Y_0$ which can be determined iteratively based on (A.7). However, there is a simpler iteration to determine Y_0^{-1}. Let us consider $Y_h = \hat{A} + \hat{B}\hat{C}\hat{B}^\mathsf{T}$, where $\hat{A} = \mathrm{diag}\{I, Y_{h+1}\}$,

$$\hat{B} = \begin{bmatrix} B^\mathsf{T} & 0 \\ 0 & Z_{h+1}^\mathsf{T} \end{bmatrix}, \; \hat{C} = \begin{bmatrix} \hat{V}_h^{-1} \Theta_{h+1} & \hat{V}_h^{-1} \\ \hat{V}_h^{-\mathsf{T}} & \gamma^{-2} D D^\mathsf{T} \hat{V}_h^{-1} \end{bmatrix},$$

then by considering the following matrix equality

$$(\hat{A} + \hat{B}\hat{C}\hat{B}^\mathsf{T})^{-1} = \hat{A}^{-1} - \hat{A}^{-1}\hat{B}(I + \hat{C}\hat{B}^\mathsf{T}\hat{A}^{-1}\hat{B})^{-1}\hat{C}\hat{B}^\mathsf{T}\hat{A}^{-1}.$$

we can determine the equality in (22), for all $h \in \mathbb{N}_0^{\tau-1}$. Moreover, this iteration starts from $Y_{\tau-1} = I + B^\mathsf{T} \hat{V}_{\tau-1}^{-1} M B$, where we can easily show that $Y_{\tau-1}^{-1} = I - B^\mathsf{T} M H^{-1} B$.

3. Proof of Theorem 1

In order to guarantee that $\gamma > \gamma_1^*$ is an ℓ_2-gain bound of the system (1), (32) and (29) (with (30) or (31)), one should satisfy $J \leqslant 0$ for all $w \in \ell_2^d$, where J is given in (8). Therefore, we can use (13) and represent it as

$$\begin{aligned} J \leqslant{}& (u_0 - u_0^*)^\mathsf{T} \Phi_1 (u_0 - u_0^*) + \textstyle\sum_{k=1}^{\infty} [(u_k - u_k^*)^\mathsf{T} \Phi_1 (u_k - u_k^*) \\ & - (Dw_{k-1} - Dw_{k-1}^*)^\mathsf{T} \Psi_1 (Dw_{k-1} - Dw_{k-1}^*)]. \end{aligned}$$

Based on the event-triggered policy (32), there is a state transmission to the controller at $k=0$, therefore $u_0 = u_0^*$. Moreover, we can change the summation in the above equation into two summations as follows

$$\begin{aligned} J \leqslant{}& \textstyle\sum_{j=0}^{\infty} \sum_{i=s_j+1}^{s_{j+1}} [(u_i - u_i^*)^\mathsf{T} \Phi_1 (u_i - u_i^*) \\ & - (Dw_{i-1} - Dw_{i-1}^*)^\mathsf{T} \Psi_1 (Dw_{i-1} - Dw_{i-1}^*)], \end{aligned}$$

where $j \in \mathbb{N}_0$ represents the number of transmissions and s_j is the time at which the j-th transmission happens. Then if u_k at all $k \in \mathbb{N}_0$ follows (29) for the data transmission scheduling policy (32), we have

$$\begin{aligned} J \leqslant{}& \textstyle\sum_{j=0}^{\infty} \sum_{i=s_j+1}^{s_{j+1}} [(\hat{u}_i - u_i^*)^\mathsf{T} \Phi_1 (\hat{u}_i - u_i^*) \\ & - (Dw_{i-1} - D\hat{w}_{i-1}^*)^\mathsf{T} \Psi_1 (Dw_{i-1} - D\hat{w}_{i-1}^*)]. \end{aligned}$$

for $\hat{u}_i = K\bar{x}_{i|i}$ and $\hat{w}_i^* = S(Ax_i + B\hat{u}_i)$. Based on the scheduling policy (32), and the fact that at data transmission times $\hat{u}_{s_j} = u_{s_j}^*$, for every $s_j \in \mathbb{N}$, we have

$$\begin{aligned} G(\hat{U}_{s_{j+1}}, \hat{W}_{s_{j+1}}) ={}& \textstyle\sum_{i=s_j+1}^{s_{j+1}} [(\hat{u}_{s_j} - u_i^*)^\mathsf{T} \Phi_1 (\hat{u}_{s_j} - u_i^*) \\ & - (Dw_{i-1} - D\hat{w}_{i-1}^*)^\mathsf{T} \Psi_1 (Dw_{i-1} - D\hat{w}_{i-1}^*)] \leqslant 0. \end{aligned}$$

Therefore, we can guarantee $J \leqslant 0$. Furthermore, by substituting $\hat{u}_i$ into $\hat{w}_i^*$, we arrive at $\hat{w}_i^* = SAx_i + SBK\bar{x}_{i|i}$. Moreover, $SBK = (-S+L)A$, which results in $\hat{w}_i^* = SAx_i + (L-S)A\bar{x}_{i|i}$.

Stability when $w = (0, 0, \dots)$: Let us take $V(x_\nu) = x_\nu^\top M x_\nu$ as a Lyapunov function candidate. Then, based on (A.2), when we have $u_k = \hat{u}_k$ and $w_k^* = \hat{w}_k^*$ for all $k \in \mathbb{N}_0$, and $w = (0, 0, \dots)$,

$$\hat{\Delta} V(x_\nu) := V(x_{\nu+1}) - V(x_0) = -\sum_{k=0}^{\nu} [x_k^\top Q x_k + \hat{u}_k^\top \hat{u}_k - (\hat{u}_k - u_k^*)^\top \Phi_1 (\hat{u}_k - u_k^*) + \hat{w}_k^{*\top} D^\top \Psi_1 D \hat{w}_k^*],$$

at every $\nu \in \mathbb{N}_0$. Moreover, based on the event-triggered scheduling law (32), when $w = (0, 0, \dots)$,

$$\sum_{k=0}^{\nu} [-(\hat{u}_k - u_k^*)^\top \Phi_1 (\hat{u}_k - u_k^*) + \hat{w}_k^{*\top} D^\top \Psi_1 D \hat{w}_k^*] \geqslant 0,$$

at every $\nu \in \mathbb{N}_0$. Therefore, $\hat{\Delta} V(x_\nu) \leqslant 0$ at every $\nu \in \mathbb{N}_0$, which indicates the Lyapunov stability of the control loop for the proposed ETC, when $w = (0, 0, \dots)$. Then similar to the proof of Lemma 1, the observability of $(Q^{\frac{1}{2}}, A)$ and the boundedness of the performance index (8), i.e., $J \leqslant x_0^\top M x_0$, guarantees the convergence of the state to zero as time goes to infinity. Therefore, we can conclude the global asymptotic stability of the control loop. Therefore, the proposed ETC is an ℓ_2-consistent ETC according to Definitions 3 and 4.

4. Proof of Theorem 2

The proof procedure is similar to the one presented for Theorem 1. However, here, in order to satisfy $J \leqslant 0$ for all $w \in \ell_2^d$, where J is given in (8), we just need to consider (21). We just need to simplify the disturbance input as it is given in (19) when the control policy follows (33). For $k = \iota\tau + \tau - 1$, we have $\hat{w}_k^* = \bar{S}_{\tau-1}(A x_k + B \hat{u}_k)$, where $\hat{u}_k = K (H^{-1} A)^{\tau-1} \bar{x}_{\iota\tau|\iota\tau}$. Then similar to what we did in the proof of Theorem 1, we can show that $\bar{S}_{\tau-1} B K = (L - \bar{S}_{\tau-1}) A$, which results in $\hat{w}_k^* = \bar{S}_{\tau-1} A x_k + (L - \bar{S}_{\tau-1}) A (H^{-1} A)^{\tau-1} \bar{x}_{\iota\tau|\iota\tau}$. Now when $k \in \mathbb{N}_{\iota\tau}^{\iota\tau+\tau-2}$, then $\hat{w}_k^* = S_h x_k + \gamma^{-2} D^\top [\Theta_{h+1} V_h^{-1} B \quad V_h^{-\top} Z_{h+1}] \hat{U}_k$, where $h = k - \iota\tau$ and

$$\hat{U}_k = -\begin{bmatrix} B^\top M H^{-1} A \\ Y_{h+1}^{-1} Z_{h+1}^\top H^{-1} A \end{bmatrix} (H^{-1} A)^h \bar{x}_{\iota\tau|\iota\tau}.$$

Moreover, we can show that

$$\begin{bmatrix} \Theta_{h+1} V_h^{-1} B & V_h^{-\top} Z_{h+1} \end{bmatrix} \begin{bmatrix} B^\top M H^{-1} A \\ Y_{h+1}^{-1} Z_{h+1}^\top H^{-1} A \end{bmatrix} = \Theta_{h+1} V_h^{-1} A - M H^{-1} A.$$

Then by substitution, $w_k^* = \bar{S}_h x_r + (L - \bar{S}_h) A (H^{-1} A)^h \hat{x}_{\iota\tau|\iota\tau}$, where $\bar{S}_h = \gamma^{-2} D^\top V_h^{-1} \Theta_{h+1} A$ and $L = \gamma^{-2} D^\top M H^{-1} A$.

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1 Auxiliary propositions

Proposition 1 *For the system* (1) *and* (6), *a given* $\gamma>\gamma_1^*$ *and* $\tau\in\mathbb{N}_{>1}$, *there is a closed-loop control policy that meets condition (ii) of Definition 2 if and only if the value* J^* *of the minimax problem* (9) *is non-positive for the given value of* γ *and* $x_0=0$. □

Proof: First, let us denote $J(\bar{u},\bar{w})$ as the value of J in (8) for the given policies of $\bar{u}$ and $\bar{w}$ following the information sets (5) and (2), respectively. Moreover, consider

$$J(\bar{u},\bar{w}^*):=\max_{\{w_k=\mathcal{T}_k(\mathcal{E}_k)\}_{k\in\mathbb{N}_0}} J(\bar{u},w),$$

for all $\gamma\in\mathbb{R}_{>0}$ and a given control policy $\bar{u}$ following the information set (5). Then, the inequality

$$J^*\leqslant J(\bar{u},\bar{w}^*), \tag{A.15}$$

always holds for all $\gamma\in\mathbb{R}_{>0}$ and any given control policy $\bar{u}$ following the information set (5). Now suppose there is a control policy $\bar{u}$ for a given γ, where $J(\bar{u},w)\leqslant 0$ for $x_0=0$ and all $w\in\ell_2^d$, based on Definition 2 part ii. Then we can infer $J(\bar{u},\bar{w}^*)\leqslant 0$. This indicates that J^* is non-positive for $x_0=0$ based on (A.15).

Now we show that it is necessary for J^* to be non-positive in order that for a given $\gamma\in\mathbb{R}_{>0}$, and $x_0=0$, a controller exists, which satisfies (ii) in Definition 2. Suppose that for a given $\gamma\in\mathbb{R}_{>0}$, $J^*>0$, for $x_0=0$. Then based on (A.15), for the given γ, $0<J(\bar{u},\bar{w}^*)$ for all control input policies $\bar{u}$. This means that there exists no control input policy $\bar{u}$, where $J(\bar{u},w)\leqslant 0$ for all $w\in\ell_2^d$ and that specific γ. Therefore, the non-positiveness of J^* for a given $\gamma\in\mathbb{R}_{>0}$ is a necessary and sufficient condition for the existence of a controller, which satisfies condition (ii) of Definition 2.

Proposition 2 *The series of inequalities in* (A.10) *is equivalent to* $\gamma^2 I-\bar{D}_\tau^\mathsf{T}\bar{M}_\tau\bar{D}_\tau>0$, *where* $\bar{M}_\tau$ *and* $\bar{D}_\tau$ *are given in part i of Lemma 2.* □

Proof: For $\tau=1$ the statement holds. Let us assume $\tau\in\mathbb{N}_{>1}$, then for an arbitrary $h\in\mathbb{N}_1^{\tau-1}$ we have

$$\begin{aligned}\Lambda_h(\gamma)=&\gamma^2 I-D^\mathsf{T}\Theta_h D=\gamma^2 I-D^\mathsf{T}(Q+A^\mathsf{T}\Theta_{h+1}A)D\\ &-D^\mathsf{T}(A^\mathsf{T}\Theta_{h+1}D(\gamma^2 I-D^\mathsf{T}\Theta_{h+1}D)^{-1}D^\mathsf{T}\Theta_{h+1}A)D>0\end{aligned}$$

and $\Lambda_{h+1}(\gamma)=\gamma^2 I-D^\mathsf{T}\Theta_{h+1}D>0$. By using the Schur complement, these two inequalities are equivalent to the following inequality

$$\begin{bmatrix}\gamma^2 I-D^\mathsf{T}(Q+A^\mathsf{T}\Theta_{h+1}A)D & -D^\mathsf{T}A^\mathsf{T}\Theta_{h+1}D\\ -D^\mathsf{T}\Theta_{h+1}AD & \gamma^2 I-D^\mathsf{T}\Theta_{h+1}D\end{bmatrix}>0. \tag{A.16}$$

Now if $\tau=2$, then for $h=1$, $\Theta_{h+1}=M$ and we can rearrange (A.16) as

$$\gamma^2\begin{bmatrix}I&0\\0&I\end{bmatrix}-\begin{bmatrix}D^\mathsf{T}&D^\mathsf{T}A^\mathsf{T}\\0&D^\mathsf{T}\end{bmatrix}\begin{bmatrix}Q&0\\0&M\end{bmatrix}\begin{bmatrix}D&0\\AD&D\end{bmatrix}>0,$$

which is equivalent to $\gamma^2 I-\bar{D}_2^\mathsf{T}\bar{M}_2\bar{D}_2>0$ and the argument is proved for $\tau=2$. Now by an induction, assume that for a given $\tau=\hat{\tau}$ the inequality $\bar{\Lambda}_{\hat{\tau}}(\gamma)=\gamma^2 I-\bar{D}_{\hat{\tau}}^\mathsf{T}\bar{M}_{\hat{\tau}}\bar{D}_{\hat{\tau}}>0$ is equivalent to the set of inequalities in (A.10). Let us partition $\bar{D}_{\hat{\tau}}$ as

$$\bar{D}_{\hat{\tau}}=\begin{bmatrix}\bar{D}_{\hat{\tau}-1}&0\\ A\bar{S}_{\hat{\tau}-1}&D\end{bmatrix},$$

where $\bar{S}_{\hat{\tau}-1}=[A^{\hat{\tau}-2}D\dots D]$ and $\bar{M}_{\hat{\tau}}=\mathrm{diag}(\bar{Q}_{\hat{\tau}-1},\Theta_{\hat{\tau}})$ for $\bar{Q}_{\hat{\tau}-1}=I_{\hat{\tau}-1}\otimes Q$. Then

$$\bar{\Lambda}_{\hat{\tau}}(\gamma)=\begin{bmatrix}\gamma^2 I-\bar{R}_{\hat{\tau}-1} & -\bar{S}_{\hat{\tau}-1}^\mathsf{T}A^\mathsf{T}\Theta_{\hat{\tau}}D\\ -D^\mathsf{T}\Theta_{\hat{\tau}}A\bar{S}_{\hat{\tau}-1} & \gamma^2 I-D^\mathsf{T}\Theta_{\hat{\tau}}D\end{bmatrix}>0, \tag{A.17}$$

in which $\bar{R}_{\hat{\tau}-1}=\bar{D}_{\hat{\tau}-1}^\mathsf{T}\bar{Q}_{\hat{\tau}-1}\bar{D}_{\hat{\tau}-1}+\bar{S}_{\hat{\tau}-1}^\mathsf{T}A^\mathsf{T}\Theta_{\hat{\tau}}A\bar{S}_{\hat{\tau}-1}$. Now if $\tau=\hat{\tau}+1$, then we can substitute $\Theta_{\hat{\tau}}$ in (A.17) based on (A.11) which results in

$$\bar{T}-C^\mathsf{T}\Lambda_{\hat{\tau}+1}^{-1}(\gamma)C>0, \tag{A.18}$$

for $C=-D^\mathsf{T}\Theta_{\hat{\tau}+1}\begin{bmatrix}A^2&A\end{bmatrix}\bar{S}_{\hat{\tau}-1}$ and $\bar{T}=[\bar{T}_{1\leqslant i,j\leqslant 2}]$, where

$$\begin{aligned}\bar{T}_{11}=&\gamma^2 I-\bar{D}_{\hat{\tau}-1}^\mathsf{T}\bar{Q}_{\hat{\tau}-1}\bar{D}_{\hat{\tau}-1}-\bar{S}_{\hat{\tau}-1}^\mathsf{T}A^\mathsf{T}Q_{\hat{\tau}}A\bar{S}_{\hat{\tau}-1}\\ &-\bar{S}_{\hat{\tau}-1}^\mathsf{T}A^{2\mathsf{T}}\Theta_{\hat{\tau}+1}A^2\bar{S}_{\hat{\tau}-1},\\ \bar{T}_{21}=&\bar{T}_{12}^\mathsf{T}=-D^\mathsf{T}(Q_{\hat{\tau}}+A^\mathsf{T}\Theta_{\hat{\tau}+1}A)A\bar{S}_{\hat{\tau}-1},\\ \bar{T}_{22}=&\gamma^2 I-D^\mathsf{T}(Q_{\hat{\tau}}+A^\mathsf{T}\Theta_{\hat{\tau}+1}A)D.\end{aligned}$$

Moreover, (A.18) along with $\Lambda_{\hat{\tau}+1}(\gamma)>0$ are equivalent to the following inequality

$$\begin{bmatrix}\bar{T}&C^\mathsf{T}\\C&\Lambda_{\hat{\tau}+1}\end{bmatrix}=\gamma^2 I-\bar{D}_{\hat{\tau}+1}^\mathsf{T}\bar{M}_{\hat{\tau}+1}\bar{D}_{\hat{\tau}+1}>0,$$

which indicates that the assumption of the induction is correct and the argument is proved.

Proposition 3 *The control input policy* $U_\iota^*=\bar{K}_\tau x_\ell$ *at all* $\ell=\iota\tau\in\mathbb{N}_0$, *where* $\bar{K}_\tau$ *follows* (A.8), *is equivalent to* (16) *and* (17). □

Proof: First, we simplify the control gain (A.8). Let us consider $Y_0 = \hat{A} + \hat{B}^\top \hat{C} \hat{B}$ and $Z_0^\top = \hat{B}^\top \hat{C} A$ where

$$\hat{A} = \begin{bmatrix} I & 0 \\ 0 & Y_1 - Z_1^\top \Theta_1^{-1} Z_1 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} B & \Theta_1^{-1} Z_1 \end{bmatrix}, \ \hat{C} = \hat{V}_0^{-1} \Theta_1. \tag{A.19}$$

Moreover, we have

$$\hat{C}^{-1} + \hat{B}\hat{A}^{-1}\hat{B}^\top = \Theta_1^{-1}\hat{V}_0 + BB^\top + \Theta_1^{-1} Z_1 (Y_1 - Z_1^\top \Theta_1^{-1} Z_1)^{-1} Z_1^\top \Theta_1^{-\top} = \Theta_1^{-1} - \gamma^{-2} DD^\top + BB^\top + M^{-1} - \Theta_1^{-1} = HM^{-1},$$

and

$$\hat{A}^{-1}\hat{B}^\top = \begin{bmatrix} B^\top \\ (Y_1 - Z_1^\top \Theta_1^{-1} Z_1)^{-1} Z_1^\top \Theta_1^{-\top} \end{bmatrix} = \begin{bmatrix} B^\top \\ Y_1^{-1} Z_1^\top M^{-1} \end{bmatrix},$$

where the second equality comes from (A.9). Then, we can simplify the control gain based on the following matrix inversion lemma

$$\bar{K}_\tau = -(\hat{A} + \hat{B}^\top \hat{C} \hat{B})^{-1} \hat{B}^\top \hat{C} A = -\hat{A}^{-1}\hat{B}^\top (\hat{C}^{-1} + \hat{B}\hat{A}^{-1}\hat{B}^\top)^{-1} A = -\begin{bmatrix} B^\top \\ Y_1^{-1} Z_1^\top M^{-1} \end{bmatrix} MH^{-1}A = -\begin{bmatrix} B^\top MH^{-1}A \\ Y_1^{-1} Z_1^\top H^{-1} A \end{bmatrix}. \tag{A.20}$$

We can repeat the same simplification procedure for every $Y_h^{-1} Z_h^\top$, where $h \in \mathbb{N}_1^{\tau-1}$ which finally results in

$$\bar{K}_\tau = -\begin{bmatrix} B^\top MH^{-1}A \\ B^\top MH^{-1}AH^{-1}A \\ \vdots \\ B^\top MH^{-1}A(H^{-1}A)^{\tau-1} \end{bmatrix}.$$

This clearly shows that (16) and (17) are equivalent to $U_\iota^* = \bar{K}_\tau x_\ell$.