

Rollout Event-Triggered Control: Beyond Periodic Control Performance

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Abstract—Cyber-Physical Systems (CPSs) resulting from the interconnection of computational, communication, and control (cyber) devices with physical processes are wide spreading in our society. In several CPS applications it is crucial to minimize the communication burden, while still providing desirable closed-loop control properties. To this effect, a promising approach is to embrace the recently proposed event-triggered control paradigm, in which the transmission times are chosen based on well-defined events, using state information. However, few general event-triggered control methods guarantee closed-loop improvements over traditional periodic transmission strategies. Here, we provide a new class of event-triggered controllers for linear systems which guarantee better quadratic performance than traditional periodic time-triggered control using the same average transmission rate. In particular, our main results explicitly quantify the obtained performance improvements for quadratic average cost problems. The proposed controllers are inspired by rollout ideas in the context of dynamic programming.

Index Terms—Approximate dynamic programming, control over communications, event-triggered control, Markov processes, stochastic optimal control.

I. INTRODUCTION

CYBER devices capable of sensing, processing, and communicating information of interest are wide spreading in our society, creating new opportunities to make our physical processes operate exceedingly better. In fact, the number of applications in which communication, computation and control elements (the cyber part) go hand in hand with motion, energy, climate, and human processes (the physical part) is steadily growing in intelligent transportation, smart buildings, energy networks, healthcare, and robotics (see, e.g., [2]–[6], respectively). To meet the challenges arising in many of these applications the traditional separation-of-concerns principle in designing control, communication, and computational algorithms must be abandoned in favor of an integrated approach. This can lead to dramatic communication and computation savings in control applications, which is crucial to prevent

overloading existing and future communication networks, to extend the battery life of cyber devices, and to enable cost-efficient control solutions (see, e.g., [7], [8]).

A research area providing integrated communication and control algorithms that deal with the need to reduce the communication load in (networked) control systems, while at the same time guaranteeing desirable stability and performance properties, is that of event-triggered control (ETC). The key idea of ETC is that transmission times in a networked control loop are triggered based on *events* (using, e.g., state or output information), as opposed to being *time*-triggered as in traditional periodic control.

Extensive research has been conducted on ETC over the past few years leading to various types of ETC strategies; see [9] for a recent overview. For instance, [10] proposes that transmissions should only be triggered when needed to guarantee a certain decrease condition for a Lyapunov function; [11], [12] analyze, in different contexts, the case in which transmissions are triggered only when the loop tracking error exceeds a given threshold; in [13] transmissions are triggered when the error between the measured state and the state of a model-based estimator used by a control input generator is large. Several related problems have been studied in the literature, including self-triggered implementations [14]–[18], co-design [19], [20], discrete-time variants [21]–[24], and periodic event-triggered control [25]. Another line of research formulates ETC in the scope of optimal control by considering cost functions that penalize transmissions [26]–[31]. Some recent works, e.g., [32]–[36], propose model predictive control methods to address related optimal event-triggered control problems. See also [37] for an early work using model predictive control to minimize bandwidth utilization.

Although the large majority of the works on ETC show very promising results, there are few ETC methods which guarantee better closed-loop performance/average transmission rate trade-offs than traditional periodic control. The works [12], [38], [39] proposed event-triggered control laws which have this property, considering a quadratic performance index, but the analysis is restricted to first-order systems. Recently, [40] extended the ideas of [12] to a class of second-order systems, formally establishing the desired ETC performance improvement property over periodic control. However, as acknowledged in [40], it is difficult to extend the results for the considered class of event-triggered controllers to higher order systems. Also in the context of first-order systems, [41], [42] optimally solve estimation and control problems, respectively, in which a quadratic cost is to be minimized, subject to constraints on the number of samples. Yet, in general, it is extremely difficult to obtain optimal event-triggered controllers for higher order systems, although several structural properties of optimal event-triggered controllers can still be inferred (see [26], [27]–[31]).

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In the present paper we present a novel class of event-triggered controllers for linear systems of *arbitrary (finite) order* which achieve better performance than periodic strategies using the same average transmission rate. Performance is measured by a quadratic cost as in the well-known Linear Quadratic Regulator (LQR) and Linear Quadratic Gaussian (LQG) problems (see, e.g., [43]–[45]). Our method, inspired by rollout ideas in the context of dynamic programming [43], consists in choosing, in a receding horizon fashion, optimal control inputs and transmission decisions over a horizon assuming that a base policy, conveniently picked as the optimal periodic control strategy, is used after the horizon. Note that we address the co-design problem, since we consider the problem of simultaneously designing the control input and the transmission times laws. For this new ETC scheme, we show that, under mild conditions, a *strict* performance improvement with respect to periodic control can be guaranteed both for average and discounted quadratic costs using the same average transmission rate. For the average cost problem we explicitly quantify these performance improvements. As quantifying the performance improvements of rollout algorithms is a hard problem,¹ this latter result is the main technical contribution of the paper.

We illustrate the applicability of our event-triggered control method in the problem of controlling a mass-spring linear system. The results show that our method can achieve a closed-loop performance significantly beyond the performance of periodic control using the same average transmission rate.

The remainder of the paper is organized as follows. Section II formulates the problem, and Section III describes the new rollout ETC method. Our main results addressing the performance properties of the proposed method are presented in Section IV. Section V discusses how to extend the main ideas to other networked control configurations. A numerical example is given in Section VI while Section VII provides concluding remarks. The proofs of the main results are given in Section VIII.

Notation: The $n \times m$ zero matrix is denoted by $0_{n \times m}$ and the n -dimensional identity matrix is denoted by I_n . When clear from the context, we omit the subscripts and write 0 and I. The trace of a square matrix A is denoted by $\text{tr}(A)$.

II. PROBLEM FORMULATION

Consider a continuous-time plant modeled by the following stochastic differential equation:

$$dx_C = (A_C x_C + B_C u_C) dt + B_\omega d\omega, \quad x_C(0) = x_0, \quad t \in \mathbb{R}_{\geq 0} \quad (1)$$

where $x_C(t) \in \mathbb{R}^{n_x}$ is the state and $u_C(t) \in \mathbb{R}^{n_u}$ is the control input at time $t \in \mathbb{R}_{\geq 0}$, and ω is an n_ω -dimensional Wiener process with incremental covariance $I_{n_\omega} dt$ (cf. [44]). Performance is measured by the discounted cost

$$\int_0^\infty \mathbb{E} [e^{-\alpha_C t} g_C(x_C(t), u_C(t))] dt \quad (2)$$

¹As stated in [43, p. 338]: ‘Empirically, it has been observed that the rollout policy typically produces considerable (and often dramatic) cost improvement over the base policy. However, there is no solid theoretical support for this observation.’

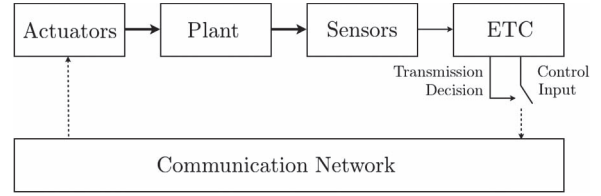


Fig. 1. Setup: the plant operates in continuous-time (continuous-time connections are indicated by thick solid lines); the event-triggered controller operates at discrete times $\{t_k\}_{k \in \mathbb{N}_0}$ (discrete-time connections are indicated by thin solid lines); transmissions over the communication network occur only at times $\{t_k | \sigma_k = 1, k \in \mathbb{N}_0\}$ (connections are indicated by thin dashed lines). The event-triggered controller periodically samples the state of the plant and decides the transmission times $\{t_k | \sigma_k = 1, k \in \mathbb{N}_0\}$ at which it computes the control input and transmits it to the actuators; at these times the actuators receive the control input enforcing it in the plant.

where $g_C(x, u) := x^\top Q_C x + u^\top R_C u$, for positive semi-definite matrices Q_C and R_C , and $\alpha_C \in \mathbb{R}_{\geq 0}$. To guarantee that (2) is bounded we assume that α_C may only take the value $\alpha_C = 0$ if $B_\omega = 0$. For the undiscounted case $\alpha_C = 0$ in which (1) is disturbed by Gaussian noise ($B_\omega \neq 0$) performance is measured by the following average cost:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbb{E} [g_C(x_C(t), u_C(t))] dt. \quad (3)$$

Performance indexes (2) and (3) are widely used in control problems. In particular, when $\alpha_C = 0$ the problems of designing a feedback strategy for the control input u_C to minimize (2) and (3) can be seen as versions of the well-known LQR and LQG problems, respectively. The LQR and LQG problems are also considered in the context of sampled-data systems [46], in which case u_C is a staircase signal updated *periodically* and designed to minimize discrete-time equivalents of (2) and (3), respectively. The main motivation of the present work is to show that, by properly choosing the actuation update times (which shall coincide with transmission times in networked control settings) in a non-periodic fashion, one can achieve better performance indexes as considered in the LQR and LQG problems, using the same average actuation (or transmission) rate.²

For ease of exposition, we assume that a scheduler-controller pair is collocated with the plant sensors and that it is connected to the actuators by a communication network. The scheduler-controller periodically samples the state of the plant x_C and decides whether or not to compute and transmit control and measurement data over a network to the actuators, as it is common in so called periodic event-triggered control (see, e.g., [25]). The setup is depicted in Fig. 1, where the scheduler-controller is denoted by event-triggered controller (ETC). While we consider this setup for concreteness, the ideas of our proposed methods can be applied in a straightforward manner also to other configurations (cf. Section V).

We denote the sampling times by $t_k, k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, spaced by a baseline period $\tau \in \mathbb{R}_{>0}$, i.e., $t_k = k\tau, k \in \mathbb{N}_0$. We assume that the network is always available for transmitting data at times $t_k, k \in \mathbb{N}_0$, that the transmission delays are small

²In fact, while the case $\alpha_C > 0$ is interesting in its own right, here we consider a discounted cost (2) mainly for convenience and we shall be mostly interested in $\alpha_C = 0$ (cf. Section III-D).

with respect to τ , and that the probability of a packet drop is small. These assumptions are reasonable in shared networks using Time-Division Multiple Access (TDMA) protocols [47] or if the communication between the scheduler-controller and actuators is made via a point-to-point dedicated link. The scheduler-controller may wish to refrain from transmitting the state to the actuators at the available times t_k , $k \in \mathbb{N}_0$, in order to: (i) reduce power consumption [7]; (ii) allow for other (non-critical) data to be transmitted.

Assuming that the actuation is held constant between sampling times and that the transmission delays are negligible

$$u_C(t) = u_C(t_k), \quad \forall t \in [t_k, t_{k+1}). \quad (4)$$

Let $\{\sigma_k\}_{k \in \mathbb{N}_0}$ be the transmission scheduling sequence

$$\sigma_k := \begin{cases} 1, & \text{if a transmission occurs at } t_k \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, for $k \in \mathbb{N}$, let $x_k := x_C(t_k)$ and $\hat{u}_k := u_C(t_{k-1})$, and let $x_0 \in \mathbb{R}^{n_x}$ and $\hat{u}_0 \in \mathbb{R}^{n_u}$ be given initial conditions. Furthermore, let $\xi_k := [x_k^\top \hat{u}_k^\top]^\top \in \mathbb{R}^n$, $k \in \mathbb{N}_0$, $n := n_x + n_u$, and u_k be the control input sent by the controller to the actuators at times t_k , $k \in \mathbb{N}_0$, that satisfy $\sigma_k = 1$; at times t_k , $k \in \mathbb{N}_0$, that satisfy $\sigma_k = 0$ we use the notation $u_k := \emptyset$, also used in [30], to denote that u_k is not transmitted. Then, we can write

$$\xi_{k+1} = \begin{cases} A_1 \xi_k + B_1 u_k + w_k, & \text{if } \sigma_k = 1 \\ A_0 \xi_k + w_k, & \text{if } \sigma_k = 0, \quad k \in \mathbb{N}_0 \end{cases} \quad (5)$$

where, for $j \in \{0, 1\}$

$$A_j := \begin{bmatrix} \bar{A}_\tau & (1-j)\bar{B}_\tau \\ 0 & (1-j)I_{n_u} \end{bmatrix}, \quad B_1 := \begin{bmatrix} \bar{B}_\tau \\ I_{n_u} \end{bmatrix} \\ \bar{A}_\tau := e^{A_C \tau}, \quad \bar{B}_\tau := \int_0^\tau e^{A_C s} ds B_C \quad (6)$$

and w_k , $k \in \mathbb{N}_0$, is a sequence of zero-mean independent random vectors with covariance $\mathbb{E}[w_k w_k^\top] = \Phi_\tau^w$, $k \in \mathbb{N}_0$

$$\Phi_\tau^w := \begin{bmatrix} \bar{\Phi}_\tau^w & 0_{n_x \times n_u} \\ 0_{n_u \times n_x} & 0_{n_u \times n_u} \end{bmatrix}, \quad \bar{\Phi}_\tau^w := \int_0^\tau e^{A_C s} B_\omega B_\omega^\top e^{A_C^\top s} ds.$$

The expression for $\bar{\Phi}_\tau^w$ can be obtained from the arguments provided in [44, Sec. 3.10].

We are interested in the problem of finding a policy, i.e., a set of functions

$$\pi = \{(\mu_0^\sigma(l_0), \mu_0^u(l_0)), (\mu_1^\sigma(l_1), \mu_1^u(l_1)), \dots\}$$

that describe the scheduling and control inputs

$$(\sigma_k, u_k) = (\mu_k^\sigma(l_k), \mu_k^u(l_k)), \quad \forall k \in \mathbb{N}_0 \quad (7)$$

based on the information available to the scheduler-controller at time t_k

$$l_k := \{(\xi_\ell, \sigma_\ell) | 0 \leq \ell < k\} \cup \{\xi_k\}, \quad \forall k \in \mathbb{N}_0.$$

Hence, note that we consider here the problem of *co-designing* scheduling and control inputs. By keeping track of previous data in l_k , the scheduler-controller can, e.g., make decisions based on the number of previous transmissions up to time t_k or based on previous state values. Note that $(\mu_k^\sigma(l_k), \mu_k^u(l_k)) \in (\{0\} \times \{\emptyset\}) \cup (\{1\} \times \mathbb{R}^{n_u})$, $\forall k \in \mathbb{N}_0$.

The discounted cost (2) can be shown to be given, apart from an additive constant factor, by

$$J_\pi^d(\xi_0) := \sum_{k=0}^{\infty} \mathbb{E} [\alpha_\tau^k g(\xi_k, \mu_k^u(l_k), \mu_k^\sigma(l_k))] \quad (8)$$

where $\alpha_\tau := e^{-\alpha_C \tau}$, $g(\xi, u, j) := \xi^\top Q_j \xi + 2\xi^\top S_j u + u^\top R_j u$, and, for $j \in \{0, 1\}$

$$Q_j := \begin{bmatrix} \bar{Q}_\tau & (1-j)\bar{S}_\tau \\ (1-j)\bar{S}_\tau^\top & (1-j)\bar{R}_\tau \end{bmatrix}, \quad S_j := \begin{bmatrix} j\bar{S}_\tau \\ 0 \end{bmatrix}, \quad R_j := j\bar{R}_\tau$$

where

$$\begin{bmatrix} \bar{Q}_\tau & \bar{S}_\tau \\ \bar{S}_\tau^\top & \bar{R}_\tau \end{bmatrix} := \int_0^\tau e \begin{bmatrix} A_C & B_C \\ 0 & 0 \end{bmatrix}^\top s \begin{bmatrix} Q_C & 0 \\ 0 & R_C \end{bmatrix} e \begin{bmatrix} A_C & B_C \\ 0 & 0 \end{bmatrix} s e^{-\alpha_C s} ds \quad (9)$$

and the average cost (3) can be described by

$$J_\pi^a := \lim_{K \rightarrow \infty} \frac{1}{\tau K} \sum_{k=0}^{K-1} \mathbb{E} [g(\xi_k, \mu_k^u(l_k), \mu_k^\sigma(l_k))] \quad (10)$$

We denote by J_π^c , $c \in \{a, d\}$, a cost which pertains to the discounted cost if $c = d$ and to the average cost if $c = a$. The discounted cost depends on ξ_0 , whereas for the policies considered in the present paper the average cost does not, as we shall see in the sequel (cf. Remark 14 below). We omit this dependency and for two policies π and ρ we use $J_\pi^d \leq J_\rho^d$ to denote $J_\pi^d(\xi_0) \leq J_\rho^d(\xi_0)$ for every $\xi_0 \in \mathbb{R}^n$. The average transmission rate of policy π is defined as

$$R_\pi := \frac{1}{\tau} \lim_{K \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} [\mu_k^\sigma(l_k)] \quad (11)$$

which also does not depend on the initial condition ξ_0 for the policies considered here (this follows trivially from the definition of Algorithm 3 below).

Traditional periodic control can be captured in the above setup by fixing the scheduling input σ_k , $k \in \mathbb{N}_0$, to correspond to transmissions once every $q \in \mathbb{N}$ time steps ($\sigma_k = 1$, if k is an integer multiple of q , $\sigma_k = 0$ otherwise), in which case $\delta := q\tau$ is the sampling period. In fact, suppose that the following standard assumptions hold:

Assumption 1:

- (i) (A_C, B_C) is controllable and B_C has full rank.
- (ii) $(A_C, Q_C^{1/2})$ is observable.
- (iii) R_C is positive definite.
- (iv) The sampling period $q\tau$ is non-pathological, i.e., A does not have two eigenvalues with equal real parts and imaginary parts that differ by an integral multiple of $2\pi/q\tau$ (cf. [46, p. 45]).

Then, from standard optimal control arguments (cf. [43], [44]), we can obtain the *optimal* control law, which results in the combined scheduling and control policy, both for the average

and discounted cost problems, $\gamma = \{(\mu_0^\sigma, \mu_0^u), (\mu_1^\sigma, \mu_1^u), \dots\}$, given for $k \in \mathbb{N}_0$ by

$$(\mu_k^\sigma(l_k), \mu_k^u(l_k)) = \begin{cases} (1, \bar{K}_\delta x_k), & \text{if } k \text{ is an integer multiple of } q \\ (0, \emptyset), & \text{otherwise} \end{cases} \quad (12)$$

where

$$\bar{K}_\delta := -(\bar{R}_\delta + \alpha_\delta \bar{B}_\delta^\top \bar{P}_\delta \bar{B}_\delta)^{-1} (\alpha_\delta \bar{B}_\delta^\top \bar{P}_\delta \bar{A}_\delta + \bar{S}_\delta^\top) \quad (13)$$

and \bar{P}_δ is the unique positive definite solution to the algebraic Riccati equation

$$\begin{aligned} \bar{P}_\delta &= \alpha_\delta \bar{A}_\delta^\top \bar{P}_\delta \bar{A}_\delta + \bar{Q}_\delta \\ &- (\alpha_\delta \bar{A}_\delta^\top \bar{P}_\delta \bar{B}_\delta + \bar{S}_\delta) (\bar{R}_\delta + \alpha_\delta \bar{B}_\delta^\top \bar{P}_\delta \bar{B}_\delta)^{-1} (\alpha_\delta \bar{B}_\delta^\top \bar{P}_\delta \bar{A}_\delta + \bar{S}_\delta^\top). \end{aligned} \quad (14)$$

This policy has an average transmission rate (11) of $1/\delta$, a discounted cost

$$J_{\text{per},\delta}^d := x_0^\top \bar{P}_\delta x_0 + \frac{\alpha_\delta}{1 - \alpha_\delta} \text{tr}(\bar{P}_\delta \bar{\Phi}_\delta^w) \quad (15)$$

and an average cost

$$J_{\text{per},\delta}^a := \frac{1}{\delta} \text{tr}(\bar{P}_\delta \bar{\Phi}_\delta^w) \quad (16)$$

(cf. [43, Vol. II, p. 142 and 273]). The main focus of this paper is to design combined scheduling/control policies which achieve (strictly) better performance than traditional periodic control using the same average transmission rate. This design problem can be formally written as follows.

Problem 2: Given a desirable transmission rate $1/q\tau$, for some $q \in \mathbb{N}$, find a policy π for which $R_\pi = 1/q\tau$ and

$$J_\pi^c < J_{\text{per},q\tau}^c \quad (17)$$

where $c = d$ if the performance is measured by (2) (discounted cost problem) and $c = a$ if the performance is measured by (3) (average cost problem). \square

A natural additional challenge after designing a policy that guarantees (17) is to quantify how much is the performance improvement expressed in (17). In Section IV-D, we address this challenge for the average cost problem ($c = a$).

III. ROLLOUT EVENT-TRIGGERED CONTROL

The proposed method is a receding horizon algorithm. At a given step of the algorithm, m transmission decisions over a horizon of h possible scheduling decisions are chosen, based on which transmission pattern would lead to a lower cost, assuming that after the horizon an optimal periodic control policy would be used, also using m transmissions in each block of h scheduling decisions (see Fig. 2). Since periodic transmission belongs to the options of the optimization procedure at each step, we will be able to prove in the sequel that this strategy outperforms periodic control. We formalize the algorithm by (i) defining the admissible transmission scheduling decisions over the horizon h in Section III-A; (ii) determining the optimal

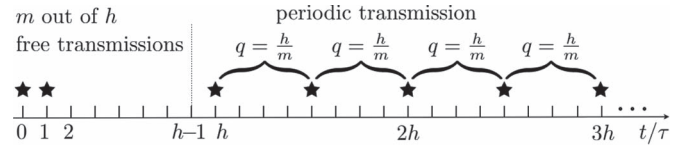


Fig. 2. Illustration of a scheduling option at scheduling decision steps.

control policy and associated cost for each of these scheduling sequences in Section III-B; and (iii) specifying the execution of the algorithm in Section III-C. We consider a discounted cost framework with $\alpha_C > 0$ for convenience in Section III-A-C and in Section III-D we consider the case $\alpha_C = 0$ which includes the average cost problem. The implementation of the proposed method is discussed in Section III-E.

A. Admissible Scheduling Sequences

Let \mathcal{T} denote the set of transmission scheduling sequences with m transmissions in the first h time steps $0, \tau, \dots, (h-1)\tau$, where h is an integer multiple of m , and that conform with periodic transmission with period $q\tau$, $q := h/m$, in the subsequent time steps $h\tau, (h+1)\tau, \dots$, starting with a transmission at $h\tau$ (see Fig. 2). The parameters h and q can be viewed as tuning knobs of the proposed ETC algorithm. Formally, there are

$$n_{\mathcal{T}} := \frac{h!}{(h-m)!m!} \quad (18)$$

scheduling sequences $\{\sigma_k^i\}_{k \in \mathbb{N}_0} \in \mathcal{T}$, $i \in \mathcal{M}$, $\mathcal{M} := \{1, \dots, n_{\mathcal{T}}\}$, characterized by

$$\sigma_k^i = \nu_k^i, \quad k \in \{0, 1, \dots, h-1\}, \quad i \in \mathcal{M} \quad (19)$$

where $\nu^i = (\nu_0^i, \dots, \nu_{h-1}^i) \in \mathcal{I}$, $i \in \mathcal{M}$, with

$$\mathcal{I} := \left\{ \nu \in \{0, 1\}^h \mid \sum_{k=0}^{h-1} \nu_k = m \right\} \quad (20)$$

and by

$$\sigma_k^i = \begin{cases} 1, & \text{if } k \text{ is an integer multiple of } q \\ 0, & \text{otherwise, } k \in \mathbb{N}_{\geq h}, \quad i \in \mathcal{M} \end{cases}$$

We assume that $q \geq 2$ and without loss of generality, we arbitrate that the schedules $\nu^1 \in \mathcal{I}$ are described by

$$\nu_\kappa^1 = \begin{cases} 1, & \text{if } \kappa = 0 \text{ or if } \kappa \text{ is an integer multiple of } q \\ 0, & \text{otherwise} \end{cases} \quad (21)$$

for $0 \leq \kappa \leq h-1$. The associated scheduling sequence in \mathcal{T} corresponds to periodic transmission with period $q\tau$.

B. Optimal Policy and Cost for Each Scheduling Sequence

Our proposed method is based on solutions to optimal control subproblems in which the transmission scheduling sequence is fixed and belongs to the set \mathcal{T} . Here, we describe the optimal control input policy that minimizes (8) for a fixed scheduling sequence in \mathcal{T} , labeled by $i \in \mathcal{M}$, which can be derived by standard optimal control arguments (cf. [43], [44]), under Assumption 1.

Let P_i be the first matrix $W_{0,i}$ of the backward recursion

$$W_{h,i} = \begin{bmatrix} \bar{P}_{q\tau} & 0 \\ 0 & 0_{n_u \times n_u} \end{bmatrix}$$

$$W_{\kappa,i} = F_{\nu_\kappa^i}(W_{\kappa+1,i}), \quad 0 \leq \kappa \leq h-1 \quad (22)$$

where $\bar{P}_{q\tau}$ can be obtained as the solution to (14) (with δ replaced by $q\tau$) and

$$F_0(P) := \alpha_\tau A_0^\top P A_0 + Q_0$$

$$F_1(P) := \alpha_\tau A_1^\top P A_1 + Q_1$$

$$- (S_1 + \alpha_\tau A_1^\top P B_1) (R_1 + \alpha_\tau B_1^\top P B_1)^{-1}$$

$$\times (\alpha_\tau B_1^\top P A_1 + S_1^\top).$$

Then the optimal control input policy corresponding to the scheduling sequence $\{\sigma_k^i\}_{k \in \mathbb{N}_0}$ is described by

$$u_k = \begin{cases} K_{k,i} x_k, & \text{if } \nu_k^i = 1 \\ \emptyset, & \text{otherwise} \end{cases}$$

for $k \in \{0, 1, \dots, h-1\}$, where for $\nu_k^i = 1$ the gains $K_{k,i}$ are given by

$$K_{k,i} := - (R_1 + \alpha_\tau B_1^\top W_{k+1,i} B_1)^{-1}$$

$$\times (\alpha_\tau B_1^\top W_{k+1,i} A_1 + S_1^\top) [1_{n_x} \ 0_{n_x \times n_u}]^\top \quad (23)$$

and for $k \in \mathbb{N}_{\geq h}$

$$u_k = \begin{cases} \bar{K}_{q\tau} x_k, & \text{if } k \text{ is an integer multiple of } q \\ \emptyset, & \text{otherwise} \end{cases}$$

where $\bar{K}_{q\tau}$ is described by (13). The discounted cost (8) for this policy is given by

$$\xi_0^\top P_i \xi_0 + c_i + b, \quad i \in \mathcal{M} \quad (24)$$

where $b := (\alpha_\tau^h \alpha_{q\tau} / (1 - \alpha_{q\tau})) \text{tr}(\bar{P}_{q\tau} \bar{\Phi}_{q\tau}^w)$ and

$$c_i := \sum_{\kappa=1}^h \alpha_\tau^\kappa \text{tr}(W_{\kappa,i} \bar{\Phi}_\tau^w), \quad i \in \mathcal{M}.$$

Note that when $i \in \mathcal{M}$ corresponds to periodic control, $i = 1$, cost (24) equals (15), which implies that

$$P_1 = \begin{bmatrix} \bar{P}_{q\tau} & 0 \\ 0 & 0 \end{bmatrix} \quad (25)$$

and

$$c_1 = \sum_{\kappa=1}^m \alpha_{q\tau}^\kappa \text{tr}(\bar{P}_{q\tau} \bar{\Phi}_{q\tau}^w). \quad (26)$$

C. Algorithm

The proposed rollout method, described next, finds at each scheduling time, in a receding horizon fashion, the scheduling sequence in \mathcal{T} that would optimize (8) if this scheduling sequence would be used thereafter, along with a corresponding optimal policy for the control input.

Algorithm 3

(i) At scheduling times $\ell := jh, j \in \mathbb{N}_0$, compute³

$$\iota(\xi_\ell) = \underset{i \in \mathcal{M}}{\text{argmin}} \xi_\ell^\top P_i \xi_\ell + c_i. \quad (27)$$

In view of (24), $\iota \in \mathcal{M}$ corresponds to a scheduling sequence from the set \mathcal{T} which would lead to the smallest cost (8) if this fixed scheduling sequence would be used from time $\ell = jh$ onwards and an associated optimal policy would be chosen for the control input.

(ii) For times $k \in \{jh, jh+1, \dots, (j+1)h-1\}$ pick the schedules $\sigma_k = \nu_{k-jh}^{\iota(\xi_\ell)}$ and the control inputs

$$u_k = \begin{cases} K_{k-jh, \iota(\xi_\ell)} x_k, & \text{if } \sigma_k = 1 \\ \emptyset, & \text{otherwise.} \end{cases} \quad (28)$$

Repeat (i) and (ii) at scheduling time $(j+1)h$. \square

Note that at time jh step (i) fixes the scheduling actions and the control *policy* (the feedback gains) to be taken in the interval $k \in \{jh, jh+1, \dots, (j+1)h-1\}$, but not the control *actions*. The latter are computed from (28) based on the actual state x_k of the plant at times $k \in \{jh, jh+1, \dots, (j+1)h-1\}$ with $\sigma_k = 1$.

In the terminology of Section II, Algorithm 3 corresponds to a family of policies described by $\rho = \{(\mu_0^\sigma, \mu_0^u), (\mu_1^\sigma, \mu_1^u), \dots\}$

$$(\mu_k^\sigma(1_k), \mu_k^u(1_k)) = \left(\nu_{k-jh}^{\iota(\xi_{jh})}, K_{k-jh, \iota(\xi_{jh})} x_k \right)$$

$$jh \leq k < (j+1)h, \quad j \in \mathbb{N}_0. \quad (29)$$

D. Average Cost Problem, $\alpha_C = 0$

Considering $\alpha_C > 0$ in the previous section was convenient since for the average cost problem, the costs (24) are unbounded (the constant b tends to infinite as $\alpha_C \downarrow 0$). However, since Algorithm 3 does not depend on b , we can still consider the algorithm for the average cost problem ($\alpha_C = 0$ and $B_\omega \neq 0$). In this case, Algorithm 3 can be viewed as a suboptimal method for designing a combined scheduling and control policy for the average cost problem, obtained by taking the limit as α_C tends to zero of the suboptimal method derived for the discounted cost problem. Note that in the case $B_\omega = 0$ and $\alpha_C = 0$ in (24) we have $b = 0$ and $c_i = 0, \forall i \in \mathcal{M}$, and one can also consider Algorithm 3.

E. Implementation

Although Algorithm 3 relies on receding horizon ideas, it does not require any on-line optimization. This resembles explicit model predictive control [48]. In fact, Algorithm 3 requires only computing the explicit functions (27) and (28). For each recursion of the algorithm, computing (28) requires at most $m n_u n_x$ multiplications, whereas computing (27) for a state $v = \xi_\ell$ with components $v_i, 1 \leq i \leq n$, requires at most $(n_{\mathcal{T}} + 1)(n(n+1)/2)$ multiplications since each of the

³We arbitrate that if the minimum argument in (27) is achieved by two or more indexes $i_1, i_2 \in \mathcal{M}$ the smallest index is selected, although this is not relevant in the results that follow. Hence, by the argmin function in (27) we mean $\underset{i \in \mathcal{M}}{\text{argmin}} \xi_\ell^\top P_i \xi_\ell + c_i := \min(h^{-1}(\min_{i \in \mathcal{M}} h(i)))$, where $h(i) = \xi_\ell^\top P_i \xi_\ell + c_i$.

$n_{\mathcal{T}}$ quadratic functions can be computed in terms of linear combinations of the $n(n+1)/2$ products $v_i v_j$, $1 \leq i, j \leq n$ (these products are computed once at each scheduling decision time and the $n_{\mathcal{T}}$ linear combinations are computed after). In the numerical example of Section VI, we consider the following parameters: $h = 6$, $m = 2$, which results from (18) in $n_{\mathcal{T}} = 15$. Note that additions and other operations as taking the minimum in (27) typically have a negligible computational burden with respect to multiplications.

IV. MAIN RESULTS

In Section IV-A we establish that the proposed rollout algorithm (Algorithm 3) performs no worse than periodic control both for the average cost and the discounted cost problems. Obtaining strict performance improvement results requires additional technical assumptions and these results are presented in Section IV-B. In Section IV-C we discuss the stability properties of our proposed method and in Section IV-D we quantify the performance improvements for the average cost problem. The proofs are deferred to Section VIII.

A. Performance Improvement

We start with the following performance improvement result which requires only the basic Assumption 1. Let $J_{\rho, q\tau}^c$, $c \in \{a, d\}$, denote the discounted cost (2) of the policy ρ , described by (29), when $c = d$ and the average cost (3) of the policy ρ when $c = a$.

Theorem 4: Consider Algorithm 3 for $\tau \in \mathbb{R}_{>0}$, $q \in \mathbb{N}$, $m \in \mathbb{N}$, and $\alpha_C \geq 0$, and suppose that Assumption 1 holds. Then

$$J_{\rho, q\tau}^d \leq J_{\text{per}, q\tau}^d. \quad (30)$$

Moreover, if $\alpha_C = 0$, then

$$J_{\rho, q\tau}^a \leq J_{\text{per}, q\tau}^a. \quad (31)$$

□

It is clear, from the construction in Algorithm 3, that policy (29) yields an average transmission rate (11) equal to $1/q\tau$. Thus Theorem 4, establishes that policy (29) performs no worse than the traditional periodic strategy with a corresponding transmission rate $1/q\tau$. In fact, in most situations policy (29) performs strictly better, thus providing a solution to Problem 2. However, this is often hard to guarantee formally [43, p. 338]. In the next section we will prove formally that, under given assumptions, policy (29) performs strictly better than the traditional periodic strategy. Still, such assumptions do not encompass important cases (e.g. $B_\omega = 0$ and $\alpha_C = 0$) captured by Theorem 4, which is hence interesting in its own right.

B. Strict Performance Improvement

Consider the following assumptions:

Assumption 5:

- (i) (A_C, B_ω) is controllable.
- (ii) The following matrix has full rank

$$\alpha_s \bar{A}_s^\top \bar{P}_{q\tau} \bar{B}_s + \bar{S}_s \quad (32)$$

for every $s \in \{k\tau | k \in \{1, \dots, (h-m+1)\}\}$.

□

Assumption 5(i) guarantees that all the states of (1) are affected by the disturbance input. Assumption 5(ii) is a mild technical assumption to simplify the proof of our main results and, along with Assumption 5(i), it is used to guarantee that (5) driven by policy (29) (described by (35) below) is not concentrated in some lower dimensional subset of the state space \mathbb{R}^n (see Remark 15 below). Assumption 5(ii) is rather mild. In Lemma 11 we prove that Assumption 5(ii) always holds for sufficiently small τ . Moreover, as discussed in Remark 15 below, Assumption 5(ii) is not necessary for the theorems stated in the sequel to hold (Theorems 7, 9, and 10).

In addition to Assumption 5, to obtain strict performance improvement of the rollout ETC method for the discounted cost problem ($\alpha_C > 0$) we need the following assumption.

Assumption 6: $\bar{K}_{q\tau} \neq \bar{K}_{q\tau}(\bar{A}_{q\tau} + \bar{B}_{q\tau}\bar{K}_{q\tau})$.

Assumption 6 is equivalent to the optimal periodic control inputs $u_k = \bar{K}_{q\tau}x_k$, $k \in \mathbb{N}_0$ (see (12)) not being equal to a constant signal, which may occur (pathologically) for a given $\alpha_C > 0$. Note that, since $(\bar{A}_{q\tau} + \bar{B}_{q\tau}\bar{K}_{q\tau})$ is Hurwitz when $\alpha_C = 0$ (cf. [43]), Assumption 6 holds for $\alpha_C = 0$, which is the case we are mostly interested in.

We state next the strict performance improvement result.

Theorem 7: Consider Algorithm 3 for $\tau \in \mathbb{R}_{>0}$, $q \in \mathbb{N}$, $m \in \mathbb{N}$ and $\alpha_C \geq 0$. Then, if Assumptions 1, 5 and 6 are satisfied, the following holds:

$$J_{\rho, q\tau}^d < J_{\text{per}, q\tau}^d. \quad (33)$$

Moreover, if Assumptions 1 and 5 are satisfied and $\alpha_C = 0$ then

$$J_{\rho, q\tau}^a < J_{\text{per}, q\tau}^a. \quad (34)$$

□

C. Stability

Here we investigate the implications of the performance improvement results Theorems 4, 7 for the stability of the closed-loop when the rollout ETC method is used. We restrict ourselves to $\alpha_C = 0$ since if $\alpha_C > 0$ it might be the case that (2) is bounded but the state grows unbounded even for the optimal periodic controller, in the absence of plan disturbances.

In this setting ($\alpha_C = 0$), consider first that no disturbances act on the plant ($B_\omega = 0$) and hence stability is simply defined as the state converging to zero. Then, as shown in the next result, (exponential) stability follows readily from the performance improvement result (30).

Theorem 8: Consider Algorithm 3 for $\tau \in \mathbb{R}_{>0}$, $q \in \mathbb{N}$, $m \in \mathbb{N}$ and $\alpha_C = 0$, and suppose that Assumption 1 holds and $B_\omega = 0$. Then there exists $c \in \mathbb{R}_{>0}$ and $\alpha \in \mathbb{R}_{>0}$, $0 < \alpha < 1$, such that $\|\xi_k\| \leq c\alpha^k \|\xi_0\|$, $\forall k \in \mathbb{N}_0$. □

We consider next the case $B_\omega \neq 0$ (and $\alpha_C = 0$). We shall establish a stability property (ergodicity) for the Markov chain (see [49]) obtained by considering (5) driven by policy (29) along a period h . In fact, let $\bar{\xi}_\ell := \xi_{\ell h}$ and

$$\bar{w}_\ell := [w_{\ell h}^\top \ w_{\ell h+1}^\top \ \dots \ w_{\ell h+h-1}^\top]^\top, \quad \ell \in \mathbb{N}_0.$$

Then (5) driven by policy (29) along a period h can be described by the Markov chain

$$\bar{\xi}_{\ell+1} = \Phi_{\iota(\bar{\xi}_\ell)} \bar{\xi}_\ell + \Psi_{\iota(\bar{\xi}_\ell)} \bar{w}_\ell, \quad \ell \in \mathbb{N}_0 \quad (35)$$

where $\iota(\bar{\xi}_\ell)$ is described by (27), and the matrices Φ_j and Ψ_j , $j \in \mathcal{M}$, are given by

$$\Phi_j := \Pi_{s=h-1}^0 \Theta_{s,j} = \Theta_{h-1,j} \Theta_{h-2,j} \dots \Theta_{0,j}$$

and

$$\Psi_j := [\Pi_{s=h-1}^1 \Theta_{s,j} \Pi_{s=h-1}^2 \Theta_{s,j} \dots \Theta_{h-1,j} \mathbf{I}_n]$$

where for $0 \leq \kappa \leq h-1$ and $j \in \mathcal{M}$,

$$\Theta_{\kappa,j} = \begin{cases} A_0, & \text{if } \nu_{\kappa}^j = 0 \\ A_1 + B_1 [K_{\kappa,j} \ 0_{n_u \times n_u}], & \text{if } \nu_{\kappa}^j = 1. \end{cases} \quad (36)$$

Let

$$P^\ell(y, A) := \text{Prob}[\bar{\xi}_\ell \in A | \bar{\xi}_0 = y] \quad (37)$$

be the probability that the chain (35) is in a set A at $\ell \in \mathbb{N}$ given that it starts at time zero in state $y \in \mathbb{R}^n$. In addition, recall that a probability measure $\chi_{\text{inv}} : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, 1]$, where $\mathcal{B}(\mathbb{R}^n)$ denotes the collection of Borel sets in \mathbb{R}^n , is said to be an invariant probability distribution for (35) if $\int_{\mathbb{R}^n} P^1(\xi, A) \chi_{\text{inv}}(d\xi) = \chi_{\text{inv}}(A)$ for every $A \in \mathcal{B}(\mathbb{R}^n)$ (cf. [49, Ch.10]). We state next that when Algorithm 3 corresponds to $\alpha_C = 0$, the Markov chain (35) is ergodic [49, Ch. 13].

Theorem 9: Consider Algorithm 3 for $\tau \in \mathbb{R}_{>0}$, $q \in \mathbb{N}$, $m \in \mathbb{N}$, and $\alpha_C = 0$, and suppose that Assumptions 1 and 5 hold. Then, there exists a unique invariant measure for the Markov chain (35), denoted by χ_{inv} , and (35) is ergodic, i.e.

$$\lim_{\ell \rightarrow \infty} \sup_{A \in \mathcal{B}(\mathbb{R}^n)} |P^\ell(y, A) - \chi_{\text{inv}}(A)| = 0. \quad (38)$$

□

Ergodicity is a crucial property to quantify the performance improvements obtained with the rollout method for the average cost problem.

D. Quantifying the Performance Improvements

In the following result we explicitly quantify the performance improvement obtained with the rollout method over optimal periodic control for average cost problems. Due to the difficulty in obtaining such results (cf. [43, Ch. 6]), the following is one of the main results of the paper.

Theorem 10: Consider Algorithm 3 for $\tau \in \mathbb{R}_{>0}$, $q \in \mathbb{N}$, $m \in \mathbb{N}$ and $\alpha_C = 0$. Then, if Assumptions 1 and 5 are satisfied

$$J_{\rho, q\tau}^a = J_{\text{per}, q\tau}^a - g_a \quad (39)$$

where g_a is a strictly positive constant given by

$$g_a = \frac{1}{\tau h} \int_{\mathbb{R}^n} f(\xi) \chi_{\text{inv}}(d\xi) \quad (40)$$

with

$$f(\xi) := \xi^\top (P_1 - P_{\iota(\xi)}) \xi + c_1 - c_{\iota(\xi)} \quad (41)$$

and χ_{inv} is the unique invariant measure of the Markov chain (35). □

Note that g_a is nonnegative since the integrand $f(\xi)$ is nonnegative due to (27). Theorem 10 states that g_a is actually

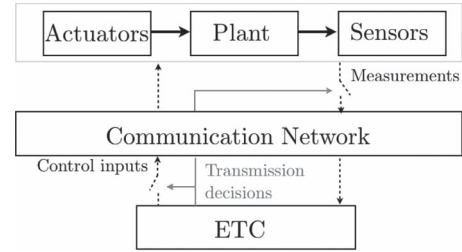


Fig. 3. Setup with remote controller. At scheduling times jh , $j \in \mathbb{N}_0$ the controller sends to the plant scheduling decisions for the interval $\{jh + 1, \dots, (j+1)h - 1\}$ (indicated by gray thin lines).

strictly positive. The integrand $f(\xi)$ should be seen as the performance gain at state ξ obtained by performing a single step optimization over the horizon h assuming periodic control is used after the horizon h , i.e., the gain obtained at a single scheduling time in Algorithm 3 by making decision (27). The overall gain g_a , described by (40), is obtained by repeating the process at every scheduling time step, according to Algorithm 3. It has the following interpretation: it is the (scaled) expected value of these single step optimization gains $f(\xi)$ with respect to the invariant probability measure (also a limiting measure according to (38)) of the Markov chain (35), induced by using Algorithm 3. Thus, if Algorithm 3 picks options different from that corresponding to periodic control ($\iota = 1$ in (27)) with large single step optimization gains $f(\xi)$, for states ξ likely to be visited asymptotically, then one may expect a large overall gain g_a . Contrarily, if $\iota = 1$ in (27) for a large region (likely to be visited asymptotically) in the state-space, then g_a is small. Numerical methods to estimate χ_{inv} can be found, e.g., in [50], [51]. In the example of Section VI we obtain a good approximation of g_a by running Monte-Carlo simulations.

V. OTHER NETWORKED CONTROL CONFIGURATIONS

Our ideas can be adapted to other network configurations. In this section we briefly discuss two examples.

A. Remote Controller

Consider first the configuration depicted in Fig. 3, in which a remote controller sends control inputs and receives state measurements from the plant through a communication network. To guarantee that the controller can make scheduling decisions at times jh , the plant *must* transmit the state to the controller at times jh , which can be directly used to compute u_{jh} . Thus, the free transmission times to be chosen at scheduling time jh are restricted to the interval $\{jh + 1, \dots, (j+1)h - 1\}$, i.e., the set \mathcal{I} , described in (20), is adapted to

$$\mathcal{I} = \left\{ \nu \in \{0, 1\}^h \mid \sum_{k=0}^{h-1} \nu_k = m \text{ and } \nu_0 = 1 \right\}$$

where we assume now that $m \in \mathbb{N}_{\geq 2}$. Considering that the network-induced delays are negligible when compared to the baseline period τ we can assume that at scheduling times jh the controller receives state measurements, makes scheduling decisions for the next $h-1$ possible transmission times $\{jh + 1, \dots, (j+1)h - 1\}$, characterized by ι and computed according to (27), and sends these scheduling decisions along with the

control input at time jh to the plant computed according to (28). At times $k \in \{jh + 1, \dots, (j + 1)h - 1\}$ such that $\nu_{k-jh}^t = 1$, the plant sends again state measurements to the controller, the controller computes the control input according to (28) and sends it to the actuators. In this manner the scheme works for the setup of Fig. 3 as well, and can be easily implemented in networks based on TDMA.

B. Model-Based Predictor at the Actuators

In the setup considered in Section II, while the actuators can update $\hat{u}_k = u_C(t_k)$ at the sampling rate $1/\tau$, this only occurs if a new transmission occurs at time t_k . An alternative configuration, considered in several works (see, e.g., [13], [30]), is to assume that the actuators use a predictor to update the control input even if no transmission occurs. Consider the following predictor-based control update

$$\hat{x}_{k+1} = \bar{A}_\tau \hat{x}_k + \bar{B}_\tau \hat{u}_k, \quad \hat{u}_k = \bar{K}_\tau \hat{x}_k, \quad \forall k \in \mathbb{N}_0 \quad (42)$$

where \hat{x}_k starts at time zero with an initial estimate of the state, denoted by \hat{x}_0 , and resets its state to the transmitted state each time a transmission occurs, i.e.

$$\hat{x}_k = x_k, \text{ when } \sigma_k = 1, \quad \forall k \in \mathbb{N}_0. \quad (43)$$

Note that we assume here that the full state is sent from the event-triggered controller collocated with the sensors to the actuators, which run (42). Moreover, since the control policy is already determined by (42), only the scheduling decisions (to send the state) need to be determined. A base policy for the scheduling is to transmit periodically with period $q\tau$ for some $q \geq 1$. An alternative rollout method, which is a straightforward adaptation of the ideas presented in Section III is described next.

The equations for the process and predictor take now the form

$$\eta_{k+1} = L_{\sigma_k} \eta_k + \omega_k, \quad k \in \mathbb{N}_0$$

where $\eta_k := [x_k^\top, \hat{x}_k^\top]^\top$ and

$$L_0 = \begin{bmatrix} \bar{A}_\tau & \bar{B}_\tau \bar{K}_\tau \\ 0 & (\bar{A}_\tau + \bar{B}_\tau \bar{K}_\tau) \end{bmatrix}, \quad L_1 = \begin{bmatrix} (\bar{A}_\tau + \bar{B}_\tau \bar{K}_\tau) & 0 \\ (\bar{A}_\tau + \bar{B}_\tau \bar{K}_\tau) & 0 \end{bmatrix}$$

and the covariance matrix of ω_k is given by

$$\Psi_\tau^w := \begin{bmatrix} \bar{\Phi}_\tau^w & 0_{n_x \times n_x} \\ 0_{n_x \times n_x} & 0_{n_x \times n_x} \end{bmatrix}.$$

The discounted cost (2) takes the form

$$\sum_{k=0}^{\infty} \mathbb{E} [\alpha_\tau^k \eta_k^\top X_{\sigma_k} \eta_k] \quad (44)$$

apart from an additive constant factor, where

$$X_0 = \begin{bmatrix} \bar{Q}_\tau & \bar{S}_\tau \bar{K}_\tau \\ \bar{K}_\tau^\top \bar{S}_\tau^\top & \bar{K}_\tau^\top \bar{R}_\tau \bar{K}_\tau \end{bmatrix}$$

$$X_1 = \begin{bmatrix} \bar{Q}_\tau + \bar{S}_\tau \bar{K}_\tau + \bar{K}_\tau^\top \bar{S}_\tau^\top + \bar{K}_\tau^\top \bar{R}_\tau \bar{K}_\tau & 0 \\ 0 & 0 \end{bmatrix}.$$

Using similar arguments to the ones used in Section III-B, the discounted cost (44) for a scheduling sequence taken from the set \mathcal{T} , labeled by $i \in \mathcal{T}$, can be shown to be given by

$$\eta_0^\top Z_i \eta_0 + z_i + d$$

where $Z_i, i \in \mathcal{T}$, are positive semi-definite matrices and $z_i, i \in \mathcal{T}$, and d are positive constants. The expressions are omitted for the sake of brevity. Scheduling decisions at each step $\ell = jh, j \in \mathbb{N}_0$ are obtained by computing

$$\iota(\eta_\ell) = \operatorname{argmin}_{i \in \mathcal{M}} \eta_\ell^\top Z_i \eta_\ell + z_i$$

which determine the scheduling decisions in the interval $\{jh, \dots, (j + 1)h - 1\}$, given by

$$(\nu_0^{\iota(\xi_\ell)}, \dots, \nu_{j-1}^{\iota(\xi_\ell)})$$

for $\nu^i \in \mathcal{I}, i \in \mathcal{M}$. Note that the scheduler needs also to run the model-based estimator (42) to make decisions based on \hat{x}_k . Similar performance improvements results can be obtained paralleling the ones in Section IV.

VI. EXAMPLE

Consider two unitary masses on a frictionless surface connected by an ideal spring and moving along a one-dimensional axis. The control input is a force acting on the first mass. The state vector is $x_C = [x_1 \ x_2 \ v_1 \ v_2]^\top$, where x_i, v_i are the displacements and velocities of the mass $i \in \{1, 2\}$, respectively, and the plant model (1) is described by

$$A_C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\kappa_m & \kappa_m & 0 & 0 \\ \kappa_m & -\kappa_m & 0 & 0 \end{bmatrix}, \quad B_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (45)$$

where κ_m is the spring coefficient. We set the initial state to $x_0 = [-1 \ 1 \ 0 \ 0]^\top$, meaning that the masses start with zero velocity and at opposite distances from their equilibrium values. The matrix A_C has two eigenvalues at zero and two complex conjugates eigenvalues at $\pm \sqrt{2\kappa_m}j$. The free response hence has oscillations with a period $2\pi/\sqrt{2\kappa_m}$. We normalize time so that one time unit $t = 1$ corresponds to one period of these oscillations, which results in $\kappa_m = 2\pi^2$. This implies that the sampling period must be different from the pathological sampling periods $0.5\kappa, \kappa \in \mathbb{N}$, so that the discretization of the plant remains controllable [46].

We start by assuming that there are no disturbances acting on the plant, i.e.

$$\text{Case I: } B_\omega = 0$$

and by considering the following cost:

$$\text{Case I: } \int_0^\infty x_1(t)^2 + x_2(t)^2 + 0.1u_C(t)^2 dt \quad (46)$$

which takes the form (2) with $\alpha_C = 0$. Using standard optimal control theory (cf., e.g., [45, Ch. 3]) we can compute the optimal continuous-time feedback law that minimizes (46) which

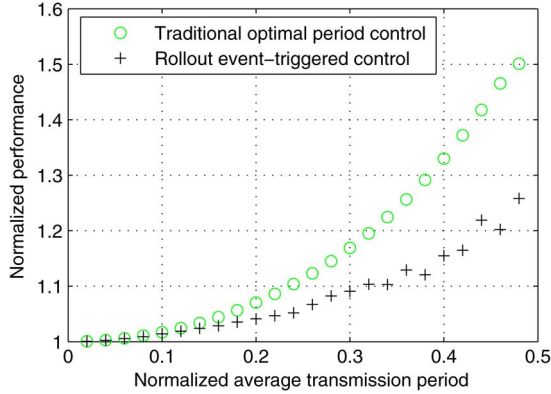


Fig. 4. Case I: no disturbances, LQR-type cost.

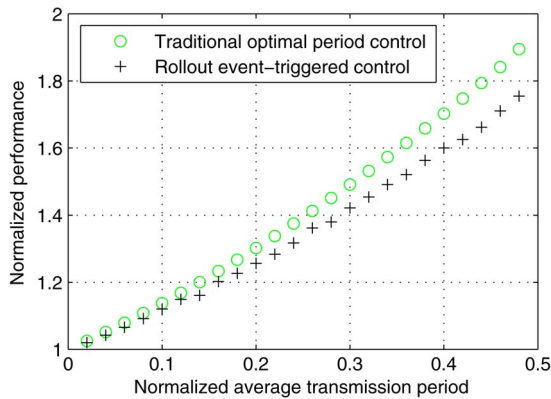


Fig. 5. Case II: Wiener disturbances, LQG-type average cost.

is a state-feedback law $u_C(t) = K_C x_C(t)$ yielding a cost (46) given by $x_0^T P_C x_0$ where P_C is the solution to the Riccati (76) given in Section VIII. For the numerical values given above this gives

$$x_0^T P_C x_0 = 5.7411 \tag{47}$$

and the eigenvalues of $A_C + B_C K_C$ are given by $-0.1775 \pm 6.2857i$, $-1.0564 \pm 1.0566i$, resulting in a lightly damped closed-loop system. Fig. 4 plots the (normalized) performance (46) obtained with the traditional periodic control strategy and with the rollout ETC strategy described by Algorithm 3 in the setup of Fig. 3 with parameters $h = 6$, $m = 2$, $q = 3$, for several values of the average transmission period $q\tau$ in the range $[0, 0.5]$. The performance (46) for the rollout event-triggered control method is obtained via simulating (5) for (7), (29) for a large time ($t \in [0, 500]$) and computing the cost (8) resulting from the parameters in (46). This method can also be used to obtain the cost of the optimal control strategy (12) to confirm the expression (15), which is used to plot the values of Fig. 5 pertaining to periodic control. The performance values in Fig. 4 are normalized with respect to the optimal LQR performance achievable by a continuous-time controller (47). The time evolution of the actuation u_C and the position x_1 of the first mass for the considered initial state and for $t \in [0, 30]$ are shown in Fig. 6 when the average transmission rate is 0.4. Note that a faster convergence to zero of these signals is obtained for the rollout method, due to the extra degree of

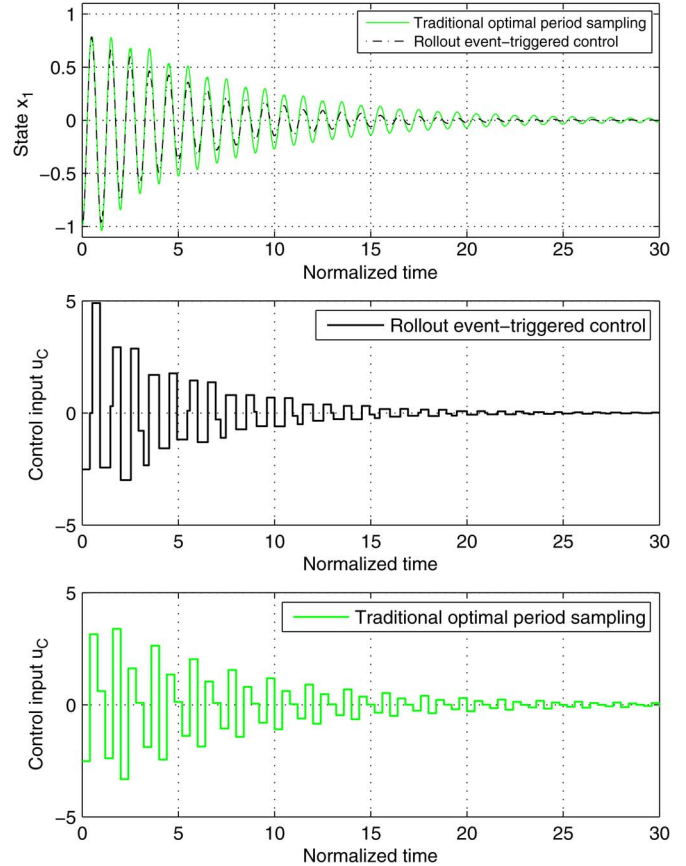


Fig. 6. Time evolution of state x_1 and control input u_C for the periodic and rollout methods when the average transmission rate is 0.4.

freedom of choosing different actuation pattern than periodic update times.

We consider next the case in which disturbances are acting on the plant characterized by the injection matrix

$$\text{Case II : } B_w = [0 \ 0 \ 0.5 \ 0]^T.$$

Performance is measured by the following cost:

$$\text{Case II : } \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T [x_1(t)^2 + x_2(t)^2 + 0.1u_C(t)^2] dt \right] \tag{48}$$

which takes the form (3). Fig. 5 plots the (normalized) performance (48) obtained with the traditional periodic control strategy and with the same rollout ETC method as in Case I. The cost (48) is estimated via Monte-Carlo simulations with 300 trials simulating (5) for (7), (29) for a large time ($t \in [0, 1500]$) and computing the cost (10) resulting from the parameters in (48). This method can also be used to obtain the cost of the optimal periodic control strategy (12) to confirm the expression (15), which is used to plot the values of Fig. 5 pertaining to periodic control. The performance values in Fig. 5 are normalized with respect to the optimal LQG performance achievable by a continuous-time controller, which is given by $\text{tr}(P_C B_w B_w^T)$ where P_C is the solution to the Riccati equation (76) given in Section VIII. For the numerical values given above $\text{tr}(P_C B_w B_w^T) = 0.06170$.

Both Fig. 4 and Fig. 5 show that for small average transmission periods the methods perform very closely. In fact, this is natural as periodic control approaches the optimal performance (2) achievable by a continuous-time controller when the sampling period tends to zero. As such, there is little room for improvements. However, for larger transmission periods the rollout strategy in Case I obtains significant performance improvements over traditional periodic control. This is a clear illustration of the main theorems in this paper and shows the effectiveness of the novel ETC strategy proposed in this paper. On the other hand, for Case II the gains are less pronounced. A possible explanation is the fact that we have considered Wiener disturbances. As discussed in [52] the performance gains of ETC strategies with respect to periodic control may be much larger considering classes of stochastic disturbances different from Wiener disturbances. A topic for future research is to incorporate such models in the setting of the present paper.

VII. CONCLUSION

In this paper, we proposed a novel ETC strategy called rollout ETC, that guarantees a performance improvement over traditional periodic control. The key to our method is to select at given scheduling times control and scheduling decisions over a given horizon assuming that periodic optimal control is used afterwards. Under mild assumptions, for the new class of ETC strategies, we showed that strict performance improvements could be formally guaranteed with respect to the performance of periodic controllers with the same average transmission rate. We illustrated by a numerical example that the proposed ETC strategy can significantly outperform periodic control.

While we have focused on basic models for the process and for the communication network, the obtained numerical results encourage pursuing various research directions for extending such models. These directions include scenarios in which (i) the full state of the plant is not available; (ii) multiple control loops are closed over the communication network; (iii) the noise model is different from Wiener processes; (iv) packet drops are taken into account in the model of the communication network.

VIII. PROOFS

Theorems 4 and 8 are proved in Section VIII-A. The proof of Theorem 8 builds upon some of the statements used in the proof of Theorem 4. Theorems 7, 9 and 10 are proved in Section VIII-C, building upon two key lemmas established in Section VIII-B.

A. Proof of Theorems 4 and 8

Before we prove Theorem 4, we note that we can think of $\lambda := \{\iota(\bar{\xi}_0), \iota(\bar{\xi}_1), \dots\}$ as a stationary policy for (35). We can then write (8) and (10) when $\pi = \rho$, where ρ is the rollout policy (29), as

$$\bar{J}_\lambda^d(\bar{\xi}_0) := \sum_{\ell=0}^{\infty} \mathbb{E} [\alpha_\tau^{h\ell} \bar{g}(\bar{\xi}_\ell, \iota(\bar{\xi}_\ell), \bar{w}_\ell)] \quad (49)$$

and

$$\bar{J}_\lambda^a = \lim_{L \rightarrow \infty} \frac{1}{\tau h L} \sum_{\ell=0}^{L-1} \mathbb{E} [\bar{g}(\bar{\xi}_\ell, \iota(\bar{\xi}_\ell), \bar{w}_\ell)]$$

respectively, where for $\bar{w} = (w_0, \dots, w_{h-1})$

$$\bar{g}(\bar{\xi}, i, \bar{w}) := \sum_{\kappa=0}^{h-1} \alpha_\tau^\kappa g(y_\kappa, \hat{K}_{\kappa,i} y_\kappa, \nu_\kappa^i)$$

the y_κ are defined recursively

$$y_{\kappa+1} = \Theta_{\kappa,i} y_\kappa + w_\kappa, \quad y_0 = \bar{\xi}, \quad \kappa \in \{0, 1, \dots, h-1\}$$

and

$$\hat{K}_{\kappa,i} = \begin{cases} [K_{\kappa,i} \ 0_{n_u \times n_u}], & \text{if } \nu_\kappa^i = 1 \\ [0_{n_u \times n_x} \ I_{n_u \times n_u}], & \text{otherwise.} \end{cases} \quad (50)$$

That is, $J_{\rho, q\tau}^a = \bar{J}_\lambda^a$ and $J_{\rho, q\tau}^d(\bar{\xi}_0) = \bar{J}_\lambda^d(\bar{\xi}_0)$, for every $\bar{\xi}_0 \in \mathbb{R}^n$.

Proof. (of Theorem 4): To establish (30) we start by defining the following policies $\zeta^r = (\psi_0^r, \psi_1^r, \dots)$, $r \in \mathbb{N}_0$, for (35)

$$\psi_j^r(\bar{\xi}_\ell) = \begin{cases} \iota(\bar{\xi}_\ell), & \text{if } 0 \leq j < r \\ 1, & \text{if } j \geq r. \end{cases}$$

obtained by applying policy (27) to (35) until iteration r and afterwards using always periodic control ($\iota = 1$). Note that $\lim_{r \rightarrow \infty} \bar{J}_{\zeta^r}^d(\xi_0) = \bar{J}_\lambda^d(\xi_0) = J_{\rho, q\tau}^d(\xi_0)$ for every $\xi_0 \in \mathbb{R}^n$ and $\bar{J}_{\zeta^0}^d(\xi_0) = J_{\text{per}, q\tau}^d(\xi_0)$ for every $\xi_0 \in \mathbb{R}^n$. From the definition of ι in (27) we have that

$$\bar{J}_{\zeta^1}^d(\xi_0) \leq \bar{J}_{\zeta^0}^d(\xi_0), \quad \text{for every } \xi_0 \in \mathbb{R}^n. \quad (51)$$

Since the cost (49) is additive along stages, we can write

$$\bar{J}_{\zeta^{r+1}}^d(\bar{\xi}_0) = \sum_{\ell=0}^{r-1} \mathbb{E} [\alpha_\tau^{h\ell} \bar{g}(\bar{\xi}_\ell, \iota(\bar{\xi}_\ell), \bar{w}_\ell)] + \alpha_\tau^{hr} \mathbb{E} [\bar{J}_{\zeta^1}^d(\bar{\xi}_r)] \quad (52)$$

for $r \in \mathbb{N}$, and

$$\bar{J}_{\zeta^r}^d(\bar{\xi}_0) = \sum_{\ell=0}^{r-1} \mathbb{E} [\alpha_\tau^{h\ell} \bar{g}(\bar{\xi}_\ell, \iota(\bar{\xi}_\ell), \bar{w}_\ell)] + \alpha_\tau^{hr} \mathbb{E} [\bar{J}_{\zeta^0}^d(\bar{\xi}_r)] \quad (53)$$

for $r \in \mathbb{N}_0$. From (51), we conclude that

$$\mathbb{E} [\bar{J}_{\zeta^1}^d(\bar{\xi}_r)] \leq \mathbb{E} [\bar{J}_{\zeta^0}^d(\bar{\xi}_r)] \quad (54)$$

for every $r \in \mathbb{N}$, where the expectations are taken with respect to $\bar{w}_0, \dots, \bar{w}_{r-1}$ that determine $\bar{\xi}_r$ by (35). Using this latter inequality in (52) and taking into account (53) we conclude that

$$\bar{J}_{\zeta^{r+1}}^d(\xi_0) \leq \bar{J}_{\zeta^r}^d(\xi_0) \quad (55)$$

for every $r \in \mathbb{N}_0$ and $\xi_0 \in \mathbb{R}^n$. Thus, for a given $\xi_0 \in \mathbb{R}^n$

$$J_{\rho, q\tau}^d = \lim_{r \rightarrow \infty} \bar{J}_{\zeta^r}^d \leq \dots \leq \bar{J}_{\zeta^2}^d \leq \bar{J}_{\zeta^1}^d \leq \bar{J}_{\zeta^0}^d = J_{\text{per}, q\tau}^d \quad (56)$$

establishing (30).

To establish (31) for the average cost ($c = a$), we define

$$V(\xi) := \xi^\top P_1 \xi, \quad \xi \in \mathbb{R}^n \quad (57)$$

and take the limit as $\alpha_C \downarrow 0$ in (26) (see Section III-D) obtaining

$$c_1 = m \text{tr} (\bar{P}_{q\tau} \bar{\Phi}_{q\tau}^w). \quad (58)$$

Taking into account (25) and the definition of P_i in Section III-B one can conclude that at iteration ℓ

$$\mathbb{E} [V(\bar{\xi}_{\ell+1}) + \bar{g}(\bar{\xi}_\ell, \nu(\bar{\xi}_\ell), \bar{w}_\ell) | \bar{\xi}_\ell] = \bar{\xi}_\ell^\top P_{\nu(\bar{\xi}_\ell)} \bar{\xi}_\ell + c_{\nu(\bar{\xi}_\ell)}. \quad (59)$$

Thus

$$\begin{aligned} & \mathbb{E} [V(\bar{\xi}_{\ell+1}) + \bar{g}(\bar{\xi}_\ell, \nu(\bar{\xi}_\ell), \bar{w}_\ell) | \bar{\xi}_\ell] - V(\bar{\xi}_\ell) \\ &= \bar{\xi}_\ell^\top (P_{\nu(\bar{\xi}_\ell)} - P_1) \bar{\xi}_\ell + c_{\nu(\bar{\xi}_\ell)} \\ &= c_1 - f(\bar{\xi}_\ell) \end{aligned} \quad (60)$$

where f is described by (41). Adding (60) for $\ell = 0, 1, \dots, L-1$, dividing by $\tau h L$, and taking expectations we obtain

$$\begin{aligned} \frac{1}{\tau h L} \mathbb{E} \left[\sum_{\ell=0}^{L-1} \bar{g}(\bar{\xi}_\ell, \nu(\bar{\xi}_\ell), \bar{w}_\ell) \right] &= \frac{c_1}{\tau h} - \frac{1}{\tau h L} \mathbb{E} \left[\sum_{\ell=0}^{L-1} f(\bar{\xi}_\ell) \right] \\ &+ \frac{1}{\tau h L} (V(\bar{\xi}_0) - \mathbb{E} [V(\bar{\xi}_L) | \bar{\xi}_0]). \end{aligned} \quad (61)$$

Provided that we prove that $\mathbb{E}[V(\bar{\xi}_L) | \bar{\xi}_0]$ remains bounded as $L \rightarrow \infty$ we can take the limit as $L \rightarrow \infty$ in (61), use the fact that the left-hand side converges to $\bar{J}_\lambda^a = J_{\rho, q\tau}^a$, and use (58) to obtain⁴

$$J_{\rho, q\tau}^a = \frac{1}{q\tau} \text{tr}(\bar{P}_{q\tau} \bar{\Phi}_{q\tau}^w) - \lim_{L \rightarrow \infty} \frac{1}{\tau h L} \mathbb{E} \left[\sum_{\ell=0}^{L-1} f(\bar{\xi}_\ell) \right]. \quad (62)$$

Then, (31) follows from (16) and the fact that f , described by (41), is a nonnegative function due to $i = \nu(\xi_\ell)$ and (27).

To prove that $\mathbb{E}[V(\bar{\xi}_L) | \bar{\xi}_0]$ remains bounded as $L \rightarrow \infty$, we use the fact that

$$\mathbb{E} [\bar{g}(\bar{\xi}_\ell, \nu(\bar{\xi}_\ell), \bar{w}_\ell) | \bar{\xi}_\ell] \geq a_1 \bar{x}_\ell^\top \bar{x}_\ell \quad (63)$$

for some sufficiently small $a_1 > 0$, where we used the decomposition $\bar{\xi}_\ell = [\bar{x}_\ell^\top \bar{u}_\ell^\top]^\top$, $\bar{x}_\ell := x_{\ell h}$, $\bar{u}_\ell := \hat{u}_{\ell h}$. Equation (63) can be proved using the positive semi-definite assumption on Q_C , the assumption that the pair $(A_C, Q_C^{1/2})$ is observable, and the assumption that R_C is positive definite. Moreover, choosing b_1 such that $b_1 > a_1 > 0$ and $\bar{P}_{q\tau} \prec b_1 I_{n_x}$, and taking into account (25) we conclude that

$$V(\bar{\xi}) \leq b_1 \bar{x}^\top \bar{x} \quad (64)$$

for $\bar{\xi} = [\bar{x}^\top \bar{u}^\top]^\top$. Using (60), (63), (64) we conclude that for $\ell \in \mathbb{N}_0$

$$\mathbb{E} [V(\bar{\xi}_{\ell+1}) | \bar{\xi}_\ell] \leq d_1 V(\bar{\xi}_\ell) + c_1$$

where $d_1 := 1 - (a_1/b_1) < 1$, which in turn implies that for $L \in \mathbb{N}$ and $d_2 = \sum_{s=0}^{L-1} d_1^s c_1$

$$\mathbb{E} [V(\bar{\xi}_L) | \bar{\xi}_0] \leq d_1^L V(\bar{\xi}_0) + d_2$$

⁴In the proof of Theorem 7 we shall establish that the limit in the right-hand side of (62) exists. Then the limit in the left-hand side of (62), described in (10), also exists.

leading to the conclusion that $\mathbb{E}[V(\bar{\xi}_L) | \bar{\xi}_0]$ is bounded as $L \rightarrow \infty$. ■

We prove Theorem 8 next.

Proof. (of Theorem 8): If we consider the case $B_\omega = 0$ and $\alpha_C = 0$, we conclude from (60) that

$$V(\bar{\xi}_{\ell+1}) - V(\bar{\xi}_\ell) \leq -\bar{g}(\bar{\xi}_\ell, \nu(\bar{\xi}_\ell), 0) \quad (65)$$

where V is described by (57) and we used the fact that $c_1 = 0$ in this case and f is a nonnegative function. As in (63) we can conclude that

$$\bar{g}(\bar{\xi}_\ell, \nu(\bar{\xi}_\ell), 0) \geq a_2 \bar{x}_\ell^\top \bar{x}_\ell \quad (66)$$

for sufficiently small $a_2 > 0$, where again we used the decomposition $\bar{\xi}_\ell = [\bar{x}_\ell^\top \bar{u}_\ell^\top]^\top$. From (65) and (66) and taking into account (25) we can conclude that

$$\bar{x}_{\ell+1}^\top \bar{P}_{q\tau} \bar{x}_{\ell+1} - \bar{x}_\ell^\top \bar{P}_{q\tau} \bar{x}_\ell \leq -a_2 \bar{x}_\ell^\top \bar{x}_\ell. \quad (67)$$

Thus

$$\bar{x}_\ell^\top \bar{P}_{q\tau} \bar{x}_\ell \leq \left(1 - \frac{a_2}{c}\right)^\ell \bar{x}_0^\top \bar{P}_{q\tau} \bar{x}_0$$

where c is a sufficiently large constant such that $\bar{P}_{q\tau} \prec c I_{n_x}$ and $(1 - (a_2/c))$ is positive. Since, under Assumption 1, $\bar{P}_{q\tau}$ is positive definite, this implies that \bar{x}_ℓ converges to zero exponentially fast, which in turn implies that x_k converges to zero exponentially fast. Moreover, since the control input is a hold version of (28) this implies that the control input also converges to zero exponentially fast and hence also ξ_k . ■

B. Two Key Lemmas

We need two preliminary lemmas to prove Theorems 7, 9, and 10. For each option $i \in \mathcal{M}$ for the scheduling vector ν_k^i , $k \in \{0, 1, \dots, h-1\}$ in (19), let

$$\bar{k}^i \in \{m-1, m, m+1, \dots, h-1\} \quad (68)$$

be the largest k such that ν_k^i equals one, i.e., \bar{k}^i is uniquely determined by $\nu_{\bar{k}^i}^i = 1$ and $\nu_k^i = 0$, if $k \in \{\bar{k}^i + 1, \dots, h-1\}$.

Lemma 11: Suppose that Assumption 1 holds and consider Algorithm 3 for $\tau \in \mathbb{R}_{>0}$, $q \in \mathbb{N}$ and $m \in \mathbb{N}$. Then:

- (i) Assumption 5(ii) holds for sufficiently small τ .
- (ii) if Assumption 5(ii) holds, then $K_{\bar{k}^i, i}$, obtained from (23), has full rank for every $i \in \mathcal{M}$.
- (iii) if Assumption 6 holds, there exist $\xi \in \mathbb{R}^n$ and $i \in \mathcal{M} \setminus \{1\}$ such that

$$\xi^\top P_i \xi + c_i < \xi^\top P_1 \xi + c_1. \quad (69)$$

□

Note that (iii) assures that for at least one state the choice in (27) is different from (21), which corresponds to periodic scheduling, i.e., there always exists a state in \mathbb{R}^n for which the periodic scheduling option is not chosen. The proof of Lemma 11(iii) needs the following proposition.

Proposition 12: Suppose that Assumptions 1 and 6 hold and consider the unique solutions \bar{P}_τ and $\bar{P}_{q\tau}$ to (14) when δ is replaced by τ and $q\tau$ respectively, $\tau \in \mathbb{R}$, $q \in \mathbb{N}$, $q \geq 2$. Then

$$\bar{P}_\tau \preceq \bar{P}_{q\tau} \quad (70)$$

and

$$\exists x \in \mathbb{R}^{n_x} : x^\top \bar{P}_\tau x < x^\top \bar{P}_{q\tau} x. \quad (71)$$

□

Proof: By construction $\bar{P}_{q\tau}$ is such that $x_0^\top \bar{P}_{q\tau} x_0$ is the cost of the following optimal control problem:

$$\min_{\{u_k, k \in \mathbb{N}_0\}} \int_0^\infty e^{-\alpha_C t} g_C(x_C(t), u_C(t)) dt \quad (72)$$

s.t. $x_C(0) = x_0$, where x_C and u_C satisfy (1) for $B_\omega = 0$, and u_C is given by

$$u_C(t) = u_k, \quad t \in [t_k, t_{k+1}) \quad (73)$$

for $t_k = j\tau k$, $k \in \mathbb{N}_0$, when $j = q$. Let $u_k^{*,j}$ denote the optimal solution corresponding to $j\tau$, for a given $j \in \mathbb{N}$, which equals

$$u_k^{*,j} := \bar{K}_{j\tau} (\bar{A}_{j\tau} + \bar{B}_{j\tau} \bar{K}_{j\tau})^k x_0 \quad (74)$$

since the control input is described by (12) and there are no disturbances acting on the plant. If for $j = 1$, we make u_k in (73) emulate the optimal control input corresponding to $q\tau$, $q \geq 2$

$$u_k = u_{\lfloor \frac{k}{q} \rfloor}^{*,q} \quad (75)$$

where $\lfloor a \rfloor$ denotes the floor of a (largest integer less or equal than a), then the cost (72) for these (not necessarily optimal) control inputs equals $x_0^\top \bar{P}_{q\tau} x_0$. Then the (optimal) control inputs $u_k^{*,1}$ will yield a cost $x_0^\top \bar{P}_\tau x_0$ smaller than $x_0^\top \bar{P}_{q\tau} x_0$ for every $x_0 \in \mathbb{R}^n$ which implies (70).

To prove (71) it suffices to prove that there exists one initial condition x_0 for the problem (72) with $j = 1$ for which (75) is not the optimal solution, since the optimal solution to the problem (72) is unique (cf. [43]) and hence will lead to a strictly smaller cost. To this effect, suppose that for a given initial condition x_0 , (75) is the optimal solution. In particular, the first q controls are the same $u_0 = u_1 = \dots = u_{q-1}$. Due to Bellman's principle of optimality [43], if the system would start at time $k = 1$ with initial condition $\hat{x}_0 = x_1 = \bar{A}_\tau x_0 + \bar{B}_\tau u_0$ the optimal control inputs would be shifted, i.e., the first control would be u_1 , the second u_2 , etcetera. However, such optimal control input does not take the form (75), unless (74) is constant, which is excluded by Assumption 6. Hence, for such initial condition \hat{x}_0 the optimal control input is different than (74), thereby concluding the proof. ■

Proof. (of Lemma 11): We start by recalling that the following Riccati equation

$$\left(A_C - \frac{\alpha_C}{2} I\right)^\top P_C + P_C \left(A_C - \frac{\alpha_C}{2} I\right) - P_C B_C R_C^{-1} B_C^\top P_C + Q_C = 0 \quad (76)$$

has a unique positive definite solution P_C if R_C is positive definite, and the pairs $(A_C - (\alpha_C/2)I, B_C)$ and $(A_C - (\alpha_C/2)I, Q_C^{1/2})$, are controllable and observable, respectively (see [45, Ch. 3]), which holds due to the assumption that (A_C, B_C) and $(A_C, Q_C^{1/2})$, are controllable and observable,

respectively. This latter fact can be seen from the characterization of controllability of the pair A_C, B_C (and observability using duality): $[(A_C - \lambda I) B_C]$ has full rank for all $\lambda \in \mathbb{C}$ (cf. [53, p. 47]). We recall also that the optimal controller that minimizes the discounted cost (2) without communication restrictions (providing a continuous-time input $u_C(t)$, $t \in \mathbb{R}_{\geq 0}$, based on full access to the state $x_C(t)$, $t \in \mathbb{R}_{\geq 0}$) yields a cost $x_C(0)^\top P_C x_C(0)$ (see [45, Ch. 3]). Then, it is clear that $\lim_{\delta \rightarrow 0} P_\delta = P_C$, i.e., the optimal continuous-time performance is recovered as the sampling period of periodic control tends to zero (see [46, Sec. 9.4]). Using this latter fact, and taking into account the expressions (6) and (9), we can obtain that

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau} (\alpha_\tau \bar{A}_\tau^\top \bar{P}_{q\tau} \bar{B}_\tau + \bar{S}_\tau) = P_C B_C. \quad (77)$$

Since P_C is positive definite and B_C has full rank (cf. Assumption 1(i)), we can conclude that $P_C B_C$ has full rank. Hence, in first approximation $\alpha_\tau \bar{A}_\tau^\top \bar{P}_{q\tau} \bar{B}_\tau + \bar{S}_\tau$ approaches a full-rank matrix $\tau P_C B_C$, which allows to conclude (i).

To prove (ii) we use the fact that

$$K_{\bar{k}^i, i} = -(\bar{R}_s + \alpha_s \bar{B}_s^\top \bar{P}_{q\tau} \bar{B}_s)^{-1} (\alpha_s \bar{B}_s^\top \bar{P}_{q\tau} \bar{A}_s + \bar{S}_s^\top) \quad (78)$$

where $s = (h - \bar{k}^i)\tau$, and \bar{k}^i is defined in (68). This fact can be obtained directly from (23). The derivation is straightforward but lengthy and therefore it is omitted. The matrix \bar{R}_s is positive definite (since R_C is positive definite) for every positive s and hence the inverse in (78) exists. Note that $1 \leq (h - \bar{k}^i) \leq h - m + 1$. Then, Assumption 5(ii) implies that $K_{\bar{k}^i, i}$ is the product of an invertible matrix and a full rank matrix and hence it is full rank.

To prove (iii) we notice that if there exists $i \in \mathcal{M}$ such that $c_i < c_1$ then (69) holds for such $i \in \mathcal{M}$ and $\xi = 0$. If $c_i \geq c_1$ for every $i \in \mathcal{M}$, to establish (69) it suffices to prove that there exist $\bar{\xi}$ and $i \in \mathcal{M}$ such that

$$\bar{\xi}^\top P_i \bar{\xi} < \bar{\xi}^\top P_1 \bar{\xi} \quad (79)$$

since then (69) holds for $\xi = a\bar{\xi}$ and sufficiently large $a \in \mathbb{R}$.

To prove (79), we start by noticing that, by construction, $\xi_0^\top P_i \xi_0$, $\xi_0 = [x_0^\top \hat{u}_0^\top]^\top$, $i \in \mathcal{M}$, is the cost of the following:

$$\min_{\{u_0, \dots, u_{h-1}\} \in \mathcal{U}_i} \int_0^{h\tau} e^{-\alpha_C t} g_C(x_C(t), u_C(t)) dt + e^{-\alpha_C h\tau} x_C(h\tau)^\top \bar{P}_{q\tau} x_C(h\tau) \quad (80)$$

s.t. $x_C(0) = x_0$, x_C and u_C satisfy (1)

$$u_C(t) = u_C(t_k), \quad t \in [t_k, t_{k+1}), \quad u_C(t_0^-) = 0_{n_u}$$

$$u_C(t_k) = \begin{cases} u_k & \text{if } \nu_k^i = 1, \\ u_C(t_k^-), & \text{otherwise} \end{cases}, \quad k \in \{0, \dots, h-1\} \quad (81)$$

where $u_C(t_0^-) := \hat{u}_0$

$$\mathcal{U}_i := \{(u_0, \dots, u_{h-1}) \in \mathcal{R} \mid u_k = \emptyset \text{ if } \nu_k^i = 0\}$$

and $\mathcal{R} := (\mathbb{R}^{n_u} \cup \{\emptyset\}) \times \dots \times (\mathbb{R}^{n_u} \cup \{\emptyset\})$. Note that there are m free control inputs in \mathcal{U}_i for the optimization (80) and recall that [see (25)]

$$\xi_0^\top P_1 \xi_0 = x_0^\top \bar{P}_{q\tau} x_0. \quad (82)$$

Let Ω be the subset of $i \in \mathcal{M}$ such that ν^i differs from ν^1 , described by (21), *only* at the first schedule, and consequently also for another schedule, e.g., if $m = 2$, $q = 2$, $\nu^1 = (1, 0, 1, 0)$ and the remaining vector of schedules corresponding to Ω are $(0, 1, 1, 0)$, and $(0, 0, 1, 1)$. For a given arbitrary non-zero $x_0 \in \mathbb{R}^n$ let

$$\xi_0^i := \begin{bmatrix} x_0 \\ u^{*,i} \end{bmatrix} \quad (83)$$

where

$$u^{*,i} = \operatorname{argmin}_{u_0 \in \mathbb{R}^{n_u}} [x_0^\top \ u_0^\top] P_i \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}. \quad (84)$$

Then, if $i \in \Omega$, clearly

$$\xi_0^{i\top} P_i \xi_0^i \leq \xi_0^\top P_1 \xi_0^i \quad (85)$$

since choosing (83), (84) is equivalent to solving problem (80) for optimization variables $(u_0, \dots, u_{h-1}) \in \mathcal{U}_i$ in a new set containing $m + 1$ free control inputs

$$\bar{\mathcal{U}}_i := \{(u_0, \dots, u_{h-1}) \in \mathcal{R} \mid u_k = \emptyset \text{ if } k \neq 0 \text{ and } \nu_k^i = 0\}$$

i.e., u_0 is also a free variable in the equivalent optimization problem. To prove that (85) cannot hold with equality for every $x_0 \in \mathbb{R}^{n_x}$ and every $i \in \mathcal{M}$, and therefore (79) holds for some $i \in \mathcal{M}$ and $\bar{\xi} = \xi_0^i$, we argue by contradiction. If (85) would hold with equality for every $i \in \Omega$, and a fixed arbitrary $x_0 \in \mathbb{R}^{n_x}$, then by uniqueness of the optimal solution to the problem (80) (cf. [43]), and Assumption 1 this would mean that adding extra control input degrees of freedom (implicit in the set $\bar{\mathcal{U}}_i$) to the optimization problem (80) when $i = 1$ would not change the optimal control input solution. However, since the cost in the problem (80) is a quadratic function of u_0, \dots, u_{h-1} which must be convex due to uniqueness of the optimal solution, this would actually imply that having all the control input degrees of freedom $(u_0, \dots, u_{h-1}) \in \mathcal{R}$ would not change the optimal control input solution. Thus, $\xi_0^{i\top} P_1 \xi_0^i$ would be equal to [using (82) and making $i = 1$ in (80)]

$$\begin{aligned} x_0^\top \bar{P}_{q\tau} x_0 &= e^{-\alpha_C h\tau} x_C(h\tau)^\top \bar{P}_{q\tau} x_C(h\tau) \\ &+ \min_{\{u_0, \dots, u_{h-1}\} \in \mathbb{R}^{n_u} \times \dots \times \mathbb{R}^{n_u}} \int_0^{h\tau} e^{-\alpha_C t} g_C(x_C(t), u_C(t)) dt. \end{aligned} \quad (86)$$

We can use (86) to obtain an expression for $x_C(kq\tau)^\top \bar{P}_{q\tau} x_C(kq\tau)$, $k \in \mathbb{N}$ and recursively replace it

in the right-hand side of (86). By doing this and taking the limit of the recursion we obtain

$$x_0^\top \bar{P}_{q\tau} x_0 = \min_{\{u_k \in \mathbb{R}^{n_u}, k \in \mathbb{N}\}} \int_0^\infty e^{-\alpha_C t} g_C(x_C(t), u_C(t)) dt. \quad (87)$$

But the right-hand side of (87) equals $x_0^\top \bar{P}_\tau x_0$ and since x_0 is arbitrary this would mean $\bar{P}_\tau = \bar{P}_{q\tau}$ which is a contradiction due to (71). ■

We state next the second of the two key lemmas. Let $B_\epsilon(x) := \{y \in \mathbb{R}^n \mid \|y - x\| < \epsilon\}$ for $\epsilon > 0$ denote the ball of radius ϵ around $x \in \mathbb{R}^n$.

Lemma 13: Suppose that Assumptions 1 and 5 hold and consider Algorithm 3 with m transmissions along a period h . Then, the following hold.

- (i) If $m \geq 2$, then for every $\zeta \in \mathbb{R}^n$ and for every open set $A \subseteq \mathbb{R}^n$

$$P^\kappa(\zeta, A) > 0 \quad (88)$$

for every $\kappa \geq 1$. Moreover, if $m = 1$, then (88) holds for every $\kappa \geq 2$.

- (ii) For every $\zeta \in \mathbb{R}^n$ and every $A \in \mathcal{B}(\mathbb{R}^n)$, there exist a *continuous* non-negative function $T(\cdot, A) : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and a constant $\epsilon > 0$ such that for every $y \in B_\epsilon(\zeta)$

$$P^1(y, A) \geq T(y, A) \quad (89)$$

and

$$T(y, \mathbb{R}^n) > 0. \quad (90)$$

□

Proof: We start by noticing that Assumption 5(i) implies that $\bar{\Phi}_\tau^\omega > 0$ for every $\tau \in \mathbb{R}_{>0}$, which in turn implies that

$$\operatorname{Prob}[x_{k+1} \in B \mid \xi_k = y] > 0 \quad (91)$$

for every $y \in \mathbb{R}^n$, every open set $B \subseteq \mathbb{R}^{n_x}$, and every $k \in \mathbb{N}_{\geq 0}$, where $\xi_k = [x_k^\top \ \hat{u}_k^\top]^\top$. Suppose that $m \geq 2$ and fix a given $j \geq 1$. Notice that $k^t(\xi_{(j-1)h})$, defined in (68), is the largest time step k smaller than jh for which $\sigma_k = 1$ and thus belongs to the interval $\{(j-1)h + 1, \dots, jh - 1\}$ (since $m \geq 2$ there are at least two transmissions between the time steps $(j-1)h$ and $jh - 1$). Due to (91) we have $\operatorname{Prob}[x_{k^t(\xi_{(j-1)h})} \in B \mid \xi_{(j-1)h} = y] > 0$ for every $y \in \mathbb{R}^n$ and every open set $B \subseteq \mathbb{R}^{n_x}$. Taking into account (28), and the fact that under Assumption 5(ii) the gain matrix $K_{k^t(\xi_{(j-1)h}) - (j-1)h, t(\xi_{(j-1)h})}$ is full rank (cf. Lemma 11(ii)) this implies

$$\operatorname{Prob}\left[\bar{u}_{k^t(\xi_{(j-1)h})} \in C \mid \xi_{(j-1)h} = y\right] > 0 \quad (92)$$

for every $y \in \mathbb{R}^n$ and every open set $C \subseteq \mathbb{R}^{n_u}$, which follows directly from (5) and (28). Moreover (91) implies that

$$\operatorname{Prob}[x_{jh} \in D \mid \xi_{(j-1)h} = y] > 0 \quad (93)$$

for every $y \in \mathbb{R}^n$ and every open set $D \subseteq \mathbb{R}^{n_u}$. Noticing that $\bar{\xi}_j = \xi_{jh} = [x_{jh}^\top \ u_{k^t(\xi_{(j-1)h})}^\top]^\top$, (92) and (93) imply that $\operatorname{Prob}[\xi_{jh} \in D \times C \mid \xi_{(j-1)h} = y] > 0$ which implies (88). A

similar reasoning can be used for the case $m = 1$ and $j \geq 2$ using the fact that there are at least two transmissions between the time steps $(j - 2)h$ and $jh - 1$.

To prove (ii) we start by defining the set

$$S := \{\xi \in \mathbb{R}^n \mid \exists_{i,j \in \mathcal{M}, i \neq j} : \xi^\top P_i \xi + c_i = \xi^\top P_j \xi + c_j\}.$$

The complement S , denoted by S^c , is an open set. From the linearity of the Markov chain (35) (and in particular linearity with respect to initial condition) and the fact that the noise w_k , $k \in \mathbb{N}_0$, is Gaussian (results from the discretization of a Wiener process) it is clear that for every $y \in S^c$, $P^1(z, A)$ is a continuous function of z for z in a neighborhood of y which implies (89) and (90) (make $T(z, A) = P^1(z, A)$ for every $z \in \mathbb{R}^n$ and for every $A \subseteq \mathbb{R}^n$). In fact, from (35) we conclude that for $z \in S^c$ in a small neighborhood of y , $P^1(z, A) = \text{Prob}(\Phi_i z + \Psi_i \bar{w}_0 \in A)$ where \bar{w}_0 is Gaussian and $i = \iota(y) = \iota(z)$ (note that $\iota(y)$ may be different from $\iota(z)$ if $y \in S$ resulting in a discontinuity). However, if $y \in S$ more than one option $i \in \mathcal{M}_y$, $\mathcal{M}_y := \{i \in \mathcal{M} \mid \xi^\top P_i \xi + c_i = \xi^\top P_j \xi + c_j, \text{ for } j = \iota(y)\}$ can be chosen in an arbitrarily small neighborhood of y . Still in this case (89) and (90) are satisfied for

$$T(z, A) = \min \left\{ \tilde{P}_\kappa^1(z, A) \mid \kappa \in \mathcal{M}_y \right\}$$

for z in a small neighborhood of y where $\tilde{P}_\kappa^1(y, A) := \text{Prob}[\bar{\xi}_1 \in A \mid \bar{\xi}_0 = y]$, $\kappa \in \mathcal{M}$, when $\iota(y) = \iota(\bar{\xi}_0)$ is replaced by κ in (35), i.e., $\bar{\xi}_1 = \Phi_\kappa \bar{\xi}_0 + \Psi_\kappa \bar{w}_0$. In fact, each $\tilde{P}_\kappa^1(\cdot, A)$ is a continuous function of $y = \bar{\xi}_0$ and the minimum of continuous functions is continuous. Moreover, the fact that $\tilde{P}_\kappa^1(y, \mathbb{R}^n) > 0$ for every $y \in \mathbb{R}^n$ and $\kappa \in \mathcal{M}$ implies (90). ■

C. Proof of Theorems 9, 7, and 10

With the two key lemmas established in Section VIII-B available, we are ready to prove Theorem 9, which uses several results for Markov chains given in [49].

Proof. (of Theorem 9): First, we notice that Lemma 13(i) implies that (35) is an open set irreducible Markov chain (cf. [49, Sec. 6.1.2]) and also that it is an aperiodic chain (cf. [49, Sec. 5.4]). Second, we notice that Lemma 13(ii) implies that (35) is a so-called T-chain (cf. [49, Ch. 6]), which follows from [49, Props. 6.2.3, 6.2.4]). Then, it suffices to find a positive coercive function W ($\{\xi \mid W(\xi) \leq r\}$ are precompact for each $r > 0$, [49, Sec. 9.4]) such that, for (35)

$$\mathbb{E} [W(\bar{\xi}_{\ell+1}) \mid \bar{\xi}_\ell] - W(\bar{\xi}_\ell) \leq -1, \quad \forall \bar{\xi}_\ell \in \mathbb{R}^n \setminus C \quad (94)$$

for some compact neighborhood of the origin C and such that $\mathbb{E}[W(\bar{\xi}_{\ell+1}) \mid \bar{\xi}_\ell] - W(\bar{\xi}_\ell)$ is bounded if $\bar{\xi}_\ell \in C$. In fact, then we conclude that the chain (35) is a so-called Harris recurrent chain [49, Th. 9.2.2(ii) and Th. 9.4.1] which implies that there exists a unique invariant measure [49, Th. 10.0.1]. The fact that such invariant measure has finite total mass (in which case (35) is a so-called positive Harris chain) and hence can be made a probability distribution follows also from (94) (see [49, Th. 11.0.1]) and ergodicity follows then from the aperiodic ergodic theorem [49, Th. 13.0.1].

We use (60) to establish (94). However we cannot make $W = V$ in (94) since V in (60) is not precompact. Hence we add a

regularization term considering a coercive function

$$W(\bar{\xi}) = \bar{x}^\top \bar{P}_{q\tau} \bar{x} + \epsilon \bar{u}^\top \bar{u}$$

for $\bar{\xi} = [\bar{x}^\top, \bar{u}^\top]^\top$ and show that such W satisfies (94) for sufficiently small ϵ . To prove this we need the fact, established below, that

$$\mathbb{E} [\bar{u}_{\ell+1}^\top \bar{u}_{\ell+1} \mid \bar{\xi}_\ell] \leq a \bar{x}_\ell^\top \bar{x}_\ell + d \quad (95)$$

for every $\bar{\xi}_\ell = [\bar{x}_\ell^\top, \bar{u}_\ell^\top]^\top \in \mathbb{R}^n$ for given positive constants a and d . Then, from (60), (63), (95) we conclude that

$$\begin{aligned} \mathbb{E} [W(\bar{\xi}_{\ell+1}) \mid \bar{\xi}_\ell] - W(\bar{\xi}_\ell) &\leq -a_1 \bar{x}_\ell^\top \bar{x}_\ell + c_1 + \epsilon \mathbb{E} [\bar{u}_{\ell+1}^\top \bar{u}_{\ell+1} \mid \bar{\xi}_\ell] - \epsilon \bar{u}_\ell^\top \bar{u}_\ell \\ &\leq (-a_1 + \epsilon a) \bar{x}_\ell^\top \bar{x}_\ell - \epsilon \bar{u}_\ell^\top \bar{u}_\ell + \epsilon d + c_1. \end{aligned}$$

Picking $\epsilon = a_1/2a$ and C equal to

$$\left\{ \bar{\xi} = [\bar{x}^\top, \bar{u}^\top]^\top \in \mathbb{R}^n \mid \frac{a_1}{2} \bar{x}^\top \bar{x} + \frac{a_1}{2a} \bar{u}^\top \bar{u} \leq \frac{a_1 d}{2a} + c_1 + 2 \right\}$$

we conclude (94). It is also clear that $\mathbb{E}[W(\bar{\xi}_{\ell+1}) \mid \bar{\xi}_\ell] - W(\bar{\xi}_\ell)$ is bounded if $\bar{\xi}_\ell \in C$.

It remains to prove (95). To this effect, we notice that

$$\bar{u}_{\ell+1} = \left[K_{\bar{k}^\iota(\bar{\xi}_\ell), \iota(\bar{\xi}_\ell)} \quad 0_{n_u \times n_u} \right] \left(\Phi_{\iota(\bar{\xi}_\ell)} \bar{\xi}_\ell + \Psi_{\iota(\bar{\xi}_\ell)} \bar{w}_\ell \right) \quad (96)$$

where $K_{\bar{k}^\iota, i}$ is described in (78) and Φ_j, Ψ_j , $j \in \mathcal{M}$ can be derived in a similar way to analogous matrices in (35). In particular $\Phi_j = \Pi_{s=\bar{k}^j-1}^0 \Theta_{s,j}$ from which conclude that (96) is not a function of \bar{u}_ℓ if $\nu_0^i = 1$ for $i = \iota(\bar{\xi}_\ell)$, due to the structure of $\Theta_{0,i}$ in (36). This is the case if $\bar{\xi}_\ell$ lies in the set

$$\left\{ \bar{\xi} = [\bar{x}^\top, \bar{u}^\top]^\top \in \mathbb{R}^n \mid \bar{u}^\top \bar{u} \geq \gamma \bar{x}^\top \bar{x} + d \right\} \quad (97)$$

for given sufficiently large positive constants γ and d . In fact, one can see that the matrices resulting from (22) take the form

$$P_i = \begin{cases} \begin{bmatrix} X_i & 0 \\ 0 & 0 \end{bmatrix}, & \text{if } \nu_0^i = 1 \\ \begin{bmatrix} Y_i & * \\ * & Z_i \end{bmatrix}, & \text{if } \nu_0^i = 0 \end{cases}$$

for positive-definite matrices X_i, Y_i, Z_i with dimension $n_x \times n_x$, $n_x \times n_x$, and $n_u \times n_u$, respectively. Positive-definiteness of these matrices can be established using Assumption 1. Then, if $\bar{\xi}_\ell$ belongs to the set (97) for sufficiently large γ and d , then it is clear that (27) will correspond to an option $i = \iota(\bar{\xi}_\ell)$ such that $\nu_0^i = 1$. Then (95) holds for $\bar{\xi}_\ell$ in the set (97) since then (96) is not a function of \bar{u}_ℓ ; for $\bar{\xi}_\ell$ in the complement of (97), the norm of \bar{u}_ℓ is bounded by the norm of \bar{x}_ℓ plus a constant, which allows to obtain (95) taking into account (96). ■

We present next the proofs of Theorems 7, 10 which build upon the proofs of Theorems 4 and 9.

Proof. (of Theorem 7 and 10): To prove (33) consider the $\xi \in \mathbb{R}^n$ and $i \in \mathcal{M}$ characterized in (69) of Lemma 11, under Assumptions 5(i) and 5(ii), and define the following:

$$\bar{C} := \left\{ y \in \mathbb{R}^n \mid y^\top P_i y + c_i - (y^\top P_1 y + c_1) < -\frac{\bar{c}}{2} \right\}$$

where $\bar{c} := \xi^\top P_1 \xi + c_1 - \xi^\top P_i \xi - c_i > 0$. Note that \bar{C} is an open set and Lemma 13(i) implies that $P^r[z, \bar{C}] > 0$ for every $z \in \mathbb{R}^n$ and $r \geq 2$. If Assumptions 5(i) and 5(ii) hold then (54) holds with strict inequality for $r \geq 2$ since for a fixed initial condition $\bar{\xi}_0 \in \mathbb{R}^n$

$$\begin{aligned} & \mathbb{E} [J_{\bar{\xi}_0}^d(\bar{\xi}_r) - \bar{J}_{\bar{\xi}_0}^d(\bar{\xi}_r)] \\ &= \mathbb{E} \left[\bar{\xi}_r^\top P_1 \bar{\xi}_r + c_1 - \left(\bar{\xi}_r^\top P_{\iota(\bar{\xi}_r)} \bar{\xi}_r + c_{\iota(\bar{\xi}_r)} \right) \right] \\ &\geq \frac{\bar{c}}{2} P^r[\bar{\xi}_0, \bar{C}] > 0. \end{aligned}$$

We can then replace the inequalities in (55) and (56) for $r \geq 2$ by strict inequalities and obtain (33).

To prove (39) we note that from (94) and the fact that $f(\xi) \leq V(\xi)$ for every $\xi \in \mathbb{R}^n$ we can conclude that there exists a positive constant α such that

$$\mathbb{E} [W(\bar{\xi}_{\ell+1}) | \bar{\xi}_\ell] - W(\bar{\xi}_\ell) \leq -\alpha f(\bar{\xi}_\ell), \quad \forall \bar{\xi}_\ell \in \mathbb{R}^n - C \quad (98)$$

which implies that (35) is f -ergodic ([49, Ch. 14]). Thus, from [49, Th. 14.0.1], we conclude that

$$\lim_{\ell \rightarrow \infty} \mathbb{E} [f(\bar{\xi}_\ell)] = \lim_{\ell \rightarrow \infty} \int_{\mathbb{R}^n} f(\xi) P^\ell(\bar{\xi}_0, d\xi) = \int_{\mathbb{R}^n} f(\xi) \chi_{\text{inv}}(d\xi) \quad (99)$$

and

$$\begin{aligned} \lim_{L \rightarrow \infty} \mathbb{E} \left[\frac{1}{L} \sum_{l=0}^{L-1} f(\bar{\xi}_l) \right] &= \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{l=0}^{L-1} \mathbb{E} [f(\bar{\xi}_l)] \\ &= \int_{\mathbb{R}^n} f(w) \chi_{\text{inv}}(dw). \quad (100) \end{aligned}$$

Then (39) follows from (61). Moreover, due to Lemma 13(i) we have that $\chi_{\text{inv}}(A) > 0$, for every open set A . This fact that can be proved from the characterization of the unique invariant distribution given in [49, Th. 10.0.1], whose interpretation is the following (c.f., [49, p. 246]): for a fixed measurable set B in \mathbb{R}^n (which we can assume to be open), with $\chi_{\text{inv}}(B) > 0$, $\chi_{\text{inv}}(A)$ is proportional to the amount of time spent in A between visits to B , provided that the chain starts in B with a special distribution. Then, noticing that Lemma 13(i) assures that any open set is reached with positive probability from any initial state we conclude that $\chi_{\text{inv}}(A) > 0$ for every open set A . Then $g_a > 0$ since $\int_{\mathbb{R}^n} f(w) \chi_{\text{inv}}(dw) \geq \chi_{\text{inv}}(\bar{C})(c/2) > 0$, which implies (34). ■

Remark 14: The fact that (35) is a positive Harris recurrent Markov chain implies that the average costs do not depend on the initial condition (cf. [49, Ch.13]). □

Remark 15: Note that Assumption 5(ii) simplified significantly the proof of Lemma 13 by guaranteeing that the gains $K_{\bar{k}^i, i}$, described by (23) and (68), have full rank. Using this fact, we obtained a simple argument for (88) which enabled the proofs of Theorem 7, 9 and 10. Although Assumption 5(ii) is mild in the sense that it holds except in possible pathological cases, we make the following two remarks. First, since we only need to take into account $i \in \mathcal{M}$ such that there exists $\xi \in \mathbb{R}^n$ for which $\iota(\xi) = i$, i.e., scheduling decisions that can be chosen by Algorithm 3 in (27), then $K_{\bar{k}^i, i}$ may only need

to be full rank for a subset of $i \in \mathcal{M}$. Thus it may be the case that one does not need to test (32) for every value $s \in \{k\tau | k \in \{1, \dots, (h-m+1)\}\}$; Second, and most importantly, even if Assumption 5(ii) does not hold we may still be able to prove (88). Indeed under Assumption 5, which guarantee that the noise always influences every state $x_k \in \mathbb{R}^n$ after a single step k , we may still be able to prove that the noise can influence \hat{u}_k even if Assumption 5(ii) does not hold. □

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REFERENCES

- [1] D. Antunes, W. Heemels, and P. Tabuada, "Dynamic programming formulation of periodic event-triggered control: Performance guarantees and co-design," in *Proc. 51st IEEE Conf. Decision Control (CDC)*, Maui, HI, USA, Dec. 2012, pp. 7212–7217.
- [2] S. Oncu, N. van de Wouw, W. Heemels, and H. Nijmeijer, "String stability of interconnected vehicles under communication constraints," in *Proc. 51st IEEE Conf. Decision Control (CDC)*, Maui, HI, USA, Dec. 2012, pp. 2459–2464.
- [3] F. Oldewurtel, A. Parisio, C. N. Jones, D. Gyalistras, M. Gwerder, V. Stauch, B. Lehmann, and M. Morari, "Use of model predictive control and weather forecasts for energy efficient building climate control," *Energy Buildings*, vol. 45, pp. 15–27, 2012.
- [4] T. Namerikawa and T. Kato, "Distributed load frequency control of electrical power networks via iterative gradient methods," in *Proc. 51st IEEE Conf. Decision Control (CDC)*, Maui, HI, USA, Dec. 2012, pp. 7723–7728.
- [5] H. Meenink, R. Hendrix, G. Naus, M. Beelen, H. Nijmeijer, M. Steinbuch, E. Oosterhout, and M. Smet, "Robot-assisted vitreoretinal surgery," in *Medical Robotics: Minimally Invasive Surgery*. Cornwall, U.K.: Woodhead Publishing Limited, 2012, pp. 185–209.
- [6] R. Oung and R. D'Andrea, "The distributed flight array," *Mechatronics*, vol. 21, no. 6, pp. 908–917, 2011.
- [7] N. C. de Castro, C. C. de Wit, and K. Johansson, "On energy-aware communication and control co-design in wireless networked control systems," in *Proc. 2nd IFAC Workshop Distrib. Estim. Control Netw. Syst. (NecSys'10)*, 2010, [CD-ROM].
- [8] C. Lozoya, P. Mart, M. Velasco, J. M. Fuertes, and E. X. Martn, "Resource and performance trade-offs in real-time embedded control systems," *J. Real-Time Syst.*, vol. 49, no. 3, pp. 267–307, May 2013.
- [9] W. Heemels, K. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *Proc. 51st IEEE Conf. Decision Control (CDC)*, Maui, HI, USA, Dec. 2012, pp. 3270–3285.
- [10] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1680–1685, Sep. 2007.
- [11] W. P. M. H. Heemels, J. H. Sandee, and P. P. J. Van Den Bosch, "Analysis of event-driven controllers for linear systems," *Int. J. Control*, vol. 81, no. 4, pp. 571–590, 2008.
- [12] K. Astrom and B. Bernhardsson, "Comparison of Riemann and Lebesgue sampling for first order stochastic systems," in *Proc. 41st IEEE Conf. Decision Control*, Dec. 2002, vol. 2, pp. 2011–2016.
- [13] J. Lunze and D. Lehmann, "A state-feedback approach to event-based control," *Automatica*, vol. 46, no. 1, pp. 211–215, 2010.
- [14] M. Velasco, J. Fuertes, and P. Marti, "The self-triggered task model for real-time control systems," in *Proc. IEEE Real-Time Syst. Symp.*, 2003, pp. 67–70.
- [15] X. Wang and M. Lemmon, "Self-triggered feedback control systems with finite-gain stability," *IEEE Trans. Autom. Control*, vol. 54, no. 3, pp. 452–467, Mar. 2009.
- [16] M. Mazo, A. Anta, and P. Tabuada, "On self-triggered control for linear systems: Guarantees and complexity," in *Proc. 48th IEEE Eur. Control Conf.*, Aug. 2009.
- [17] Q. C. J. Almeida, C. Silvestre, and A. M. Pascoal, Observer Based Self-Triggered Control of Linear Plants With Unknown Disturbances, pp. 5688–5693, Montréal, Canada, Jun. 27–29 2012, 27–29.

- [18] T. Gommans, D. Antunes, T. Donkers, P. Tabuada, and M. Heemels, "Self-triggered linear quadratic control," *Automatica*, vol. 50, no. 4, pp. 1279–1287, Apr. 2014.
- [19] A. Anta and P. Tabuada, "On the minimum attention and anytime attention problems for nonlinear systems," in *Proc. 49th IEEE Conf. Decision Control (CDC)*, Dec. 2010, pp. 3234–3239.
- [20] M. Donkers, P. Tabuada, and W. Heemels, "Minimum attention control for linear systems," *Discrete Event Dyn. Syst.*, pp. 1–20, 2012.
- [21] J. Yook, D. Tilbury, and N. Soparkar, "Trading computation for bandwidth: Reducing communication in distributed control systems using state estimators," *IEEE Trans. Control Syst. Technol.*, vol. 10, no. 4, pp. 503–518, Jul. 2002.
- [22] A. Eqtami, D. Dimarogonas, and K. Kyriakopoulos, "Event-triggered control for discrete-time systems," in *Proc. Amer. Control Conf. (ACC'10)*, Jul. 2010, pp. 4719–4724.
- [23] J. Wu, Q.-S. Jia, K. Johansson, and L. Shi, "Event-based sensor data scheduling: Trade-off between communication rate and estimation quality," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 1041–1046, 2013.
- [24] W. Heemels and M. Donkers, "Model-based periodic event-triggered control for linear systems," *Automatica*, vol. 49, no. 3, pp. 698–711, 2013.
- [25] W. Heemels, M. Donkers, and A. Teel, "Periodic event-triggered control for linear systems," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 847–861, 2013.
- [26] Y. Xu and J. Hespanha, "Optimal communication logics in networked control systems," in *Proc. 43rd IEEE Conf. Decision Control (CDC)*, Dec. 2004, vol. 4, pp. 3527–3532.
- [27] R. Cogill, S. Lall, and J. Hespanha, "A constant factor approximation algorithm for event-based sampling," in *Proc. Amer. Control Conf. (ACC'07)*, Jul. 2007, pp. 305–311.
- [28] R. Cogill, "Event-based control using quadratic approximate value functions," in *Proc. 48th IEEE Conf. Decision Control, Held Jointly With 28th Chinese Control Conf. (CDC/CCC)*, Dec. 2009, pp. 5883–5888.
- [29] A. Molin and S. Hirche, "On LQG joint optimal scheduling and control under communication constraints," in *Proc. 48th IEEE Conf. Decision Control, Held Jointly With 28th Chinese Control Conf. (CDC/CCC)*, Dec. 2009, pp. 5832–5838.
- [30] A. Molin and S. Hirche, "Structural characterization of optimal event-based controllers for linear stochastic systems," in *Proc. 49th IEEE Conf. Decision Control (CDC)*, Dec. 2010, pp. 3227–3233.
- [31] C. Ramesh, H. Sandberg, L. Bao, and K. Johansson, "On the dual effect in state-based scheduling of networked control systems," in *Proc. Amer. Control Conf. (ACC'11)*, Jul. 2011, pp. 2216–2221.
- [32] A. Eqtami, D. Dimarogonas, and K. Kyriakopoulos, "Novel event-triggered strategies for model predictive controllers," in *Proc. 50th IEEE Conf. Decision Control & Eur. Control Conf. (CDC-ECC)*, Dec. 2011, pp. 3392–3397.
- [33] E. Henriksson, D. E. Quevedo, H. Sandberg, and K. H. Johansson, "Self-triggered model predictive control for network scheduling and control," in *Proc. IFAC Int. Symp. Adv. Control Chem. Processes (ADCHEM)*, 2012, [CD-ROM].
- [34] J. D. J. B. Berglind, T. M. P. Gommans, and W. Heemels, "Self-triggered MPC for constrained linear systems and quadratic costs," in *Proc. IFAC Conf. Nonlin. Model Predictive Control*, Noordwijkerhout, The Netherlands, Aug. 2012, pp. 342–348.
- [35] P. Varutti and R. Findeisen, "Event-based NMPC for networked control systems over UDP-like communication channels," in *Proc. Amer. Control Conf. (ACC'11)*, Jul. 2011, pp. 3166–3171.
- [36] D. Bernardini and A. Bemporad, "Energy-aware robust model predictive control based on noisy wireless sensors," *Automatica*, vol. 48, no. 1, pp. 36–44, 2012.
- [37] G. Goodwin, H. Haimovich, D. Quevedo, and J. Welsh, "A moving horizon approach to networked control system design," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1427–1445, 2004.
- [38] M. Rabi, "Packet Based Inference and Control," Ph.D. dissertation, University of Maryland, Baltimore, 2006.
- [39] T. Henningson, E. Johansson, and A. Cervin, "Sporadic event-based control of first-order linear stochastic systems," *Automatica*, vol. 44, no. 11, pp. 2890–2895, 2008.
- [40] X. Meng and T. Chen, "Optimal sampling and performance comparison of periodic and event based impulse control," *IEEE Trans. Autom. Control*, vol. 57, no. 12, pp. 3252–3259, Dec. 2012.
- [41] O. Imer and T. Basar, "Optimal estimation with limited measurements," in *Proc. 44th IEEE Conf. Decision Control & Eur. Control Conf.*, Dec. 2005, pp. 1029–1034.
- [42] O. Imer and T. Basar, "Optimal control with limited controls," in *Proc. Amer. Control Conf.*, Jun. 2006, p. 6.
- [43] D. P. Bertsekas, *Dynamic Programming and Optimal Control*, vol. 1 and 2, 3rd ed. Boston, MA: Athena Scientific, 2005.
- [44] K. Astrom, *Introduction to Stochastic Control Theory*. New York: Elsevier, 1970.
- [45] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1989.
- [46] T. Chen and B. Francis, *Optimal Sampled-Data Control Systems*. New York: Springer-Verlag, 1995.
- [47] V. Cionca, T. Newe, and V. Dardarlat, "TDMA protocol requirements for wireless sensor networks," in *Proc. 2nd Int. Conf. Sensor Technol. Appl. (SENSORCOMM'08)*, 2008, pp. 30–35.
- [48] A. Alessio and A. Bemporad, "A survey on explicit model predictive control," in *Nonlinear Model Predictive Control*, vol. 384, ser. *Lecture Notes in Control and Information Sciences*, L. Magni, D. Raimondo, and F. Allgwer, Eds. Berlin, Germany: Springer, 2009, pp. 345–369.
- [49] S. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, 2nd ed. New York, USA: Cambridge Univ. Press, 2009.
- [50] K. B. Athreya and M. Majumdar, "Estimating the stationary distribution of a markov chain," *Economic Theory*, vol. 21, pp. 729–742, 2003.
- [51] G. Pags and F. Panloup, "Approximation of the distribution of a stationary markov process with application to option pricing," *Bernoulli*, vol. 15, no. 1, pp. 146–177, 02 2009.
- [52] D. Antunes, "Event-triggered control under Poisson events: The role of sporadicity," in *Proc. 4th IFAC Workshop Distrib. Estim. Control Networked Syst.*, Koblenz, Germany, Sep. 2013, pp. 269–276.
- [53] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, NJ: Prentice-Hall, 1995.



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