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Root Locus Analysis for Randomly Sampled Systems

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Abstract— **Root locus analysis is a graphical method to determine how the roots of the characteristic equation of a linear time-invariant feedback loop change with the loop gain. In this paper we show that a similar analysis can be carried out for randomly sampled systems, i.e., controlled linear systems sampled at random times spaced by independent and identically distributed time-varying intervals. For such systems, the roots of a characteristic equation determine the behavior of expected values of signals in the loop. The root locus analysis in this context is especially useful for positive systems, for which (almost sure) stability can be concluded if the roots of the characteristic equation have a negative real part, and it is particularly simple when the distribution of the intervals between sampling times is exponential or Erlang.**

I. INTRODUCTION

Originally proposed by Evans [1], root locus analysis consists of a graphical method to determine how the poles of a linear single-input single-output feedback loop change with a given parameter, typically the loop gain. It is taught worldwide in introductory control courses [2, Ch. 5] and thus it is widespread in industrial control design. However, it is limited to *time-invariant systems*, either continuous-time or discrete-time/periodic sampled-data, since the concept of pole does not easily extend to *time-varying systems*. As a result, root-locus analysis is not immediately applicable to emergent networked control applications (remote surgery, smart grids, automotive industry, etc.) in which sensors, actuators and controllers are connected by communication networks introducing *time-varying* sampling intervals and delays (see, e.g., [3]–[14]). The absence of root locus analysis and other classical analysis tools (Nyquist criterion, sensitivity analysis, etc.) in these contexts is a bottleneck in the process of streamlining the analysis and design of networked control systems (NCSs).

In this paper we show that an analogous method to root-locus analysis can be carried out for studying a closed loop model capturing important features of NCSs. We consider randomly sampled systems, i.e., systems in which the sampling times at which the loop is closed are spaced by independent and identically distributed time-varying intervals. Randomness in the sampling time intervals models the fact that communication networks (such as wireless and Ethernet local area networks) implement protocols involving stochastic features such as packet collisions, drops, back-off times, etc (see [15]).

Our analysis builds upon the fact, established in [16], [17], that one can obtain Laplace transform expressions for the expected value of signals in a randomly sampled loop. Such expressions reveal that the behavior of these expected values is determined by the roots of a characteristic equation, so-called characteristic exponents, which thus play the role of poles in the context of randomly sampled systems. The main contribution of this paper is to show that the characteristic exponents can be obtained as a function of the loop gain by a graphical method that parallels classical root locus analysis.

There are two important differences with respect to the classical analysis for deterministic systems. First, the number of characteristic exponents may be infinite. Nonetheless, we show that the number of roots is finite if the distribution of the sampling intervals belongs to the class of phase-type distributions, which can arbitrarily approximate any probability distribution [18, Ch. III]. In particular, for exponential and Erlang distributions, instances of phase-type distributions, the analysis is especially simple. Second, (almost sure) stability cannot in general be assured if the characteristic exponents have a negative real part. Yet, we show that the latter condition does imply (almost sure) stability for the class of positive systems (found in economics, transportation networks, etc. [19]).

While, to the best of our knowledge, root-locus analysis has not been considered beyond the scope of deterministic time-invariant systems, we can establish connections with previous work. Early work on randomly sampled systems can be found in [20], [21] [22], [23]. One of the motivations was to capture sampling jitter in digital control. More recent work, motivated by networked control and applications in system biology, can be found in [6]–[8], [16], [17], [24]. For related work considering sampling jitter, delays, packet drops, and other network-induced control constraints, see [3]–[14]. For an eigenvalue-based approach to study deterministic time-delay systems see [25].

The paper is organized as follows. Section II provides preliminary results. Our main results are given in Section III and Section IV presents a numerical example. Section V discusses the limitations of the work motivating future research.

II. PRELIMINARIES

Consider a control loop in which the process is described by a minimal realization

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t), \qquad\qquad t \in \mathbb{R}_{\geq 0},\end{aligned}$$

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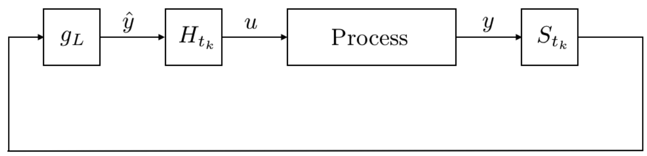


Fig. 1. SISO digital control loop; S_{t_k} is the sampler, H_{t_k} is the standard zero-order hold updated at times t_k, and g_L is the loop gain; the process block may include a controller/compensator.

where $x(t) \in \mathbb{R}^m$ is the state, $u(t) \in \mathbb{R}$ is the control input, and $y(t) \in \mathbb{R}$ is the output. The state of the process may include (and in general does include) some variables from a dynamic controller/compensator such that analyzing the control law $u(t) = -g_L y(t)$ for a *positive loop gain* g_L suffices to obtain desired behavior for the closed loop. The open loop transfer function is given by $\frac{n(s)}{d(s)}$, where

$$d(s) := \det(sI - A), \quad n(s) := C\text{adj}(sI - A)B,$$

and adj stands for adjugate of a matrix, and the closed loop poles (eigenvalues of $A - g_L BC$) coincide with the roots of the *characteristic equation*

$$d(s) + g_L n(s) = 0. \tag{1}$$

The root locus analysis (see [2, Ch. 5]) describes how the location of the closed loop poles change with g_L. It can also be applied to the standard digital control loop depicted in Figure 1, where the controller has only access to the output samples $\{y(t_k) | t_{k+1} - t_k = h, k \in \mathbb{N}_0\}$, for some sampling period h, and a standard zero order hold interface is used

$$\hat{y}(t) := y(t_k), \qquad t \in [t_k, t_{k+1}), \tag{2}$$

which results in the input $-g_L \hat{y}(t)$ to the process/compensator. The characteristic equation takes now the form

$$d_d(z) + g_L n_d(z) = 0, \tag{3}$$

where

$$n_d(z) := C\text{adj}(zI - A_d)B_d, \quad d_d(z) := \det(zI - A_d),$$

$A_d := e^{Ah}$, and $B_d := \int_0^h e^{A\tau} d\tau B$ result from the step-invariant discretization of the process.

Motivated by networked control applications, in this paper we assume that $h_k := t_{k+1} - t_k$ in the setup of Figure 1 are independent and identically distributed random variables, following a probability cumulative distribution F,

$$t_{k+1} - t_k \sim F, \quad k \in \mathbb{N}_0.$$

We assume that F can be written as $F = F_1 + F_2$, where F_1 is an absolutely continuous function $F_1(\tau) = \int_0^\tau f(s)ds$, for some density function f, and F_2 is a piecewise constant increasing function that captures possible atom points $\{a_i\}$ occurring with probability $\{w_i\}$. Then, for a matrix-valued function G, we have $\int_0^\infty G(\tau)F(d\tau) = \int_0^\infty G(\tau)f(\tau)d\tau + \sum_i w_i G(a_i)$.

To pursue and interpret a root locus analysis in this setting, we need two preliminary results. The first follows from [16], [17] and provides Laplace transforms for expected values of state variables of the randomly sampled control loop. Let $w(t) := [\, x(t)^\intercal \ \ \hat{y}(t) \,]^\intercal$ and note that

$$\begin{aligned} \dot{w}(t) &= \bar{A}w(t), & t \in \mathbb{R}_{\geq 0}, t \neq t_k, \\ w(t_k) &= \bar{J}w(t_k^-), & k \in \mathbb{N}, \end{aligned} \tag{4}$$

where

$$\bar{A} := \begin{bmatrix} A & -Bg_L \\ 0 & 0 \end{bmatrix}, \quad \bar{J} := \begin{bmatrix} I & 0 \\ C & 0 \end{bmatrix},$$

and $w(t_k^-) := \lim_{s \uparrow t_k} w(s)$. Moreover, consider the following assumption:

$$e^{\bar{\lambda}(A)t} r(t) < ce^{-\alpha_1 t} \text{ for some } c > 0,\ \alpha_1 > 0, \tag{5}$$

where $\bar{\lambda}(A)$ is the real part of the eigenvalue(s) of A with the largest real part, which is always satisfied if the open loop system is stable, or if F has bounded support.

Theorem 1: Suppose that (5) holds. Then, the Laplace transform of the expected value of the state of (4),

$$\hat{w}(s) := \int_0^\infty \mathbb{E}[w(t)]e^{-st}dt,$$

is equal to

$$\hat{w}(s) = \hat{H}(s)[I - \hat{K}(s)]^{-1} w(0), \tag{6}$$

where

$$\hat{K}(s) := \bar{J} \int_0^\infty e^{\bar{A}\tau} e^{-s\tau} F(d\tau), \ \ \hat{H}(s) := \int_0^\infty e^{\bar{A}\tau} r(\tau) e^{-s\tau} d\tau,$$

and $r(\tau) := \int_\tau^\infty F(dr)$.

□

The *characteristic exponents* are defined as the roots of the characteristic equation

$$\det(I - \hat{K}(s)) = 0 \tag{7}$$

and determine the behavior of $\hat{w}(s)$ and thus also the behavior of $\mathbb{E}[w(t)]$, which can be obtained by inverse Laplace transforms. In particular, one can conclude from the inverse Laplace transform that $\mathbb{E}[w(t)] \to 0$, as $t \to \infty$, if the characteristic exponents have a negative real part (cf. [26, Ch. 7]). The second of our two preliminary result establishes that this is sufficient for (almost sure) stability if the closed loop system is positive, in the sense that each component of $x(t)$ is positive for every $t \in \mathbb{R}_{\geq 0}$.

Proposition 2: Consider the system (4) and suppose that each component of $x(t)$ is positive for every time $t \in \mathbb{R}_{\geq 0}$, that (5) holds and that $\mathbb{E}[w(t)] \to 0$, as $t \to \infty$. Then, the closed loop system is almost surely stable, i.e., $\text{Prob}[\lim_{t\to\infty} \|x(t)\| = 0] = 1$.

□

The proof is omitted, but we mentioned that it can be obtained by analyzing the discrete-time process $\{w(t_k) | k \in \mathbb{N}_0\}$ and combining arguments from [16], [27], [28]. For general systems, other methods can be used to assure almost sure stability (see, e.g., [16], [20], [21], [27], [28]) and the root locus analysis presented next is still useful as a method to infer the behavior of the expected value of the state. We exploit this in the application example of Section IV.

III. Main results

The following is the main result of the paper.

Theorem 3: The roots of (7), i.e., the characteristic exponents, coincide with the roots of

$$d_F(s) + g_L n_F(s) = 0, \tag{8}$$

where

$$d_F(s) = \det(I - \hat{A}(s)), \quad n_F(s) = C\text{adj}(I - \hat{A}(s))\hat{B}(s),$$

and

$$\hat{A}(s) := \int_0^\infty e^{A\tau} e^{-s\tau} F(d\tau),\ \hat{B}(s) := \int_0^\infty \int_0^\tau e^{Ar} B dr e^{-s\tau} F(d\tau).$$

□

Note that (8) has the same structure as (1) suggesting that a root-locus analysis may be carried out as a function of g_L. This is in fact the case, provided that we make some adjustments to cope with the fact that, contrarily to the classical case (1), $n_F(s)$ and $d_F(s)$ are not necessarily finite complex functions for every $s \in \mathbb{C}$ (if F has unbounded support) and are not in general polynomials. In particular, the number of roots of (8) may be infinite. This is illustrated in the following example.

Example 4: Suppose that $n(s) = 1$, $d(s) = (s + a)$, $C = B = 1$, $A = -a$, $a \in \mathbb{R}_{\geq 0}$, leading to

$$d_F(s) = (1 - \hat{F}(s+a)), \quad n_F(s) = \frac{1}{a}(\hat{F}(s) - \hat{F}(s+a)), \tag{9}$$

where

$$\hat{F}(s) := \int_0^\infty e^{-st} F(dt). \tag{10}$$

For uniform F in the interval $[0, T]$, we obtain $\hat{F}(s) = \frac{(1-e^{-sT})}{sT}$ and (8) has an infinite number of roots. For example, for $g_L = 0$, the roots are given by the infinite number of complex solutions to $T(s + a) = 1 - e^{-(s+a)T}$. On the other hand, if F corresponds to an exponential distribution with rate λ (unbounded support), i.e.,

$$F(\tau) = e^{-\lambda\tau}, \quad \tau \in \mathbb{R}_{\geq 0}, \tag{11}$$

then $\hat{F}(s) = \frac{\lambda}{\lambda+s}$ and

$$n_F(s) = \frac{\lambda}{(s+\lambda)(\lambda+s+a)}, \quad d_F(s) = \frac{s+a}{\lambda+s+a}, \tag{12}$$

are both infinite when $s = -(a+\lambda)$ and $d_F(s)$ is also infinite when $s = -\lambda$.

□

A consequence of the fact that $n_F(s)$ and $d_F(s)$ are not necessarily finite (if F has unbounded support) is that, contrarily to the classical case, it does not suffice to consider the roots of $n_F(s) = 0$ and $d_F(s) = 0$ to perform a root-locus analysis based on (8). This is clear from Example 4, in which the characteristic equation (8) for (12) is equivalent to

$$(s + \lambda)(s + a) + g_L\lambda = 0. \tag{13}$$

Here $-a$ and $-\lambda$ play the role of 'open loop poles' (roots of $d(s) = 0$) in the classical analysis (1) instead of simply $-a$, which is the only root of $d_F(s) = 0$ (in this case $n_F(s) = 0$ has no roots). This illustrates the fact that, in general, if F has unbounded support, one needs to rearrange (8) to perform a root-locus analysis.

The fact that $n_F(s)$ and $d_F(s)$ are not polynomial, leading to a possible infinite number of characteristic exponents, seems to hinder a closed loop analysis in such a case. However, this can be circumvented by approximating F by a phase-type distribution (dense in the set of all probability distribution on $(0, \infty)$ [18, Th. 4.2, Ch. III]), i.e., it can accurately approximate any given probability distribution). The class of phase-type distributions [18, Ch. III] is parameterized by a $p \times p$ square matrix S and a $1 \times p$ row vector α and it is characterized by $F(x) = 1 - \alpha e^{Sx}\mathbf{1}$ and $\hat{F}(s) = -\alpha(sI - S)^{-1}S\mathbf{1}$, respectively, where $\mathbf{1}$ is a row vector of ones. Computing $n_F(s)$ and $d_F(s)$ for a phase type distribution one obtains rational functions (ratio of polynomials). Hence, the root-locus analysis can be pursued by rearranging (8) to obtain an equation as (1) for polynomials $n(s)$ and $d(s)$, like we did to obtain (13) for (12), and using standard rules [2, Ch. 5]. The problem of fitting a phase-type distribution to a given probability distribution is addressed in [18, Ch. III]. Note that from a practical point of view it is desirable to obtain a good fit with a small p (dimension of the matrix S), since in general the number of characteristic exponents increases with p.

We consider below two special cases of phase-type distributions: (i) exponential distributions in Section III-A; (ii) Erlang distributions in Section III-B, where we also illustrate how a Dirac distribution (digital control) can be approximated by phase-type distributions. In Section III-C we discuss the connection between our method and a discrete-time approach in the spirit of [20]–[23], [27].

A. Exponential distribution

When considering exponential distributions in Example 4 we obtained a characteristic equation (13) taking the form $d(s)(s + \lambda) + g_L\lambda n(s) = 0$. The next result shows that this is a general property.

Theorem 5: Suppose that F corresponds to an exponential distribution (11) with rate parameter $\lambda > \bar{\lambda}(A)$. Then, (5) holds and the characteristic equation (8) takes the form

$$d(s)(s + \lambda) + g_L\lambda n(s) = 0. \tag{14}$$

□

This result has two important implications. First, the root locus analysis as a function of g_L (for exponential intertransmission times and fixed λ) boils down to the following procedure. Consider the set of zeros and poles of the open loop transfer function $\frac{n(s)}{d(s)}$ and add a pole at location $-\lambda$; perform a classical root locus analysis to determine the closed loop poles as a function of the loop gain $\tilde{g}_L = \lambda g_L$. Then, the location of the roots of the characteristic equation of the randomly sampled system coincide with that of these closed loop poles. Second, noting that (14) is equivalent to

$$d(s)s + \lambda(d(s) + g_L n(s)) = 0,$$

a root locus analysis as a function of the intensity λ (for fixed g_L, and $\lambda > \max(\bar{\lambda}(A), 0)$) can be pursued, where the role of numerator and denominator of the open loop transfer function are played by the denominator of the closed loop transfer function and by the polynomial obtained by multiplying the denominator of the open loop transfer function by s, respectively. Note, in particular, the simplicity of determining the influence of the (average) sampling rate on the characteristic exponents when compared to an analogous analysis for traditional digital control with a deterministic sampling rate.

B. Erlang distributions

Consider now that F corresponds to an Erlang distribution with shape parameter $\kappa \in \mathbb{N}$ and mean $1/\lambda$, i.e., F has the density

$$f(x) = \frac{(\lambda\kappa)^\kappa x^{\kappa-1}}{(\kappa-1)!} e^{-\lambda\kappa x}. \tag{15}$$

Then, one can show that (5) is met if $\lambda > \frac{\bar{\lambda}(A)}{\kappa}$, and

$$\hat{F}(s) = (\frac{\lambda\kappa}{s+\lambda\kappa})^\kappa. \tag{16}$$

By varying κ, we can consider a range of distributions starting with the exponential distribution with rate λ for $\kappa = 1$ and approximating a Dirac function at $\frac{1}{\lambda}$ as $\kappa \to \infty$. In particular, as $\kappa \to \infty$, (16) converges to $e^{-\frac{s}{\lambda}}$, which is (10) for a Dirac at $\frac{1}{\lambda}$. The Dirac function coincides with the special case of digital control in the setup of Figure 1 with sampling periodic $t_{k+1} - t_k = \frac{1}{\lambda}$. As such, we can interpret considering a finite but large κ for a gamma distribution in this setting as inferring the effect of a sampling jitter in digital control.

We start by considering the simple case $\frac{n(s)}{d(s)} = \frac{1}{s+1}$ as in Example 4. For this case we can conclude from (8), (9), (16) that, for a given κ, the roots of (8) coincide with the roots of

$$\bar{d}(s) + g_L \bar{n}(s) = 0 \tag{17}$$

for polynomials

$$\bar{n}(s) := \frac{(\lambda\kappa)^\kappa}{a}\Big((s+\lambda\kappa+a)^\kappa - (s+\lambda\kappa)^\kappa\Big),$$
$$\bar{d}(s) := (s+\lambda\kappa)^\kappa\Big((s+a+\lambda\kappa)^\kappa - (\lambda\kappa)^\kappa\Big).$$

A classical root locus analysis can then be pursued based on the location of the roots of $\bar{d}(s) = 0$ and $\bar{n}(s) = 0$; $\bar{d}(s) = 0$ has κ multiple roots at $s = -\kappa\tau$ and the following roots

$$s = -a + \lambda\kappa(e^{\frac{j2\pi\ell}{\kappa}} - 1), \quad \ell \in \{0, \dots, \kappa-1\} \tag{18}$$

with an interesting geometric pattern: they lie on the circle passing through α_i and $\alpha_i - \lambda\kappa$; the roots of $\bar{n}(s) = 0$ are given by:

$$s = -\lambda\kappa + \frac{-a}{1 - e^{\frac{j2\pi\ell}{\kappa}}}, \quad \ell \in \{1, \dots, \kappa-1\}.$$

For a more general case, suppose that A is diagonalizable, $A = UDU^{-1}$, and let a_i, $i \in \{1, 2, \dots, m\}$ denote the diagonal elements of D assumed to the real, v_i the row vectors of U and w_j the row columns of U^{-1}. Then, from the expressions provided in Theorem 3 we can conclude that

$$\frac{n_F(s)}{d_F(s)} = \sum_{i=1}^{n} C v_i w_i B g_i(s, a_i),$$

where

$$g_i(s, a_i) = \frac{(\hat{F}(s-a_i) - \hat{F}(s))}{a_i(1 - \hat{F}(s-a_i))}.$$

We can again see that the roots of (8) coincide with the roots of an equation taking the form $\bar{d}(s) + g_L\bar{n}(s) = 0$ for polynomials $\bar{n}(s)$ and $\bar{d}(s)$. The roots of $\bar{d}(s) = 0$ are easy to determine: κ multiple roots are located at $-\kappa\tau$ and the remaining roots take the form (18) when a is replaced by each $-a_i$, $i \in \{1, \dots, m\}$. However, the location of the roots of $\hat{n}(s) = 0$ do not seem straightforward to determine.

C. Connection to a discrete-time approach

When F corresponds to a Dirac at h (i.e., digital control with sampling period h), we obtain $\hat{A}(s) = A_d e^{-sh}$, $\hat{B}(s) = B_d e^{-sh}$, and (8) can be written as

$$\det(I - A_d e^{-sh}) + g_L C \text{adj}(I - A_d e^{-sh}) B_d e^{-sh} = 0,$$

or equivalently as (3) for $z = e^{sh}$. Note that (3) is obtained taking a discrete-time approach, i.e., considering the system $\{w(t_k) | k \in \mathbb{N}_0\}$ or $\{x(t_k) | k \in \mathbb{N}_0\}$. This raises the question whether a discrete-time approach in the spirit of [20]–[23] can also be pursued for root-analysis of randomly sampled systems.

To this effect, let $z_k := \mathbb{E}[x(t_k)]$ and note that from (4) we obtain

$$z_{k+1} = (\tilde{A} - g_L \tilde{B} C) z_k, \tag{19}$$

where

$$\tilde{A} := \int_0^\infty e^{A\tau} F(\tau), \quad \tilde{B} := \int_0^\infty \int_0^r e^{A\tau} d\tau F(dr) B.$$

From this we can conclude that the eigenvalues are the roots of

$$d_d(z) + g_L n_d(z) = 0, \tag{20}$$

where $d_d(z) := \det(zI - \tilde{A})$ and $n_d(z) := C\text{adj}(zI - \tilde{A})\tilde{B}$, suggesting that a root-locus analysis can be carried out. The roots of $d_d(z)$ have the following relation with respect to the open loop poles of the continuous-time transfer function:

$$\{z \in \mathbb{C} | d_d(z) = 0\} = \{\int_0^\infty e^{s\tau} F(d\tau) | \det(sI - A) = 0, s \in \mathbb{C}\}. \tag{21}$$

Table III-C summarizes this relation for distributions of interest. Note that the stable continuous-time poles (Re$(s) < 0$) are always mapped onto the unit circle. As for digital control, there appears to be no easy way to describe the roots of $n_d(z)$.

A possible drawback of this approach with respect to the one provided above is that it is not easy to interpret how the location of the roots of (20) influences the behavior of the closed loop. Note that in the approach of the previous section the characteristic exponents can be interpreted as exponential

Exp.	Erlang	Unif. (support T)	Dirac at h
$z=\frac{\lambda}{s-\lambda}$	$z=(\frac{\lambda\kappa}{s-\lambda\kappa})^{\kappa}$	$z=\frac{e^{sT}-1}{sT}$	$z=e^{sh}$

TABLE I

CONNECTION BETWEEN THE ROOTS OF $\det(sI-A)=0$ AND THE ROOTS OF $\det(zI-\tilde{A})=0$ FOR SEVERAL DISTRIBUTIONS

decays/increase times rates of the expected value of the state. Here, however, they represent exponential decay/increase rates for the sampled system (19), and it does not appear to be easy to relate these with decay/increase time rates.

IV. NUMERICAL EXAMPLE

Consider the following model:

$$\begin{aligned}\dot{x}_1(t) &= -5x_1(t) + \ell_{12}(x_2(t) - x_1(t))\\ \dot{x}_2(t) &= \ell_{21}(x_1(t) - x_2(t)) + \ell_{23}(x_3(t) - x_2(t))\\ \dot{x}_3(t) &= \ell_{32}(x_2(t) - x_3(t)), \quad t \in \mathbb{R}_{\geq 0},\end{aligned} \tag{22}$$

which can be used to describe a platoon of three vehicles [29, Example 2]. As mentioned in [29, Example 2], the terms $\ell_{ij}(x_j(t) - x_i(t))$ represent position adjustments based on distance measurements between the vehicles and the term $-5x_1(t)$ reflects the fact that the first vehicle maintains his position stable but the second and third vehicles rely on the distance measurements for stabilization. These measurements are assumed to be ideally transmitted between vehicles (available for every time t). We set $\ell_{ij} = 1$ for every i, j. The model (22) is a positive system in the sense that if $x_i(0) \geq 0$ for every i, then $x_i(t) \geq 0$, for every $t \in \mathbb{R}_{\geq 0}$ and every i (see [29]). Here we analyze the introduction of additional terms representing a position adjustment between the first and the third vehicles. If the communication between these two vehicles was ideal, we would have

$$\dot{x} = (A - g_L BC)x(t),$$

where

$$A := \begin{bmatrix} -6 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

is the system matrix for (22), $B := [1\,0\,,-1]^{\intercal}$, $C := [1\,0\,-1]$, and g_L a positive gain. However, here we assume that vehicles 1 and 3 acquire each others' positions through a bidirectional transmission link. Transmissions occur at times $\{t_k | k \in \mathbb{N}_0\}$ and, in between, the last position adjustment is held constant as in Figure 1.

We address first under which conditions the closed loop model of Figure 1 for the process characterized by matrices A, B, C remains a positive system (as the open loop (22)). It is easy to conclude that between sampling times

$$x(t) = (e^{A(t-t_k)} - g_L \int_{t_k}^{t} e^{A(t-t_k)} ds BC) x(t_k),\ t \in [t_k, t_{k+1}), \tag{23}$$

and that $e^{A(t-t_k)}$ has all entries strictly positive for $t > t_k$. Suppose that the distribution F of the intervals between transmission has bounded support T. Then, for sufficiently small g_L and T, $x(t)$ will have positive entries in the interval $[t_k, t_{k+1})$ if the same holds at t_k, and by induction we conclude that the system is positive. Figure 2(a) illustrates the values of g_L, T where all the components of the matrix $e^{AT} + g_L \int_0^T e^{As} ds BC$ are positive (and thus the closed loop is positive). From Proposition 2 we conclude that for such values of g_L, T, if the characteristic exponents lie in the left half plane, we can guarantee (almost sure) stability.

However, considering a distribution with a bounded support (e.g., uniform, see Example 4) typically leads to an infinite number of characteristic exponents. Moreover, if we approximate F by a phase-type distribution, we must in general deal with a distribution with unbounded support. From (23) we can conclude that for a fixed gain g_L there is a $t > t_k$ and a $x(t_k)$ such that $x(t)$ has negative components, i.e., the system is not necessarily positive. As such, we have to resort to other methods to assert stability.

In this example we consider only the exponential (phase-type) distribution (11) with rate λ. We use the results from [16] to test mean square stability (which implies almost sure stability) for specific values of g_L, λ, which is illustrated in Fig. 2(b). Note that there may exist other values of g_L and λ for which the closed loop is almost surely stable, although not mean square stable (see [16] for the definition of mean square stability).

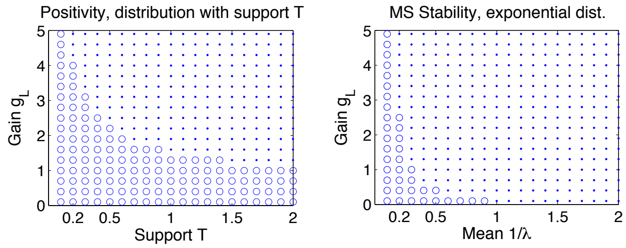
We turn now to a root-locus analysis made possible by the results of the present paper. As mentioned in Section III-A, the analysis is especially simple for exponential distributions: it suffices to add a pole at $-\lambda$ to the poles and zeros of the open loop transfer function, which for the matrices A, B, C above is given by

$$\frac{n(s)}{d(s)} = \frac{2s^2 + 11s + 10}{s^3 + 9s^2 + 18s + 5};$$

the zeros of this transfer function are at -4.35 and -1.15 and the poles at -0.33, -2.42, -6.25. Figure (3) shows the root-locus in this setting showing the characteristic exponents as a function of g_L for exponential F with rate $\lambda = 7$. A root-locus analysis based on ideal communication would simply not have the extra open loop pole at -7 and thus the closed loop pole that starts at -6.25 for $g_L = 0$ would tend to minus infinity along the real line. Intuitively, adding a pole at $-\lambda$ has a repulsive effect on the remaining poles, in the sense that they become closer to the right-half plane. This plays well with intuition, since imposing communication constraints leads to a slower convergence of the system to zero.

V. DISCUSSION

In this paper, we proposed a novel root-locus analysis framework for studying how the characteristic exponents of a randomly sampled systems change with the loop gain. One of the highlights is the simplicity of the analysis when the distribution of the sampling intervals is exponential or Erlang. Some limitations include the fact that (almost sure) stability when no right-half plane characteristic exponents exist can only be guaranteed for positive systems, and the



(a) F with support T (b) Exponential F

Fig. 2. (a) Values of T and g_L for which closed loop is positive are indicated by 'o'; Values of $1/\lambda$ and g_L for which mean-square stable are indicated by 'o'

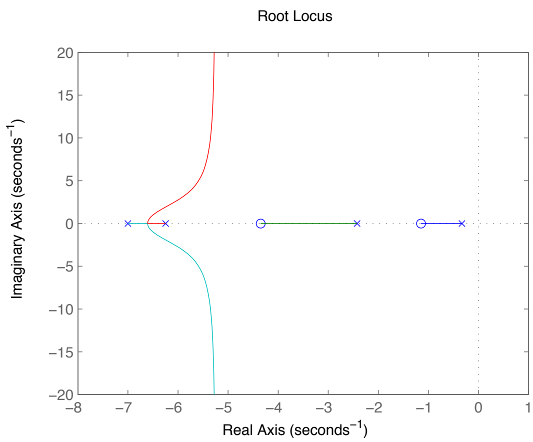


Fig. 3. Root-locus for a randomly sampled system when F is exponential with $\lambda = 7$.

fact that distributions with bounded support lead in general to an (impractical) infinite number of exponents -although this can be circumvented by using a phase-type distribution to approximate the original distribution, this may still lead to a large number of characteristic exponents.

One way to overcome the latter limitation is to use the Nyquist criterion to assure the existence of no right-half plane roots of the characteristic equation (8), a topic for future work. In order to make the stability analysis applicable to general systems (other than positive), we also plan to investigate to which extent the results carry on to an analysis of second moments as in [16].

References

[1] W. R. Evans, "Graphical analysis of control systems," *American Institute of Electrical Engineers, Transactions of the*, vol. 67, no. 1, pp. 547–551, 1948.

[2] G. F. Franklin, D. J. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*, 4th ed. Upper Saddle River, NJ, USA: Prentice Hall PTR, 2001.

[3] K.-E. Arzén, A. Cervin, and D. Henriksson, "Implementation-aware embedded control systems," in *D. Hristu-Varsakelis and W. S. Levine (Eds.): Handbook of Networked and Embedded Control Systems*. Birkhuser, 2005.

[4] D. E. Quevedo and D. Nesic, "Robust stability of packetized predictive control of nonlinear systems with disturbances and markovian packet losses," *Automatica*, vol. 48, no. 8, pp. 1803 – 1811, 2012.

[5] E. Bini and A. Cervin, "Delay-aware period assignment in control systems," in *Real-Time Systems Symposium, 2008*, Nov 2008, pp. 291–300.

[6] C.-C. Chen, S. Hirche, and M. Buss, "Sampled-data networked control systems with random time delay," in *17th IFAC World Congress*, 2008. [Online]. Available: ;

[7] A. Cuenca, J. Salt, A. Sala, and R. Piza, "A delay-dependent dual-rate pid controller over an ethernet network," *Industrial Informatics, IEEE Transactions on*, vol. 7, no. 1, pp. 18–29, Feb 2011.

[8] P. Martí, M. Velasco, R. Villa, J. M. Fuertes, and J. Ayza, "A probabilistic approach to the stability analysis of real-time control systems," in *16th IFAC World Congress, Praha, Czech Republic*, July 2005.

[9] D. Goswami, R. Schneider, and S. Chakraborty, "Relaxing signal delay constraints in distributed embedded controllers," *IEEE Transaction on Control Systems Technology*, accepted for publication.

[10] A. Seuret, "A novel stability analysis of linear systems under asynchronous samplings," *Automatica*, vol. 48, no. 1, pp. 177 – 182, 2012.

[11] L. Hetel, J. Daafouz, and C. Iung, "Stabilization of arbitrary switched linear systems with unknown time-varying delays," *Automatic Control, IEEE Transactions on*, vol. 51, no. 10, pp. 1668–1674, Oct 2006.

[12] J. Xiong and J. Lam, "Stabilization of linear systems over networks with bounded packet loss," *Automatica*, vol. 43, no. 1, pp. 80 – 87, 2007.

[13] R. Blind and F. Allgower, "A controller design for networked control systems with random delays via the jump linear system approach, which reduces the effects of the delay," in *European Control Conference*, 2009.

[14] W.-J. Ma and V. Gupta, "Input-to-state stability of hybrid systems with receding horizon control in the presence of packet dropouts," *Automatica*, vol. 48, no. 8, pp. 1920 – 1923, 2012.

[15] J. F. Kurose and K. W. Ross, *Computer Networking: A Top-Down Approach (3th Edition)*. Addison Wesley, 2005.

[16] D. Antunes, J. P. Hespanha, and C. Silvestre, "Volterra integral approach to impulsive renewal systems: Application to networked control," *Automatic Control, IEEE Transactions on*, vol. 57, no. 3, pp. 607 –619, march 2012.

[17] ——, "Stochastic hybrid systems with renewal transitions: Moment analysis with application to networked control systems with delays," *SIAM J. Control Optim.*, vol. 51, no. 2, pp. 1481,1499, 2013.

[18] S. Asmussen, *Applied Probability and Queues*. Springer, 2003.

[19] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. John Wiley & Sons, 2000.

[20] H. Kushner and L. Tobias, "On the stability of randomly sampled systems," *Automatic Control, IEEE Transactions on*, vol. 14, no. 4, pp. 319–324, 1969.

[21] R. Agniel and E. Jury, "Almost sure boundedness of randomly sampled systems," *SIAM Journal on Control*, vol. 9, no. 3, pp. 372–384, 1971. [Online]. Available: http://epubs.siam.org/doi/abs/10.1137/0309027

[22] W. L. de Koning, "Digital Optimal Control Systems with Stochastic Parameters," Ph.D. dissertation, TU Delft, 1984.

[23] W. De Koning and L. Van Willigenburg, "Randomized digital optimal control," in *Non Uniform Sampling Theory and Practice*. New York: Kluwer Acadamic/Plenum Publishers, 2001.

[24] F. R. P. Safaei, K. Roh, S. R. Proulx, and J. P. Hespanha, "Quadratic control of stochastic hybrid systems with renewal transitions," Dec. 2013, submitted to journal publications.

[25] W. Michiels and S.-I. Niculescu, *Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach*. SIAM Advances in Design and Control, 2007.

[26] G. Gripenberg, S. O. Londen, and O. Staffans, *Volterra Integral and Functional Equations*. Cambridge University Press, 1990.

[27] M. Ogura and C. F. Martin, "Stability analysis of positive semi-markovian jump linear systems with state resets," *SIAM Journal on Control and Optimization*, accepted arXiv:0706.1234 [math.OC].

[28] Y. Fang, K. A. Loparo, and X. Feng, "Stability of discrete time jump linear systems," *Journal of Mathematical Systems, Estimation and Control*, vol. 5, no. 3, p. 275321, 1995.

[29] A. Rantzer, "Distributed control of positive systems," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, 2011, pp. 6608–6611.