Event-triggered quantized control for input-to-state stabilization of linear systems with distributed output sensors

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Abstract—We study output-based stabilization of linear time-invariant (LTI) systems affected by unknown external disturbances. The plant outputs are measured by a collection of distributed sensors, which transmit their feedback information to the controller in an asynchronous fashion over different digital communication channels. Before transmission of measurements is possible quantization is needed, which is carried out by means of dynamic quantizers. To save valuable communication resources, the transmission instants of each sensor are determined by event-triggering mechanisms that only depend on locally available information. We propose a systematic methodology for the joint design of the (distributed) dynamic quantizers and the event-triggering mechanisms ensuring an input-to-state stability (ISS) property of a size-adjustable set around the origin. Moreover, the proposed approach prevents the occurrence of Zeno behaviour on the transmission instants and on the updates of the quantizer variable thereby guaranteeing that a finite number of data is transmitted within each finite time window. The tradeoff between transmissions and quantization is characterized in terms of the design parameters. The method is feasible for any stabilizable and detectable linear plant. The systematic design procedure and the effectiveness of the approach are illustrated on a numerical example.

I. INTRODUCTION

The increasing popularity of networked control systems (NCS) has motivated an extensive research effort in the last two decades. In NCS, the sensors, the controllers, and the actuators interact with each other over shared communication channels. This configuration offers several advantages compared to dedicated point-to-point connections in terms of increased flexibility, lower cost, and ease of maintenance. However, the communication resources of the network are often limited, which induces new challenges on the design of control systems [1]–[3]. In this context, event-triggered control (ETC) schemes have been proposed in the literature as an alternative to traditional time-triggered platforms. The idea of ETC is to generate transmission events based on locally available output measurements of the system instead of purely on time as in most traditional digital control setups. In this way, unnecessary access to the network can be prevented, leading to more efficient usage of the communication resources, see, e.g., [4]–[6] and the references therein. One of the main difficulties in the synthesis of event-triggering conditions is to guarantee appropriate stability/performance properties while preventing the occurrence of Zeno (an infinite number of transmissions in finite time); certainly when only the plant output is available for feedback instead of the full state [7] and/or when the control system is subject to exogenous inputs [8].

Besides reducing the amount of transmissions over the network, quantization is another important and challenging aspect in NCS [9]–[12]. The use of quantization is unavoidable due to the digital nature of the communication channel and the fact that only a finite amount of data can be transmitted over the network. Quantization requires a careful handling as well since the closed-loop stability may no longer be guaranteed when state or output measurements are quantized with insufficient number of quantization regions, see, e.g., [13] and the references therein. Most existing techniques reported in the literature are developed for static quantizers in which the quantizer range and the quantizer error bound are fixed. Therefore, to guarantee that the feedback information remains within range of the static quantizer, i.e., to make sure that the quantizer does not saturate, it is often assumed in the analysis of static quantizers that the quantizer range is infinite [12], [14]. This requirement is impractical due to the finite size of the transmitted data packages. To overcome this requirement, the authors of [15] have proposed to dynamically adjust the quantizer range and the quantizer error bound according to the available feedback information, which leads to so-called dynamic quantizers. To that end, a zoom variable is used to either increase the quantizer range to avoid saturation (referred to as the zoom-out stage) or decrease the quantizer range to extract more precise information (referred to as the zoom-in stage). As such, with dynamic quantizers, quantizer saturation can be avoided while using only a small number of quantization regions (and thus less number of bits need to be communicated), which show their advantages over static quantizers. However, the use of dynamic quantizers also introduce various design challenges. For instance, since the zoom actions are state-dependent, the accumulation of zoom instants need to be avoided. Moreover, chattering between the zoom-in and the zoom-out stages should be prevented, and the zoom variable should remain bounded, see also, e.g., [16].

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This work was supported by the Innoveration Research Incentives Scheme under the VICI grant “Wireless control systems: A new frontier in automation” (No. 11382), and research programme “Integrated design approach for safety-critical real-time automotive systems” (No. 12698), which is (partly) financed by the Netherlands Organisation for Scientific Research (NWO).
In this paper, we consider the joint design of event-triggering conditions and dynamic quantizers for the purpose of the robust stabilization of linear time-invariant (LTI) systems. The plant may be affected by external disturbances and, as in many applications, only the output of the plant can be measured and not the full state. These output measurements are assumed to be distributed, i.e., they are collected by multiple network nodes and are asynchronously transmitted over different channels. Each of these network nodes employs an event-triggering condition, which only depends on locally available information, to decide when to transmit measurement data. It is important to emphasize that when event-triggering and dynamic quantization for encoding the measured values are considered in NCS, the combined analysis becomes more complicated since the design of the event-triggering mechanism and the quantizer are directly coupled. For instance, the sampling-induced error of the feedback information is in general not reset to zero at each transmission instant, as is common in event-triggering results, e.g., [5], [17], due to the effect of quantization [9]–[11]. This issue will have negative impact on the closed-loop stability if not handled properly in the joint design. The handling of this behaviour is far from trivial. Moreover, it is desirable in practice that the combined event-triggering mechanism and the dynamic quantizer for each node are designed such that, at each transmission instant, the transmitted information is more accurate with respect to the current output measurement than the information already available at the receiving node. Obviously, the latter is important to avoid redundant usage of the network. Finally, we deal with this delicate co-design problem in a very general context with distributed output measurements with asynchronous transmissions, which requires very careful handling.

Our main contribution is the joint and systematic design of event-triggered controllers and dynamic quantizers for a general NCS setup, and we provide an approach that is feasible for any stabilizable and detectable LTI plant. To ensure that the proposed strategy is applicable in practice, the joint event-triggering and dynamic quantization design achieves the following properties:

(I) an input-to-state stability property is guaranteed for a (size-adjustable) bounded set around the origin with respect to the external disturbances and Zeno behaviour is excluded. The size of the bounded set can be made arbitrarily small by selecting the tuning parameters in the design appropriately, hence, a global practical ISS property is achieved;

(II) the size of the data packages that are transmitted is bounded (which is the main goal of quantization) so that at each transmission only a finite number of bits have to be communicated. This size strongly depends on the transmission instants, as we will reveal in our detailed analysis.

In addition to these two properties, we also show that:

(III) at each transmission event, the output measurement is within the quantizer range;

(IV) the transmitted output measurement is more accurate than the information already available at the receiving node based on the last transmission instant. Note that in conventional event-triggering setups, this is already given for free since the sampling induced error is reset to zero at each transmission instant, which is not the case in quantized event-triggered control systems and is therefore important to handle explicitly to avoid redundant transmissions.

Despite the practical importance of the addressed problem, only a few results in the literature have investigated this topic [18]–[24]. The techniques of [18]–[21] are dedicated to the case of state feedback control. The extension of these results to the case of output feedback by modifying the triggering condition is far from trivial as Zeno behaviour is likely to occur in this case, see, e.g., [7], [25], [26]. Moreover, the results of [23], [24] are developed for discrete-time plant models in which the triggering condition is only verified at discrete time instants (hereby, not having to consider the Zeno phenomenon) but not continuously verified as we consider in our approach. To the best of our knowledge, only the paper [22] is applicable for the case of output feedback control of continuous-time plant model. However, [22] does not consider the presence of disturbances, which on the one hand is crucial for practical applications while on the other hand it forms a tremendous technical challenge in event-triggered control, as also highlighted in [8]. Furthermore, the practical aspects that we consider in (I)-(IV) have not been studied in the previously mentioned works. As said, this is the first work on the design of input-to-state stabilizing event-triggered controllers with dynamic quantization of the output feedback information that deals with all the previously mentioned issues in (I)-(IV). In addition, we handle the implementation scenario where the plant outputs are distributed and transmitted in an asynchronous fashion.

The event-triggering mechanism that we construct is inspired by [5], [25], [27] (in which no quantization was considered). Before being sent to the controller, the sensor measurement is quantized by means of a dynamic quantizer associated to its node in order to send a finite number of bits over the digital channel. To prevent the accumulation of zoom actions, the quantizer is only allowed to update its range (and consequently its error bound) at the transmission instants of the respective node. To be more specific, the zoom in/out actions by the quantizer will be performed before the data is being sent over the network.

As already mentioned, the design procedure is feasible for any stabilizable/detectable LTI plant (and any stabilizing output-based LTI controller) and the required design steps (and to be used design conditions) are provided leading to a systematic design procedure. The solution of linear matrix inequality (LMI) is needed in order to select the tuning parameters in the event-triggered mechanism and the dynamic quantizer combination leading to the desired ISS result and size of the ISS set. Interestingly, the proposed design strategy reveals the intuitive tradeoff between the amount of transmissions and the number of quantization levels (and thereby the size of each transmitted data package). The effectiveness of the approach is illustrated on a numerical example.
A preliminary version of this work has been reported in [28]. Compared to [28], in this paper we handle a much more general scenario. Indeed, [28] only treats the case where the plant outputs are sent over a single network in a centralized fashion. Additionally, in this paper, we provide more insights on the problem and all technical proofs are included.

II. PRELIMINARIES

Let $\mathbb{R} := (\mathbb{R} \cup \{\infty\}, \leq, \geq, +, -)$ be the real number system. For a countable set $S$, the symbol $\times$ denotes its cardinality. A continuous function $\gamma : \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$ is of class $C$ if it is zero at zero and strictly increasing. It is of class $C_\infty$ if, in addition, $\gamma(s) \to \infty$ as $s \to \infty$. A continuous function $\gamma : \mathbb{R}_{\geq}^s \to \mathbb{R}_{\geq}^s$ is of class $C_\infty$ if for each fixed $t \in \mathbb{R}_{\geq}$, $\gamma(s, t)$ is of class $C_\infty$, and for each fixed $s \in \mathbb{R}_{\geq}$, $\gamma(s, \cdot)$ is nonincreasing and satisfies $\lim_{t \to \infty} \gamma(s, t) = 0$. A function $V : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous if for each $x \in \mathcal{X}$, there exists a neighborhood $U_x$ and a constant $M > 0$ such that $|V(y) - V(x)| \leq M|y - x|$ for all $y, z \in U_x$.

We define the minimum and maximum eigenvalues of the real symmetric square matrix $A$ as $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively. We write $A^T$ to denote the transpose of $A$, and $I_n$ stands for the identity matrix of dimension $n$. For a partitioned matrix, the symbol $\ast$ stands for symmetric blocks, e.g., $[A \ B]$ means $[A \ B]^T = [A^T \ B^T]$. We denote by $0_n$ and $I_n$ the vectors in $\mathbb{R}^n$ whose all elements are $0$ or $1$, respectively. We write $(x, y) \in \mathbb{R}^{n_x \times n_y}$ to represent the vector $[x^T, y^T]^T$ for $x \in \mathbb{R}^{n_x}$ and $y \in \mathbb{R}^{n_y}$. For a vector $x \in \mathbb{R}^{n_x}$, we denote by $|x| = \sqrt{x^T x}$ its Euclidean norm and, for a matrix $A \in \mathbb{R}^{n \times m}$, $|A| := \sqrt{\lambda_{\max}(A^T A)}$. Given a set $A \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the distance of $x$ to $A$ is defined as $|x|_A := \inf_{y \in A} |x - y|$. The symbol $\wedge$ is used to represent logical conjunction of two conditions. We use the following regularized ceiling function, for $x \in \mathbb{R}$

$$
|x| := \begin{cases} 
\min\{k \in \mathbb{Z} : k > x\}, & x \notin \mathbb{Z} \\
\{x, x + 1\}, & x \in \mathbb{Z}.
\end{cases}
$$

(1)

Note that this (set-valued) regularized ceiling function is outer semicontinuous in the sense that for each $x \in \mathbb{R}$, each sequence of points $x_j \in \mathbb{R}$ that converge to $x$, and each sequence of points $y_j \in \mathbb{R}$ that converge to $y$, $y \in [x]$. We consider hybrid systems of the following form [29], [30]

$$
\dot{x} \in F(x, w), \quad x \in C, \quad x^+ \in G(x) \quad (x \in D),
$$

(2)

where $x \in \mathbb{R}^n$, is the state, $w \in \mathbb{R}^n_w$ is an exogenous input, $C$ is the flow set, $F$ is the flow map, $D$ is the jump set and $G$ is the jump map. We assume that the vector field $F$ is continuous and $G$ is outer semicontinuous and locally bounded with respect to $D$, and the sets $C$ and $D$ are assumed to be closed, ensure that the hybrid model (2) satisfies the basic regularity conditions, see Section 6.2 in [29]. Solutions to system (2) are defined on hybrid time domains. We call a subset $E \subset \mathbb{R}_{\geq} \times \mathbb{N}$ a compact hybrid time domain if $E = \bigcup_{t_j = 0}^{T} \{t_j, t_{j+1}, \ldots\}$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \ldots \leq t_T$ and it is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. A hybrid signal $w : \mathbb{R}_{\geq} \to \mathbb{R}^{n_w}$ is called a hybrid input if $w(t, .)$ is measurable and locally essentially bounded for each $t$. A hybrid signal $x : \mathbb{R}_{\geq} \to \mathbb{R}^{n_x}$ is called a hybrid arc if $x(t, j)$ is locally absolutely continuous for each $t$. A hybrid arc is $x : \mathbb{R}_{\geq} \to \mathbb{R}^{n_x}$ and a hybrid input $w : \mathbb{R}_{\geq} \to \mathbb{R}^{n_w}$ form a solution pair $(x, w)$ to system (2) if $w(t, 0) = 0 \in \mathcal{C} \cup \mathcal{D}$, and:

(i) for all $j \in \mathbb{N}$, and almost all $t$ such that $(t, j) \in dom x$, $(x(t, j), z(t, j)) \in F(x(t, j), w(t, j))$;

(ii) for all $(t, j) \in dom x$ such that $(t, j + 1) \in dom x$, $(x(t, j), \mathcal{D})$ and $(x(t, j + 1) \in G(x(t, j)))$.

A solution pair $(x, w)$ to system (2) is nontrivial if dom $x$ contains at least two points, maximal if it cannot be extended, it is complete if its domain, dom $x$, is unbounded, it is Zeno if it is complete and sup$_{dom x < \infty}$, where sup$_{dom x := sup \{t \in \mathbb{R}_{\geq} : \exists j \in \mathbb{N}_0 \text{ such that } (t, j) \in dom x \}$, and it is $t$-complete if dom $x$ is unbounded in the $t$-direction, i.e., sup$_{dom x = \infty}$.

We use the following definition of $\mathcal{L}_\infty$-norm for hybrid signals [30], [31].

Definition 1. For a hybrid signal $w$, with domain $dom w \subset \mathbb{R}_{\geq} \times \mathbb{N}$, and a scalar $T \in \mathbb{R}_{\geq}$, the $T$-truncated $\mathcal{L}_\infty$-norm is given by

$$
\|w[T]\|_\infty := \sup_{j \in \mathbb{N}_0} \left\{ \text{ess sup}_{t \in \mathbb{R}_{\geq}} |w(t, j)| \right\}.
$$

(3)

The $\mathcal{L}_\infty$-norm of $w$ is given by

$$
\|w\|_\infty := \lim_{T \to \infty} \|w[T]\|_\infty,
$$

(4)

where $T^* := sup \{t + j : (t, j) \in dom w \}$. Moreover, we say that $w \in \mathcal{L}_\infty$ whenever the above limit exists and is finite.

We adopt the following ISS notion for hybrid systems [30].

Definition 2. Consider the hybrid system (2), a set $A \subset \mathbb{R}^{n_x}$ and a set $\mathcal{X}_0 \subset \mathbb{R}^{n_x}$. The set $A$ is input-to-state stable (ISS) w.r.t. $w$ and initial state set $\mathcal{X}_0$ if there exist $\beta \in C_\infty$ and $\psi \in C_\infty$ such that, for each $(x, 0) \in \mathcal{X}_0$ and $w \in \mathcal{L}_\infty$, each maximal solution pair $(x, w)$ is $t$-complete and satisfies for all $(t, j) \in dom x$:

$$
|x(t, j)|_A \leq \max \{\beta(|x(0, 0)|_A, t, j), \psi(|w|_\infty)\}.
$$

(5)

\[\square\]

III. PROBLEM FORMULATION

Consider the continuous-time plant model

$$
x_\mathcal{P} = A_\mathcal{P} x_\mathcal{P} + B_\mathcal{P} u + E_\mathcal{P} w, \quad y = C_\mathcal{P} x_\mathcal{P},
$$

(6)

where $x_\mathcal{P} \in \mathbb{R}^{n_\mathcal{P}}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is unknown plant disturbance, and $y \in \mathbb{R}^{n_y}$ is the measured output. The disturbance $w$ is assumed to

\[\footnote{In general, t-completeness is not required in the ISS property for hybrid systems as mentioned in [30, Remark 2.2]. However, in the context of NCSs, it is desired that all solutions are t-complete and is therefore explicitly required in this definition.}\]
be Lebesgue measurable and locally bounded. The plant is stabilized by a dynamic controller of the form
\[ \dot{x}_c = A_c x_c + B_c y_{q,i}, \quad u = C_c x_c + D_c y_{q,i}, \quad (7) \]
where \( x_c \in \mathbb{R}^{n_c} \) is the controller state and \( \hat{y}_{q,i} \in \mathbb{R}^{n_{q,i}} \) denotes the most recent quantized output measurement available at the controller, see Fig. 1. The controller (7) is designed by an emulation approach in the sense that we assume that the closed-loop system given by (6) and (7) is stable when the effects of both the quantization and the network are absent, i.e., when \( \hat{y}_q = y \).

**A. Setup description**

We consider the scenario where the controller is directly connected to the plant while the output measurement is transmitted to the controller over a digital channel. This control architecture has many practical applications in control systems based on (wireless) sensor networks such as, e.g., in heat, ventilation and air-conditioning control systems [32] and in vehicle platoons [3, Chapter 3]. In particular, we assume that the plant has \( l \) distributed output measurements \( y_1, y_2, \ldots, y_l \), which are measured by \( l \) distributed sensors. The sensors communicate with the controller at discrete time instants \( t^i_k, \ k \in \mathbb{N}, i \in \{1, 2, \ldots, l\} \). At any node \( i \in \{1, 2, \ldots, l\} \), the measured output \( y_i \in \mathbb{R}^{n_{y_i}} \) is collected, quantized, encoded and the resulting encrypted data is sent over the communication channel, see Fig. 1. This encryption is required to make sure that at each transmission, only a limited number of bits is sent.

**B. Event-triggering mechanism**

The sequence of transmission instants of each node \( i \) is generated by an independent event-triggering condition in the sense that the event-triggering condition only depends on locally available information at node \( i \). Each triggering mechanism determines the next transmission instant \( t^i_k, k \in \mathbb{N}, i \in \{1, 2, \ldots, l\} \) based on the actual values of the output measurement \( y_i \) of node \( i \) and the most recent transmitted (quantized) value \( \hat{y}_{q,i} \). The triggering mechanism at each node \( i \in \{1, 2, \ldots, l\} \) is dynamic, in the sense of [5, 25, 27], and takes the following form
\[ t^i_k+1 = \inf\{t > t^i_k + T_i \mid \eta_i(t) = 0\}, \quad (8) \]
where \( t^i_0 = 0, T_i > 0 \) and \( \eta_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is a strictly positive lower bound on the inter-transmission times of the output \( y_i \) that we enforce to prevent the occurrence of Zeno with respect to transmission events at node \( i \). The variable \( \eta_i, i \in \{1, 2, \ldots, l\} \), is the solution to the dynamical system
\[ \eta_i \in \Psi_i(\alpha_i) \quad t \in (t^i_k, t^i_{k+1}), \quad \eta_i(t) = \eta_i(\alpha_i), \quad (9) \]
where \( \alpha_i \in \mathbb{R}^{n_{y_i}} \) represents locally available information at the event-triggering mechanism. The time constant \( T_i \) and the functions \( \Psi_i \) and \( \eta_i \) are to be designed and will be specified in Section V.

**C. Dynamic quantization**

As mentioned before, at each transmission instant \( t^i_k, k \in \mathbb{N}, i \in \{1, 2, \ldots, l\} \), the value of \( y_i \) is quantized before being sent over the network. A quantizer is essentially a piecewise constant function \( q_i : \mathbb{R}^{n_{y_i}} \to Q_i \subset \mathbb{R}^{n_{y_i}} \) with \( Q_i \) a finite subset of \( \mathbb{R}^{n_{y_i}} \). As such, a quantizer induces a partition in \( \mathbb{R}^{n_{y_i}} \) consisting of card(\( Q_i \)) quantization regions described by \( \{z \in \mathbb{R}^{n_{y_i}} : q_i(z) = x\}, x \in Q_i \}. We assume that the function \( q_i \) satisfies the following assumption as proposed in [33], see also [9, 10, 16].

**Assumption 1.** [33] There exist constants \( M_i, \Delta_i, i \in \{1, 2, \ldots, l\} \) such that for all \( y_i \in \mathbb{R}^{n_{y_i}} \), it holds that
\[ |y_i| \leq M_i \quad \Rightarrow \quad |q_i(y_i) - y_i| \leq \Delta_i. \quad (10) \]
Moreover, there exists a constant \( \delta > 0 \) such that for all \( z \in \mathbb{R}^{n_{y_i}} \) with \( |z| \leq \delta \), it holds that \( q_i(z) = 0 \). \( \square \)

This assumption in essence states that the magnitude of the quantization error \( |q_i(y_i) - y_i| \) is upper bounded by \( \Delta_i \), as long as the quantizer is not saturated, i.e., the output measurement \( y_i \) is within the range of its respective quantizer. Moreover, it states that \( q_i(z) = 0 \) for \( z \) in some neighborhood around the origin (and implies \( 0 \in Q_i \)). Let us remark that Assumption 1 allows the quantizer regions to have arbitrary shapes. In the remainder of the paper, we will refer to \( M_i \) and \( \Delta_i \) as the initial quantizer range and initial error bound of node \( i \in \{1, 2, \ldots, l\} \), respectively.

In this paper, we consider dynamic quantizer functions \( q_{i}^{\mu}(t), \ i \in \{1, 2, \ldots, l\} \), which, as in [15, 16, 33], are defined as
\[ q_{i}^{\mu}(t) := \mu_{i} q_{i} \left( \frac{y_{i}}{\mu_{i}} \right), \quad (11) \]

**Fig. 1. NCS with distributed and quantized output measurements.**

Let \( \hat{y}_q = (\hat{y}_{q,1}, \hat{y}_{q,2}, \ldots, \hat{y}_{q,l}) \), where \( \hat{y}_{q,i}, i \in \{1, 2, \ldots, l\} \), denotes the most recent quantized value of \( y_i \) available at the controller. The value of \( \hat{y}_{q,i} \) is kept constant between two consecutive transmission instants of node \( i \) in a zero-order-hold (ZOH) fashion, i.e., \( \hat{y}_{q,i} = 0 \). We ignore communication and computation delays in this study, although it would be possible to include them by using the techniques in, e.g., [2, 25].
where \( \mu_i \in \mathbb{R}_{\geq 0}, i \in \{1, 2, \ldots, l\} \), are dynamic variables referred to as the zoom variables and where \( q_i \) satisfies Assumption 1 for some \( M_i, \Delta_i > 0 \) and where \( \mu_i > 0 \) is a lower-bound on \( \mu_i \) to be specified. The zoom variables \( \mu_i, i \in \{1, 2, \ldots, l\} \), are used to adjust the quantizer range initially equal to \( M_i > 0 \) and the quantizer error bound initially equal to \( \Delta_i > 0 \) of node \( i \) based on the magnitude of the output measurement \( y_i \). To be more specific, the range and the error bound of the dynamic quantizer as in (11) are given by \( M_i \mu_i \) and \( \Delta_i \mu_i \), respectively. As such, property (10) becomes \( |y_i| \leq \mu_i M_i \Rightarrow |q_i^{\mu_i}(y_i) - y_i| \leq \mu_i \Delta_i \). Note that the number of quantization regions of dynamic quantizers remains constant all the time.

In the context of NCSs, it is of importance that at each transmission instant, before data is actually transmitted, the dynamic quantizer is set such that property (III) and (IV) mentioned in Section I are satisfied. To achieve these two properties, the zoom variable \( \mu_i \) is adapted at transmission instants \( t_k^i, k \in \mathbb{N}, i \in \{1, 2, \ldots, l\} \), according to

\[
\mu_i^+ \in \Theta_i(\hat{y}_i, \mu_i) := \Theta_{\text{in},i}(\hat{y}_i, \mu_i) \Theta_{\text{out},i}(\hat{y}_i, \mu_i) \mu_i
\]

where \( \Theta_{\text{in},i} \), \( \Theta_{\text{out},i} \in \{0, 1\}, \Theta_{\text{out},i} > 1 \), are the zoom-in and zoom-out factors, respectively, and where we omitted the time arguments for sake of compactness. The functions \( \Theta_{\text{in},i}, \Theta_{\text{out},i} : \mathbb{R}^{n_y} \times \mathbb{R} > 0 \rightarrow \mathbb{N} \) determine the number of zoom-in or zoom-out actions that need to be performed at each transmission.

Let us elaborate on the dynamic adjustment strategy of \( \mu_i \), \( i \in \{1, 2, \ldots, l\} \). At any transmission instant \( t_k^i, k \in \mathbb{N}, i \in \{1, 2, \ldots, l\} \), if the magnitude of \( |y_i| \) is close to the range of the quantizer, we increase the value of \( \mu_i \) with the factor \( \sigma_{\text{out},i} \in \Theta_{\text{out},i}(\hat{y}_i, \mu_i) \) with \( \sigma_{\text{out},i} > 1 \) in order to make sure that the transmitted information is within range of the quantizer (which corresponds to property (III) stated in Section I). We refer to this action as the zoom-out event. On the other hand, if \( |y_i| \) is small with respect to the current quantizer error bound, we decrease \( \mu_i \) by means of the zoom-in-factor \( \sigma_{\text{in},i} \in \Theta_{\text{in},i}(\hat{y}_i, \mu_i) \) with \( \sigma_{\text{in},i} \in \{0, 1\} \) such that more precise information is transmitted (which corresponds to property (IV) stated in Section I). We refer to this action as the zoom-in event. See, e.g., [16] for more details on dynamic quantizers.

Let us emphasize that the zoom variable \( \mu_i \) in (12) is only updated at transmission instants \( t_k^i \), \( k \in \mathbb{N}, i \in \{1, 2, \ldots, l\} \) and held constant in between transmissions, i.e., \( \mu_i = 0 \) for \( t \in (t_k^i, t_{k+1}^i) \). Consequently, in each node \( i \in \{1, 2, \ldots, l\}, \) the time in between the zoom events is lower bounded by the minimum inter-transmission time \( T_i \) ensured by the local triggering condition (8).

To be able to successfully reconstruct the broadcast encoded information, the zoom variable \( \mu_i \) of both the encoder and the decoder at any channel should be initialized at the same value and the quantization regions \( Q_i \) and the zoom-factor \( \Theta_{\text{in},i} \) and \( \Theta_{\text{out},i} \) should be known at both the encoder and decoder, see [34], [35], and Remark 1 in [10] for an in-depth discussion on this point. Then, at each update instant \( t_k^i \), we only transmit

\[\hat{y}_i = A_1 x + B_1 e + \epsilon_1 w,\]

where \( A_1 := \begin{bmatrix} A_p + B_p D_p C_p & B_p C_p \end{bmatrix}, B_1 := \begin{bmatrix} B_p D_p \end{bmatrix}, \) and \( \epsilon_1 := \begin{bmatrix} \epsilon_p \end{bmatrix} \). Let the matrix \( C_p \) in (6) be partitioned as \( C_p = \begin{bmatrix} C_{p,1} & \cdots & C_{p,l} \end{bmatrix}^T \) with \( C_{p,i} \in \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \) such that \( y_i := C_{p,i} x p_i \in \mathbb{R}^{n_y}, i \in \{1, 2, \ldots, l\} \). Then, because of the ZOH implementation, the flow dynamics of \( \epsilon_i \) is

\[
\dot{\epsilon}_i = -\hat{y}_i = -C_p \hat{x}_p = A_2 x + B_2 e + \epsilon_2 w.
\]

where \( A_2 := \begin{bmatrix} -C_p A_p + B_p D_p C_p \cr -C_p B_p C_p \end{bmatrix}, B_2 := \begin{bmatrix} -C_p B_p D_p \end{bmatrix}, \) and \( \epsilon_2 := \begin{bmatrix} \epsilon_p \end{bmatrix} \). In view of (16), the flow dynamics of the overall \( e \) is

\[
\dot{e} = A_2 x + B_2 e + \dot{\epsilon}_2 w,
\]
where $A_2 := \begin{bmatrix} A_{21} & \cdots & A_{2l} \\ \vdots & \ddots & \vdots \\ A_{2l} & \cdots & A_{2N} \end{bmatrix}$, $B_2 := \begin{bmatrix} B_{21} \\ \vdots \\ B_{2l} \end{bmatrix}$ and $E_2 := \begin{bmatrix} E_{21} \\ \vdots \\ E_{2l} \end{bmatrix}$.

We introduce auxiliary variables $\tau_i \in \mathbb{R}_{\geq 0}$ and $p_i \in \{0, 1\}$ for $i \in \{1, 2, \ldots, l\}$. The variable $\tau_i$ represents the time elapsed since the last transmission instant of node $i$. It has the dynamics

$$\tau_i = 1 \quad t \in (t_i^0, t_i^{k+1}), \
\tau_i(t_k^i) = 0 \quad \text{for } k \in \mathbb{N}.$$  

(18)

The variable $p_i$ is a boolean that keeps track of whether at the next event, the zoom-variable is updated (before $y_i$ is transmitted). Let $\xi := (x, e, \mu, \tau, \eta, p) \in \mathbb{X}$ with $\mathbb{X} = \mathbb{R}^n \times \mathbb{R}^n \times \{\pm \} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \{0, 1\}$ be the concatenation of the state variables, where $\mu := (\mu_1, \ldots, \mu_l) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_{\geq 0}$, $\eta := (\eta_1, \ldots, \eta_l) \in \mathbb{R}^l_{\geq 0}$, and $p := (p_1, \ldots, p_l) \in \{0, 1\}^l$. Then, in view of (8) and (9), the flow set $C$ and the jump set $D$ are given by

$$C := \{\xi \in \mathbb{X} : p = 0\}, \quad D := \bigcup_{i=1}^l D_i.$$  

(19)

with $D_i := \{\xi \in \mathbb{X} : (\eta_i = 0 \land \tau_i = T_i) \land p_i = 1\}$. Note that the triggering condition related to $\eta_i(t) = 0$, $i \in \{1, 2, \ldots, l\}$, in (8) is embedded in the flow set $C$ via the definition of $\mathbb{X}$ ($\xi \in \mathbb{X}$ implies $\eta_i \geq 0$ for each $i \in \{1, 2, \ldots, l\}$). Given (19), we obtain the hybrid system

$$\xi \in F(\xi, w) := \begin{cases} (A_1 x + B_1 e + E_1 w, 0_1) \\ (A_2 x + B_2 e + E_2 w, 1_1) \\ (\Psi(o), 0_1) \end{cases} \quad \xi \in C$$

$$\xi^+ \in G(\xi) \quad \xi \in D,$$

(20)

where $\Psi(o) := (\Psi_1(o_1), \ldots, \Psi_l(o_l))$ with $o := (y_i, e_i, \tau_i, \eta_i) \in \mathbb{R}^{n_i}$. In view of the update conditions of $\eta_i$ in (9), $\mu_i$ in (12), $e_i$ after (13), $\tau_i$ in (18) and $p_i$ after (18), the jump map is given by $G(\xi) := \bigcup_{i=1}^l G_i(\xi)$, where

$$G_i(\xi) := \begin{cases} G_i^0(\xi) & \text{for } \xi \in D_i \land p_i = 0 \\ G_i^1(\xi) & \text{for } \xi \in D_i \land p_i = 1 \\ \emptyset & \text{for } \xi \notin D_i \end{cases}$$

(21)

with

$$G_i^0(\xi) := \begin{cases} x \\ e \\ \Lambda_i \Theta_i(y_i, \mu_i) + (\bar{I}_i - \bar{A}_i) \mu \\ \tau \\ \eta \\ \Lambda_i \mathbf{1}_l + (\bar{I}_i - \bar{A}_i) p \end{cases}$$

$$G_i^1(\xi) := \begin{cases} x \\ \Lambda_i e_q + (\bar{I}_n - \Lambda_i) e \\ \mu \\ (\bar{I}_i - \Lambda_i) \tau \\ \bar{\Lambda}_i \rho_0(s_1) + (\bar{I}_i - \Lambda_i) \eta \\ (\bar{I}_i - \Lambda_i) p \end{cases},$$

where $\Lambda_i := \text{diag}\{\delta_{i,1}^{\text{in}}, \ldots, \delta_{i,|y_n|}\}, i \in \{1, 2, \ldots, l\}$, $\lambda_i := \text{diag}\{\delta_{i,1}, \ldots, \delta_{i,l}\}$ with $\delta_{ij}$ the Kronecker delta, which takes the value $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$, $\eta_i(e) := (\eta_{i,1}(e_1), \ldots, \eta_{i,l}(e_l))$, $e_q := (e_q, \ldots, e_q)$, the function $\Theta_i : \mathbb{R}^{n_i} \times \mathbb{R}^n \times \mathbb{R}^n \Rightarrow \mathbb{R}^{n_i}$ as in (12) with the functions $\bar{\kappa}_{\text{in},i}, \text{out},i : \mathbb{R}^{n_i} \times \mathbb{R}^n \Rightarrow \mathbb{R}$ to be specified.

System (20) flows on $C$ as long as the triggering conditions are not satisfied and $p = 0$. When $\xi \in D_i$, $D_i$ is updated with $p_i = 0$, $\xi$ can jump according to $\xi^+ \in G_i^0(\xi)$ corresponding to an update of the quantizer settings. To be more specific, when the state jumps according to $\xi^+ \in G_i^0(\xi)$, the quantizer variable $\mu_i$, $i \in \{1, 2, \ldots, l\}$, is updated and $p_i$ is changed to 1. Consequently, $\xi^+$ lies into the jump set $D_i$ with $p_i = 1$. Since the system is not allowed to flow when $p_i = 1$ for some $i \in \{1, 2, \ldots, l\}$, since, in view of (19), $\xi \notin C$ when $p_i = 1$ for some $i \in \{1, 2, \ldots, l\}$, a transmission is generated and $p_i$ is reset to 0, in view of the jump map $G_i^1(\xi)$. As such, the boolean variable $p_i$ ensures that at transmission instant $t_i^e$, the quantizer variable $\mu_i$ is updated before transmitting $q_i^0(y_i)$, which is fundamental for realizing properties (ii) and (iii) as mentioned in the introduction.

Let us remark that the hybrid system $H$ described by (19) and (20), is well-posed, see also, [29, Chapter 6].

**Remark 1.** We note that the dynamic quantization strategy in (12), (20) and (22) involves some differences compared to related techniques in the context of quantized control systems (QCS) [16], [36], [39]. First, in the proposed scheme, the zoom actions are carried out based on the past (true) values of $y_i$ and not based on the quantized feedback information $q_i^0(y_i)$ as in [16], [36], [39] for instance. To that end, we rely on the assumption that the true values of $y_i$ can be
accessed by the corresponding encoder at node \(i\). Indeed, this requirement is relevant if the communication network is the reason of quantization, such as in e.g., [22], [40]-[42] and as we consider in this study, but not the sensor, see [39] for further discussion on this point. Second, unlike [16], [39], we do not reset the control input to zero during a zoom-out event, which allows to avoid large overshoot during the zoom-out event. □

V. MAIN RESULT

A. Design conditions for the event-triggering mechanism

We make the following design condition regarding system (20) to construct the event-triggering mechanism appropriately. This is always feasible for any plant-controller combination (6), (7), which is in absence of the network \((e = 0)\) and disturbances \((w = 0)\) is asymptotically stable (i.e., \(A_1\) is Hurwitz). Clearly, the design of such a controller is always possible if the plant (6) is stabilizable and detectable.

Condition 1. Consider system (20). There exist a positive definite symmetric real matrix \(P\), real numbers \(\varepsilon_x, \varepsilon_w > 0\) and \(\varepsilon_y, \gamma_i > 0\) for \(i \in \{1, 2, \ldots, l\}\), such that

\[
\begin{aligned}
\Sigma & \leq 0, \\
B_1^T P + B_2 B_2^T A_2 - \Gamma^2 + B_2^T B_2 & \leq 0, \\
\varepsilon_x^T P + \varepsilon_x^T A_2 & \leq \varepsilon_w^T B_2, \\
\varepsilon_x^T P + \varepsilon_x^T A_2 & \leq \varepsilon_x^T B_2,
\end{aligned}
\]  

(23)

with \(\Sigma := \begin{bmatrix} A_1^T P + P A_1 + \varepsilon_x I_{n_x} + A_2^T A_2 + C_p^T Y C_p \end{bmatrix}, \Gamma := \text{diag}\{\gamma_1 I_{n_y}, \ldots, \gamma_l I_{n_y}\}, \) and \(B_2 = \begin{bmatrix} B_{21}^T & \ldots & B_{2l}^T \end{bmatrix}^T\). The LMI condition (23) in essence establishes an \(L_2\)-gain stability property for the system \(\dot{x} = A x + B_1 e + C_1 w\) from \((e, w)\) to \((A_2 x + B_2 e + C_2 w, y)\). Indeed, if we define \(V(x) := x^T P x\) for all \(x \in \mathbb{R}^n\), then in view of (20) and the definitions of \(T, \Gamma, A\), the feasibility of (23) is equivalent to, for all \((x, e, w) \in \mathbb{R}^{n_x+n_e+n_w}\),

\[
(\nabla V(x), A x + B_1 e + C_1 w) \leq -\varepsilon_x |x|^2 - \sum_{i=1}^{l} \varepsilon_y |y_i|^2,
\]

(24)

\[-\sum_{i=1}^{l} |A_{i2} x + B_{i2} e + C_2 w|^2 - \sum_{i=1}^{l} \gamma_i^2 |e_i|^2 + \varepsilon_w |w|^2.\]

This property is needed to design the enforced minimum time \(T_{i}, i \in \{1, 2, \ldots, l\}\), on the inter-transmission times of each node and to design the dynamics of \(\eta_i, i \in \{1, 2, \ldots, l\}\), in (19) such that closed-loop stability (in an appropriate sense) is guaranteed.

As already shortly indicated above, inequality (24) can be always satisfied if the matrix \(A_1\) is Hurwitz, i.e., if the closed-loop system (6), (7) is stable in the absence of the communication network, by selecting the matrix \(P\) and the parameters \(\gamma_i, \varepsilon_w\) sufficiently large. As a consequence, since LMI (23) is equivalent to inequality (24), we deduce that LMI (23) can be made feasible for any stabilizable and detectable plant model (6) by taking any arbitrary stabilizing controller (7). Furthermore, since the required conditions are in essence Lyapunov-based conditions, this will be helpful in extending the application of our technique on nonlinear systems, which is another advantage of the approach.

To sum up, we first formulate the closed-loop system in the hybrid framework (20) and then we solve LMI (23). By doing so, we obtain the matrix \(P\) and the parameters \(\varepsilon_x, \varepsilon_w > 0\) and \(\varepsilon_y, \gamma_i > 0\) for \(i \in \{1, 2, \ldots, l\}\), which are needed to design the event-triggering mechanism and the enforced lower bound on the inter-transmission times, as explained in the next section.

The dynamics of the functions \(\eta_i, i \in \{1, 2, \ldots, l\}\), in (9) are defined by the functions \(\Psi_i\) and \(\eta_0, i\), inspired by [25], where quantization was not considered, and are given by

\[
\Psi_i(\tau_i) := \varepsilon_y \max \{ |y_i|^2, \Delta_{\eta, i}^2 \} - (1 - \omega_i(\tau_i)) \gamma_i |e_i|^2 - \theta_i \eta_i, \]

\[
\eta_0(i, \tau_i) := \gamma_i (\lambda_i - \lambda_i) |e_i|^2,
\]

(25)

where

\[
\omega_i(\tau_i) := \begin{cases}
1 \text{, for } \tau_i \in [0, T_i) \\
0 \text{, for } \tau_i = T_i \\
0 \text{, for } \tau_i > T_i,
\end{cases}
\]

(26)

and where \(\eta = (\eta_1, \eta_2, \tau_1, \eta_l, \lambda_i, \gamma_i)\), as before, and the constants \(\varepsilon_x, \gamma_i\) as in Condition 1. The constants \(\Delta_{\eta, i} > 0\) and \(\theta_i > 0\) can be arbitrarily chosen (sufficiently small). The parameter \(\gamma_i\) is given by \(\gamma_i := \gamma_i^2 + \gamma_i^2 \lambda_i^2 + 2 \gamma_i \lambda_i L_i\), with \(\lambda_i \in (0, 1)\), \(\lambda_i \in \{\lambda_i, \lambda_i^{-1}\}\), \(L_i := L_i + \nu_i\) for any \(\nu_i > 0\) sufficiently small and \(L_i := \{|b_{2l,1}|, \ldots, |b_{2l,l}|\}\) such that the constants \(\Delta_{\eta, i} > 0\), \(\eta_i > 0\), \(\nu_i > 0\) can be tuned to adjust the upper bound on the size of the set around the origin for which the ISS stability is guaranteed, as shown in Section V-C. The time constant \(T_i\) of any node \(i \in \{1, 2, \ldots, l\}\), is taken such that \(T_i = T_i(\lambda_i, \lambda_i, \gamma_i, \tilde{L}_i)\), where

\[
(\tilde{L}_i, \lambda_i, \gamma_i, \tilde{L}_i) := \begin{cases}
\frac{1}{\nu_i \tau_i} \arctan\left( \frac{\tau_i (1 - \lambda_i \tilde{L}_i)}{2 \lambda_i (1 - \lambda_i)} \right), \quad \gamma_i > \tilde{L}_i & \\
\frac{1}{\nu_i \tau_i} \arctan\left( \frac{\tau_i (1 - \lambda_i \tilde{L}_i)}{2 \lambda_i (1 - \lambda_i)} \right), \quad \gamma_i = \tilde{L}_i & \\
\frac{1}{\nu_i \tau_i} \arctan\left( \frac{\tau_i (1 - \lambda_i \tilde{L}_i)}{2 \lambda_i (1 - \lambda_i)} \right), \quad \gamma_i < \tilde{L}_i &
\end{cases}
\]

(27)

with \(r_i := \sqrt{(\frac{\tau_i}{\nu_i})^2 + 1}\). When \(\tilde{L}_i = \lambda_i\), the time \(T_i(\lambda_i, \gamma_i, \tilde{L}_i)\) corresponds to the maximally allowable transmission interval (MATI) of time-triggered controllers [43] for NCSs without quantization. Let us remark that \(\lambda_i\) also has an important role in the design of the quantizer, as we will discuss in later in Section V-D. The time-constant \(T_i\) as given in (27) is derived as the time it takes for the function \(\phi_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}\) to decrease from \(\phi_i(0) = \lambda_i^{-1}\) to \(\phi_i(T_i) = \lambda_i\), where \(\phi_i\) satisfies, see also [25], [43],

\[
\frac{d\phi_i}{dT_i} = -2 \tilde{L}_i \phi_i(\tau_i) - \gamma_i (\phi_i(\tau_i) + 1).
\]

In (25), we use \(\varepsilon_y \max \{ |y_i|^2, \Delta_{\eta, i}^2 \} - (1 - \omega_i(\tau_i)) \gamma_i |e_i|^2 - \theta_i \eta_i\) to denote the set \(\varepsilon_y \max \{ |y_i|^2, \Delta_{\eta, i}^2 \} - (1 - \omega_i(\tau_i)) \gamma_i |e_i|^2 - \theta_i \eta_i | \omega \in \omega_i(\tau_i)).\)
Note the flow and the jump dynamics of the variables \( \eta_i, \phi_i \) involve important differences compared to those defined in [25] to cope with the quantization effect.

**Remark 2.** We observe that, in view of (27) and its interpretation in terms of (28), when \( \lambda_i \in [\lambda, \lambda_i^{-1}] \) is increased, the guaranteed minimum time \( T_i \) between two transmission instants of node \( i \) will be reduced. However, by increasing \( \lambda_i \), the value of \( y_0 \) in (25) will increase. Consequently, this may lead to increase the time it takes for \( \eta \) to decrease to 0, i.e., it may enlarge the inter-transmission times. Hence, the tuning of \( \lambda_i \) may generate a tradeoff between the guaranteed minimum inter-transmission time \( T_i \) and the average inter-transmission times.

**B. Design conditions for the dynamic quantizer**

In this subsection, we specify how to design the parameters \( \Delta_i, M_i, \Omega_{in,i}, \Omega_{out,i} \) and the functions \( \kappa_{in,i}, \kappa_{out,i} \) for \( i \in \{1, 2, \ldots, l \} \), see (12). For each node \( i \in \{1, 2, \ldots, l \} \), we design the initial quantizer range \( M_i \), the initial error bound \( \Delta_i \), the zoom-in parameters \( \Omega_{in,i} \in (0, 1) \), and the zoom-out parameters \( \Omega_{out,i} > 1 \) such that

\[
\frac{M_i}{\Delta_i} \geq \left( \kappa_i + \frac{2\sqrt{\gamma_i}}{\sqrt{y_0 \Omega_{in,i}}} \right) \quad (29)
\]

\[
\kappa_i > \max \left\{ \frac{1, (\Omega_{in,i}, \Omega_{out,i} - 1) M_i + \Delta_i}{\Omega_{in,i}, \Omega_{out,i} - \Delta_i} \right\} \quad (30)
\]

Moreover, the functions \( \kappa_{in,i} \) and \( \kappa_{out,i} \) for \( i \in \{1, 2, \ldots, l \} \), are, for \( y_i \in \mathbb{R}^{n_v_i} \) and \( \mu_i \in \mathbb{R}_{>0} \), given by

\[
\kappa_{in,i} (y_i, \mu_i) := \left\lceil \log \left( \frac{-\zeta \log \left( \frac{\max \{ |y_i|, \Delta_i \} / \Omega_{in,i} \} \right)}{\Delta_i} \right) \right\rceil
\]

\[
\kappa_{out,i} (y_i, \mu_i) := \left\lceil \log \left( \frac{-\zeta \log \left( |y_i| / \Omega_{out,i} \right)}{\Delta_i} \right) \right\rceil \quad (31)
\]

where

\[
\ell_{in,i} := \Omega_{in,i} (M_i - \kappa_i \Delta_i), \quad \ell_{out,i} := M_i - \Delta_i \quad (32)
\]

with \( \kappa_i \) as in (30), \( \Omega_{in,i} \) as in (25) and where the constant \( \zeta \in (0, 1) \) can be chosen arbitrarily.

The functions \( \kappa_{in,i} \) and \( \kappa_{out,i} \) and the constants \( \ell_{in,i} \) and \( \ell_{out,i} \) are specified such that the following properties are satisfied. For all \( y_i \in \mathbb{R}^{n_v_i} \) and \( \mu_i \in \mathbb{R}_{>0} \), \( i \in \{1, 2, \ldots, l \} \), we have that

(a) \( \kappa_{in,i} (y_i, \mu_i), \kappa_{out,i} (y_i, \mu_i) \in \mathbb{N} \)

(b) \( \kappa_{in,i} (y_i, \mu_i) \neq \{0\} \Rightarrow |y_i| < \ell_{out,i} \mu_i \). Moreover, \( \kappa_{out,i} (y_i, \mu_i) = \{0\} \)

(c) \( \kappa_{out,i} (y_i, \mu_i) \neq \{0\} \Rightarrow |y_i| < \ell_{in,i} \mu_i < \max \{|y_i|, \Delta_i\} \). Moreover, \( \kappa_{out,i} (y_i, \mu_i) = \{0\} \).

Moreover, for each \( \mu_i^{+} \in \Omega_{in,i} (y_i, \mu_i) \mu_i \) with \( y_i \in \mathbb{R}^{n_v_i} \) and \( \mu_i \in \mathbb{R}_{>0} \), it holds that

(d) \( \frac{\Omega_{in,i}}{\ell_{in,i}} \max \{|y_i|, \Delta_i\} \leq \mu_i^{+} \leq \frac{\max \{|y_i|, \Delta_i\}}{\ell_{in,i}} \mu_i \)

and, for each \( \mu_i^{-} \in \Omega_{out,i} (y_i, \mu_i) \mu_i \) with \( y_i \in \mathbb{R}^{n_v_i} \) and \( \mu_i \in \mathbb{R}_{>0} \), it holds that

(e) \( \frac{\Omega_{out,i}}{\ell_{out,i}} \leq \mu_i^{-} \leq \frac{\max \{|y_i|, \Delta_i\}}{\ell_{out,i}} \mu_i \)

Property (a) follows from the fact that \( \lfloor -\zeta \rfloor = \{0\} \) for any \( \zeta \in (0, 1) \). Properties (b) and (c) are due to the fact that according to (32) and (30), \( 0 < \ell_{in,i} \mu_i < \ell_{out,i} < M_i \). At last, properties (d) and (e) follow directly from the definitions of \( \kappa_{in,i} (y_i, \mu_i) \) and \( \kappa_{out,i} (y_i, \mu_i) \), respectively. Let us remark that property (c) implies that \( \kappa_{in,i} (y_i, \mu_i) = 0 \) when \( \ell_{in,i} \mu_i < \Delta_i \).

As we will show, this property is important to ensure that the values of \( \kappa_{in,i} (y_i, \mu_i) \) and \( \kappa_{out,i} (y_i, \mu_i) \) remain finite at all time, especially when \( y_i \) crosses zero at any transmission instant \( t_k \). Similar conditions have been used in [16], [33], [36]. Moreover, observe that properties (b) and (c) imply that when the quantizer is updated, either a zoom-in or zoom-out event occurs or neither of them. It is worth mentioning, due to the fact that we only update the zoom variable \( \mu_i \) at transmission instants, that the required number of zoom actions at each transmission instant, which will be transmitted over the network to the decoder, is directly affected by the length of the inter-transmission times since the last transmission instant. In other words, it is expected that the longer the inter-transmission time the more zoom actions need to be performed at the next transmission instant.

**C. Stability result**

In this section, we present the main result. We are going to establish the following properties:

(i) there exists a bounded set that is ISS, and in fact the tuning parameters \( \Delta_{0,i}, \varsigma, \nu_i, \theta_i \) in the algorithm, can be chosen to make this set arbitrarily small;

(ii) the information transmitted over the network is bounded in each bounded time window, which implies the absence of Zeno behavior in transmission times and zoom in/out actions, i.e., the number of bits to be transmitted and the number of transmission instants are finite in each finite time window;

(iii) the output measurement of each sensor remains within the range of its associated dynamic quantizer, i.e., quantizer saturation is avoided;

(iv) the proposed approach prevents the redundant usage of the network due to quantization, in the sense that at each transmission instant it is guaranteed that the information to be transmitted from the sensor side is more accurate that what is available at the controller side.

In what follows, we will provide the corresponding formal results regarding these statements. We first define the following bounded set for which the ISS property will be ensured.

**Definition 3.** We define the following set \( A := \{ \xi \in \mathbb{R} : R(\xi) \leq c \} \), where

\[
R(\xi) := x^T P x + \sum_{i=1}^{l} \left( \gamma_i \phi_i (\tau_i) |e_i|^2 + \eta_i \right), \quad (33)
\]

\[
\tilde{\phi}_i (\tau_i) := \begin{cases} 
\phi_i (\tau_i) & \text{when } \tau < T_i \\
\phi_i (T_i) & \text{when } \tau > T_i 
\end{cases}, \quad (34)
\]

and \( c := \frac{1}{\epsilon \min \{ \gamma_i \Delta_{0,i}^2, 2 \min \{ \nu_i, \min \nu_i \} \}} \) with \( P, \gamma_i, \theta_i, \nu_i \).
as in Condition 1. ∆₀,₁ as in (12) and where ε ∈ (0, 1) can be chosen arbitrarily. □

We are ready to state the main result.

**Theorem 1.** Consider system (20) with the flow and the jump sets as in (19) with Ψ, η₀, specified in (25) and Tᵢ defined in (27). Suppose that Assumption 1 and Condition 1 are satisfied and that the dynamic quantizer is designed as in (29)-(30). Let \( \mathcal{X}_0 := \{x \in \mathbb{R} : x < 0, p_i = 0\} \). Then (i) the set \( \mathcal{A} \) as defined in Definition 3, is input-to-state stable w.r.t. \( \varepsilon \); (ii) for each maximal solution pair \((\xi, w)\) with \((\xi, 0) \in \mathcal{X}_0 \) and \( w \in \mathcal{L}_\infty \), it holds, when \( p_i(t, j) = 1 \), that \( \kappa_{\text{in}, i}(y_i(t, j), \mu_i(t, j)) \) and \( \kappa_{\text{out}, i}(y_i(t, j), \mu_i(t, j)) \) are bounded by \( 0, 1, \ldots, \kappa_{\text{in}, i}^*, \kappa_{\text{out}, i}^* \), for some positive constants \( \kappa_{\text{in}, i}^*, \kappa_{\text{out}, i}^* \).

Property (i) in Theorem 1 implies that an ISS property is guaranteed for the closed-loop system. In particular, in the absence of disturbances, the state trajectory converges to a neighbourhood of the origin whose size depends on \( ∆₀,₁ \).

This is due to the fact that \( µ_i \) does not eventually go to zero since no zoom-in occurs when \( µ_i < \frac{∆₀,₁}{\log \Omega_{\text{in}, i}} \) according to (12) in combination with (31), which is also the case in, e.g., [16], [33]. Note that the set \( \mathcal{A} \) in Definition 3, for which the ISS property is guaranteed, is bounded by the constant \( c > 0 \), which can be adjusted arbitrarily small by tuning the parameters \( ∆₀,₁, \varepsilon, \xi, ρ, \partial_i \) of the event-triggering mechanism and of the dynamic quantizer appropriately. Property (ii) in Theorem 1 shows that at transmissions, the elements of the sets \( \kappa_{\text{in}, i}(y_i, \mu_i) \) and \( \kappa_{\text{out}, i}(y_i, \mu_i), i \in \{1, 2, \ldots, l\} \), are bounded by \( \kappa_{\text{in}, i}^*, \kappa_{\text{out}, i}^* \), respectively. The latter property is important to make sure that the amount of data sent over the network per transmission is bounded. Let us remark that the quantities \( \bar{µ}_i \) and \( \bar{y}_i, \bar{y}_{i, \text{out}, i} \), \( i \in \{1, 2, \ldots, l\} \), represent upper-bounds on \( ∥\mu_i∥_{\infty} \) and \( ∥y_i∥_{\infty} \), respectively. As shown in the proof of Theorem 1, the upper bounds \( \kappa_{\text{in}, i}^*, \kappa_{\text{out}, i}^* \) are given by

\[
\kappa_{\text{in}, i}^* := \max \left( \frac{\log (\Delta_{0,1}/(\Omega_{\text{in}, i} \mu_i))}{\log \Omega_{\text{in}, i}} \right),
\]

\[
\kappa_{\text{out}, i}^* := \max \left( \frac{\log (\bar{y}_i/(\Omega_{\text{out}, i} \mu_i))}{\log \Omega_{\text{out}, i}} \right)
\]

with \( \bar{y}_i := \max \{\kappa_{\text{out}, i}^*, y_i, y_{i, \text{out}, i}(0, 0)\} \), \( \kappa_{\text{in}, i}^* := \max \{\kappa_{\text{in}, i}^*, \bar{y}_{i, \text{out}, i} \} \), and \( \mu_i := \frac{\Omega_{\text{in}, i}/\Delta_{0,1}}{\Omega_{\text{in}, i}} \).

**Corollary 1.** Consider system (20) with the flow and the jump sets as in (19) with \( \Psi, \eta_0 \), specified in (25) and \( T_i \), defined in (27). Suppose that Assumption 1 and Condition 1 are satisfied and that the dynamic quantizer is designed as in (29)-(30). Let \( \mathcal{X}_0 := \{x \in \mathbb{R} : x < 0, p_i = 0\} \). Then, for each maximal solution pair \((\xi, w)\) with \((\xi, 0) \in \mathcal{X}_0 \) and \( w \in \mathcal{L}_\infty \), it holds that, when \( p_i(t, j) = 1 \), (iii) \( |y_i(t, j)| \leq M_{\text{in}, i}(t, j) \); (iv) \( |\bar{y}_i, \bar{y}_{i, \text{out}, i}| > |\bar{y}_i, \bar{y}_{i, \text{out}, i}| \).

Property (iii) in Corollary 1 implies that at each transmission event, the output measurement \( y_i \) is within the range of the associated quantizer. As a consequence, the quantization error is always smaller than or equal to the quantization error bound at each transmission event. Property (iv) in Corollary 1 implies that at each transmission event, i.e., when \( \xi \in D_i \) and \( p_i = 1 \), \( i \in \{1, 2, \ldots, l\} \), the magnitude of the sampling-induced error is larger than the error due to quantization, which implies that the transmitted information is more accurate than the information already available at the corresponding receiving node. As such, property (iv) helps in avoiding redundant usage of the network.

**Remark 3.** We note from (38) that the constant parameter \( ∆_{0,1} > 0 \) acts to provide a lower bound \( \mu_i \) on the zoom variable \( \mu_i, i \in \{1, 2, \ldots, l\} \). This lower bound \( \mu_i \) is still needed even though the zoom variable \( \mu_i \) is only updated at transmission instants, which automatically rules out Zeno behaviour on the zoom actions thanks to the enforced minimum time \( T_i \) between two transmissions. The lower bound \( \mu_i \) is rather introduced to limit the size of the transmitted data-packages, which is the purpose of quantization. Otherwise, if the state is in equilibrium and \( \mu_i \) is infinitesimal, small disturbances on the plant may require an infinite number of zoom-out actions, which implies that the data-package that has to be transmitted is of infinite size.

**Remark 4.** It is important to emphasize that the problem stated in Section III-D cannot be solved in a straightforward manner by a direct combination of existing techniques on event-triggered control, e.g., [25] and on dynamic quantization, e.g., [16], [39], for different reasons. Firstly, the interaction between the event-triggered implementation and the dynamic quantization produces new phenomena, which can have negative impact on the stability analysis and/or the efficient utilization of the network. For instance, due to the quantization effect, the sampling-induced error \( \varepsilon_i \) at each channel \( i \in \{1, 2, \ldots, l\} \) is not necessarily reset to zero at each transmission instant \( t_{k,i} \), \( k \in \mathbb{N} \), see (14), which often forms an important argument in the stability analysis of event-triggered control systems, see, e.g., [17], [25], [44]. Hence, this issue is not trivial to handle and requires careful construction of both the design strategies, see (12), (25), (28), (29), and the Lyapunov function candidate in order to guarantee the closed-loop stability, see (52) in the proof of Theorem 1 in the Appendix. Secondly, since the zoom-out actions are output-dependent and since the plant is subject to external disturbances, it is challenging to ensure that the quantization variable remains bounded, which is necessary to achieve an
input-to-state stability property, see, e.g., [16], [39]. Note that the technique of [16] is developed for the continuous-time case and relies on the fact the full state vector can be measured while the implementation setup in [39] is different from the one we consider in this study and hence, they cannot be directly applied to our case. Moreover, since the zoom-in actions are also output-dependent, the quantization strategy must be carefully designed in order to avoid Zeno behaviour on the quantization events, in particular when the output trajectory crosses zero, as confirmed later on simulation. Given these and other reasons, the joint design of the event-triggered controller and the dynamic quantizer is intricate and its easy to make design choices in which important properties such as input-to-state stability or non-Zenoness are lost, which hamper practical applicability.

D. Design procedure of the ETM and quantizer

In this subsection, we discuss the design of the ETM and the dynamic quantizer. Note that, in view of (27), (29), that the design parameters \( \gamma_i, \varepsilon_{y_i}, \lambda_i, \lambda_i, \) \( i \in \{1,2,\ldots,l\} \), create a strong coupling between the design of the event-triggering mechanism and the design of the dynamic quantizer.

The first step of the design procedure is to find suitable \( \gamma_i, \) \( i \in \{1,2,\ldots,l\} \), by minimizing the weighted sum \( \sum_{i=1}^{l} \pi_i \gamma_i \) subject to (23) with \( \pi_i \in (0,1), \) \( i \in \{1,2,\ldots,l\} \), such that \( \sum_{i=1}^{l} \pi_i = 1 \). The selection of \( \pi_i \) allows to balance the communication resource utilization among the different nodes. The parameters \( \Omega_{\text{in},i} \) and \( \Omega_{\text{out},i}, i \in \{1,2,\ldots,l\} \), allow to balance the required number of quantization regions that is needed to satisfy (30) and the maximum number of zoom-in and zoom-out events per transmission as given in (35) and (36), respectively. The next step is to select \( \kappa_i \) as small as possible to minimize the lower-bound on \( M_i/\Delta_i \) as given in (30). The latter is desired as \( M_i/\Delta_i \) typically reflects the number of quantization regions. After \( \Omega_{\text{in},i}, \Omega_{\text{out},i} \) and \( \kappa_i, i \in \{1,2,\ldots,l\} \), are obtained, we take \( M_i/\Delta_i \) such that (30) holds. Finally, the tuning of \( \lambda_i \) can be used to obtain an intuitive tradeoff between the number of transmissions and the amount of data that needs to be transmitted per transmission. When \( \lambda_i \) is reduced, the value of \( T_i \) in (27) will increase and, depending on the choice of \( \lambda_i \), the value of \( \gamma_i \) will decrease, which may result in a reduction of the amount of transmissions. However, by reducing \( \lambda_i \), the right-hand side of (29) will also increase. Hence, the value of \( M_i/\Delta_i \) needs to be increased which typically implies that more quantization regions are required, in order to ensure that (29) holds. We will illustrate this also with an example in the next section. The design procedure is summarized in the following algorithm.

Algorithm 1 Design procedure

For each node \( i \in \{1,2,\ldots,l\} \)

1: pick \( \pi_i \in (0,1) \) such that \( \sum_{i=1}^{l} \pi_i = 1 \)
2: minimize \( \sum_{i=1}^{l} \pi_i \gamma_i \) subject to (23) and compute the parameters \( \varepsilon_{y_i}, \gamma_i \)
3: pick \( \pi_i, \gamma_i, \Delta_{i,\Omega} > 0 \) in (25) sufficiently small and take \( \lambda_i \in (0,1), \lambda_i \in [\lambda_i, \lambda_i^{-1}] \) in (27) to achieve the desired tradeoff between performance and bit length, see (33), (35), (36) and Remark 2
4: compute the time \( T_i \) and the parameters of (25) for the event-triggering mechanism (8)-(9)
5: take \( \kappa_i > 1 \) sufficiently small and \( \Omega_{\text{in},i} \in (0,1) \) in (29)
6: choose \( M_i, \Delta_i > 0 \), such that (29) is satisfied
7: take \( \Omega_{\text{out},i} > 1 \) such that (30) holds

VI. ILLUSTRATIVE EXAMPLE

We consider the following linearized model of an inverted pendulum with a cart system, taken from [45], affected by external disturbances

\[
\begin{align*}
x_p &= \begin{bmatrix} 0 & 0 & 0 & 0 & -0.1789 & 7.7447 & 0 & 0 & 0 & 0 \
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 
\end{bmatrix} x_p + \begin{bmatrix} 0 
1.7895 
0 
5.2632 
\end{bmatrix} u + \begin{bmatrix} 0 
0 
w 
\end{bmatrix}
y &= \begin{bmatrix} 1 
0 
0 
0 
0 
0 
1 
0 
\end{bmatrix} x_p,
\end{align*}
\]

where \( x_p \) is the cart position, \( x_{p_v} \) is the cart velocity, \( x_{p_s} \) is the pendulum angle from vertical, \( x_{p_a} \) is the pendulum angular velocity, \( u \) is the input force, and \( w \) is the external disturbance. The numerical values in the matrices \( A_p, B_p, C_p \) correspond to a mass cart of \( M = 0.5 \) kg, pendulum mass \( m = 0.5 \) kg, pendulum length \( l = 0.3 \) m, pendulum inertia \( I = 0.006 \) kgm\(^2\), and cart surface friction \( b = 0.1 \) N/m/s. Interestingly, the open-loop system is unstable, and the disturbances affect the states that are not directly affected by the input.

In [45], the plant is stabilized by an observer-based controller of the form

\[
\dot{x} = A_p \dot{x} + B_p u + L(y - C_p \dot{x}), \quad u = K \dot{x},
\]

where \( \dot{x} \in \mathbb{R}^4 \) is the observer state, \( L \in \mathbb{R}^{4 \times 2} \) is the observer gain, and \( K \in \mathbb{R}^{1 \times 4} \) is the controller gain. The values of \( L, K \) in [45] are given by

\[
L = \begin{bmatrix} -0.29708 & 2.1396 \
-0.51518 & 11.3189 
\end{bmatrix},
K = \begin{bmatrix} 10.6963 & -0.09913 
17.0386 & 13.0877 
-5.0520 & 9.8150 
-63.2481 
\end{bmatrix}.
\]

By considering that the plant output is \( y \) is quantized and transmitted asynchronously over two different channels and by following similar steps as in Section IV, we obtain model (15), (17) with (note from (7), (41) that \( D_e = 0 \) in this case) \( A_1 = \begin{bmatrix} A_p & B_pK \\
LC_p + B_pK - LC_p & \] \( B_1 = \begin{bmatrix} 0 
0 
\end{bmatrix} \), \( E_1 = \begin{bmatrix} 0 
0 
\end{bmatrix} \), \( A_2 = \begin{bmatrix} -C_pA_p & -C_pB_pK 
\end{bmatrix}, B_2 = \begin{bmatrix} 0 
\end{bmatrix} \), \( E_2 = \begin{bmatrix} -C_pE_p 
\end{bmatrix} \). Note that the matrix \( A_1 \) is Hurwitz by design and hence we know that the LMI (23) is feasible for some symmetric real matrix \( P \), real numbers \( \varepsilon_x, \varepsilon_w > 0 \) and \( \varepsilon_{y_i}, \gamma_i > 0 \) for \( i \in \{1,2\} \). Following Algorithm 1 and by solving the LMI (23), we obtain \( \varepsilon_y = 4.4414, \varepsilon_y = 39.7353, L_1 = L_2 = 0, \gamma_1 = 70.4173, \gamma_2 = 119.0389 \). We take \( \lambda_1 = \lambda_2 = 0.5, \lambda_1 = 0.6, \lambda_2 = 0.8, \nu_1 = \nu_2 = 0.01 \) and we compute the values of \( T_1, T_2 \) by using (27), which yields \( T_1 = 0.0076 \) and \( T_2 = 0.0035 \). Furthermore, we obtain \( \gamma_1 = 67.44 \) and
\( \gamma_2 = 2324 \). Finally, we set \( \vartheta_1 = \vartheta_2 = 0.01 \) and, hence, all the required parameters for the event-triggering functions in (25) are defined. Next, we set the range of the quantizers to be \( M_1 = 50, M_2 = 50 \) and we take \( \Delta_1 = 0.1, \Delta_2 = 0.2, \Delta_{01} = \Delta_{02} = 1 \times 10^{-6}, \Omega_{in,1} = \Omega_{in,2} = 0.5, \Omega_{out,1} = \Omega_{out,2} = 2 \) and \( \kappa_1 = \kappa_2 = 2 \), which verify conditions (29), (30) and lead to \( \ell_{in,1} = 24.9, \ell_{in,2} = 24.8, \ell_{out,1} = 49.9 \) and \( \ell_{out,2} = 49.8 \).

We run simulations for 10 seconds with the initial conditions \( x(0,0) = (2, -2, 1.5, -1.5, 1, -1, 1, -1), e(0,0) = (0,0), \eta(0,0) = (0,0), \phi(0,0) = (\lambda_1^{-1}, \lambda_2^{-1}), \mu(0,0) = (1,1) \) and with the external disturbance \( w \) satisfying \( w(t,j) = 2 \cos(2\pi t) \) for all \( (t,j) \in \text{dom} \ w \) with \( t \in [0,1] \), \( w(t,j) = 0 \) for all \( (t,j) \in \text{dom} \ w \) with \( t \in \{1,3\} \) and \( w(t,j) = 0.2 \) for all \( (t,j) \in \text{dom} \ w \) with \( t \in \{3,5\} \). The observed minimum and average inter-transmission times, respectively denoted by \( \tau_{min} \) and \( \tau_{avg} \), for node 1 are \( \tau_{min,1} = 0.0089, \tau_{avg,1} = 0.0266 \) and for node 2 are \( \tau_{min,2} = 0.0036, \tau_{avg,2} = 0.0161 \).

We note that \( \tau_{min,1} > T_1 \) while \( \tau_{min,2} \approx T_2 \), which supports the observation of Remark 2 on the choice of \( \lambda_i \) since \( \lambda_1 > \lambda_2 \) and \( \lambda_2 = \lambda_2 \). The state trajectories of the plant and the dynamic controller are shown in Fig. 2, where we note that the state converges to a small neighborhood of the origin as expected. The zoom-in/zoom-out events by the respective dynamic quantizers are shown in Fig. 3 and Fig. 4 during the first 7 seconds. We note that at any transmission instant \( t_i^k, k \in \mathbb{N} \), only a zoom-in event or a zoom-out event occurs but not both of them due to the design conditions in Section V-B that imply properties (b) and (c) below (38). Fig. 5 and Fig. 6 present the generated transmission instants and the zoom instants during the first 3 seconds for nodes 1 and 2, respectively. We observe that the zoom actions are only executed at transmission instants as described in Section III-C and that at some transmission instants, no zoom action occurs. The latter is the case when none of the zoom conditions are met at those instants. The tradeoff between transmissions and number of quantization regions is illustrated in Fig 7 and 8 for node 1 and 2, respectively. These figures are generated by varying \( \lambda_i \) in (27), (29) from 0.01 to 0.99 and by taking \( \lambda_i = \lambda_i \). Observe that larger values of \( \lambda_i \), \( i \in \{1,2\} \), which for this case implies more quantization regions, lead to larger values of the guaranteed minimum time \( T_i \) between two consecutive transmissions/zooms at the corresponding node and vice versa, as discussed in Section V-D.
VII. CONCLUSIONS

In this paper we addressed the problem of input-to-state stabilization of linear systems over digital communication networks. We considered three important features of the communication network. First of all, our problem involved quantization of the measured variables to have a finite number of bits to be transmitted in packets. Secondly, we employed a resource-aware control paradigm (in particular, event-triggered control) to utilise the bandwidth-limited communication channels only when needed. Thirdly, we studied the scenario where multiple sensor nodes transmit their information asynchronously. We provided a complete design solution for this setup for a linear sensor nodes transmit their information asynchronously. We provided a complete design solution for this setup for a linear

In order to streamline the proof of Theorem 1, we first present the proof of Corollary 1. In the proofs, we often omit the time arguments of the solution $\xi$ of hybrid system $\mathcal{H}$ and we do not mention dom $\xi$ explicitly.

**Proof of Corollary 1.** Proof of statement (iii). By recalling that $\mathcal{X}_0 := \{\xi \in \mathcal{X} : \eta_i > 0, p_i = 0\}$, we can see from (21) that $p_i(t, j) = 1$ is only possible if the system has jumped according to the jump map $G^i(t, j)$. To be more concrete, we can conclude from (21) that for all $(t, j) \in \text{dom } \xi$ for which $p_i(t, j) = 1$, there exists an $n \in \mathbb{N}$ such that $\xi(t, j - n) \in G^i_{i, j}(D_i)$ and that $\xi(t, j) \in \bigcup_{k \in \{1, \ldots, \tau_i\} \setminus \{i\}} \left(\{G^i_{k, j}(D_i)\} \cup \{G^i_{k, j}(D_i)\}\right)$ for all $j \in \{j - n + 1, \ldots, j\}$. Since $\mu_i$ is only affected by the jump map $G^i_{i, j}(D_i)$ (corresponding to a zoom-update event at node $i$), we only need to show that $|y_i(t, j)| \leq M_i\mu_i(t, j)$ for all $\xi(t, j) \in G^i_{i, j}(D_i)$ with $(t, j) \in \text{dom } \xi$. To do so, we consider the cases that a zoom-in event occurs, a zoom-out event occurs and that none of the zoom conditions are violated.

In case of a zoom-in event, i.e., when $\xi \in D_i$, and the system jumps according to $\xi \in G^i_{i, j}(\xi)$ with $\mu_i^+ \in \Omega_{\text{out}, i}(\xi, y_i, \mu_i) \mu_i$ and $y_i^+ = y_i$, we have that

$$\ell_{\text{out}, i}^+ \Omega_{\text{in}, i}^+ \max\{|y_i|, |\Delta_{\text{out}, i}|\} \geq \frac{M_i - \Delta_{\text{in}}}{\kappa_{\text{out}, i}} |y_i|. \tag{32}$$

Since according to (30), $\kappa_i > 1$, we obtain that $\ell_{\text{out}, i}^+ \Omega_{\text{in}, i}^+ \max\{|y_i|, |\Delta_{\text{out}, i}|\} \geq \frac{M_i - \Delta_{\text{in}}}{\kappa_{\text{out}, i}} |y_i|$. \tag{42}

In case of a zoom-out event, i.e., when $\xi(t, j) \in D_i$ and the system jumps according to $\xi (\xi, \mu_i) \in G^i_{i, j}(\xi)$ with $\mu_i^+ \in \Omega_{\text{out}, i}(\xi, y_i, \mu_i)$ and $y_i^+ = y_i$, it immediately follows from (31) that $\ell_{\text{out}, i}^+ \Omega_{\text{in}, i}^+ \max\{|y_i|, |\Delta_{\text{out}, i}|\} \geq M_i^+ \mu_i^+ \geq |y_i^+|$ as $\ell_{\text{out}, i}^+ \Omega_{\text{in}, i}^+ \max\{|y_i|, |\Delta_{\text{out}, i}|\} \geq M_i^+ \mu_i^+ \geq |y_i^+|$. Hence, statement (iii) in Corollary 1 holds.

**Proof of statement (iv).** Consider the following claim.

**Claim 1.** Let the hypotheses of Theorem 1 hold. Consider a solution pair $(\xi, w)$ to (20) with $\xi(0, 0) \in \mathcal{X}_0$ with $\mathcal{X}_0$ as defined in Theorem 1. Then, it holds for all $(t, j) \in \text{dom } \xi$ with $\xi(t, j) \in D_i$ that $\tilde{\gamma}_i |\xi(t, j)|^2 \geq \varepsilon_i \max\{|y_i^2(t, j)|, \Delta_{\text{out}, i}^2\}, \ i \in \{1, 2, \ldots, l\}$.

**Proof of Claim 1.** To prove this claim, we use the following lemma.

**Lemma 1.** Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ that is continuously differentiable at an open interval containing $[T_1, T_2]$ with

![Fig. 8. Tradeoff curves between number of quantization regions and transmissions for $y_2$.](image-url)
Moreover, let us define \( \tilde{E}_i := \{ (t, j) \in \text{dom} \xi : \xi(t, j) \in \mathcal{C} \cap D_i \} \) for all \( i \in \{1, \ldots, l\} \). Observe from (25) that \( \Phi_{s(t),i}(t,j) > 0 \) for all \( i \in \{1, \ldots, l\} \), \( j \in \text{dom} \xi \) for which \( \eta_i(t, j) = 0 \) and \( 0 \leq \tau_i(t, j) < T_i \). By means of the latter, and by recalling (9) and the fact that, per definition of \( \mathcal{D}_i \), \( \eta_i(0,0) > 0 \) for all \( i \in \{1, \ldots, l\} \), we obtain that \( \eta_i(t, j) > 0 \) for all \( (t, j) \in \text{dom} \xi \) for which \( \tau_i(t, j) = T_i \). Since \( \eta_i(t, j) = 0 \) and \( \tau_i(t, j) \geq T_i \), when \( (t, j) \in \mathcal{D}_i \), we can consequently conclude that for each \( (t, j) \in \tilde{E}_i \), there exists a \( t' < t \) such that \( (t', j) \in \mathcal{C} \) with \( \tau_i(t', j) > T_i \). By using the latter, the fact that, for all \( (t, j) \in \tilde{E}_i \), \( \eta_i(t, j) > 0 \), per definition of \( \chi \) and Lemma 1, we obtain that for all \( (t, j) \in \tilde{E}_i \), \( \eta_i(t, j) > 0 \). Observe from (25) that \( \tilde{\eta}_i(t, j) \geq \varepsilon_{y_i} \max \{ |y_i(t, j)|^2, \Delta_{\tilde{\eta}_i} \} - \tilde{\gamma}_i |e_i(t, j)|^2 - \eta_i(t, j) \) for all \( (t, j) \in \text{dom} \xi \). Hence, we have that for all \( i \in \{1, \ldots, l\} \), and all \( (t, j) \in \tilde{E}_i \), \( \tilde{\gamma}_i |e_i(t, j)|^2 \geq \varepsilon_{y_i} \max \{ |y_i(t, j)|^2, \Delta_{\tilde{\eta}_i} \} \).

To complete the proof, we need to show that the property \( \tilde{\gamma}_i |e_i(t, j)|^2 \geq \varepsilon_{y_i} \max \{ |y_i(t, j)|^2, \Delta_{\tilde{\eta}_i} \} \) also holds for all \( (t, j) \in \text{dom} \xi \) for which \( \xi(t, j) \in \mathcal{D}_i \setminus \mathcal{C} \). Observe that \( \xi(t, j) \in \mathcal{D}_i \setminus \mathcal{C} \) implies that \( p_i(t, j) = 1 \) for all \( (t, j) \in \text{dom} \xi \). As mentioned before, \( p_i(t, j) = 1 \) implies that there exists an \( n \in \mathbb{N} \) such that \( \xi(t, j - n) = G_i^n(\mathcal{D}_i) \) and that \( \xi(t, j) \in \bigcup_{k \in \{1, \ldots, l\} \setminus \{i\}} \{ (G_k^n(\mathcal{D}_i)) \cup \{ G_i^n(\mathcal{D}_i) \} \} \) for all \( j \in \{1, \ldots, n + 1, \ldots\} \). Since \( y_i \) and \( e_i \) are not affected after the jumps maps \( G_i^n(\mathcal{C}) \) or \( G_i^n(\xi) \), \( i \in \{1, \ldots, l\} \), \( k \in \{1, \ldots, n\} \), \( \{i\} \) are applied, Claim 1 follows.

\section{Proof of Theorem 1. Proof of statement (i)\[\]}

To prove the ISS properties of the set \( \mathcal{A} \), we show that there exists an ISS Lyapunov function \( U \) for the hybrid system \( \mathcal{H} \) described by (19) and (20). To be more specific, we aim to find a locally Lipschitz non-negative function \( U \) that satisfies the following properties. For some functions \( \alpha_1, \alpha_2 \in \mathcal{K}_\infty \) such that for all \( \chi \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D}) \), it holds that

\[ \alpha_1(|\chi|_A) \leq U(\chi) \leq \alpha_2(|\chi|_A) \]

Moreover, for each solution pair \( (\xi, w) \) with \( \xi(0,0) = \chi_0 \) and \( w \in \mathcal{L}_\infty \), there exist some functions \( \hat{p} \in \mathcal{K}_\infty \) and \( \sigma \in \mathcal{K} \), such that for all \( (t, j) \in \text{dom} \xi \), it holds that when \( \xi(t, j) \in \mathcal{D} \),

\[ U(G(\xi(t, j))) - U(\xi(t, j)) \leq 0, \]

and for almost all \( (t, j) \in \text{dom} \xi \), when \( \xi(t, j) \in \mathcal{C} \), it holds that

\[ \langle \nabla U(\xi(t, j), F(\xi(t, j), w(t, j))) \rangle \leq -\hat{p}(U(\xi(t, j))) + \sigma(|w(t, j)|). \]

Observe that the ISS conditions above are closely related to the ISS conditions presented in [47, Definition 3.2]. Note that the ISS condition in (45)-(47) are relaxed compared to [47]. This is possible as we will have that the all maximal solutions of \( \mathcal{H} \) described by (19) and (20) are t-complete, see also Proposition 3.27 in [29], Lemma III.3 in [7, 48].

Consider the function \( U(\xi) := \max\{0, R(\xi) - \epsilon\} \) with the function \( R \) as in Theorem 1. Observe that \( U(\xi) = 0 \) for \( \xi \in \mathcal{A} \). Moreover, note that, in view of (28) and (34), \( \phi_i(\tau_i(t, j)) > 0 \) for all \( (t, j) \in \text{dom} \xi \) and \( i \in \{1, \ldots, l\} \). Hence, since \( \eta_i(t, j) \geq 0 \) for all \( (t, j) \in \text{dom} \xi \) in view of the definition of \( \chi \), we deduce that the function \( U \) satisfies the inequality

\[ \ell_{m,i} \mu_i^+ \leq \max\{ |y_i|^2, \Delta_0, 1 \}, \]

which completes the proof of Claim 2. 

To complete the proof of statement (iii), we proceed with observing that in view of (29), \( M_i - \kappa_i \Delta_i \geq \frac{2\sqrt{T_i \mu_i}}{\sqrt{\mu_i} \lambda_i} \Delta_i \). Hence, in view of the definition of \( \ell_{m,i} \) in (32), we have \( 2\Delta_i \mathcal{N}_{\lambda_i, \mu_i} \leq \Omega_{m,i}(M_i - \kappa_i \Delta_i) = \ell_{m,i}. \) Hence, \( \frac{2\lambda_i}{\lambda_i} \leq \frac{2\Delta_i \mathcal{N}_{\lambda_i, \mu_i} \ell_{m,i}}{\sqrt{\mu_i} \lambda_i} \). By multiplying both sides by \( \mu_i \) we have that, for all \( (t, j) \in \mathcal{D} \) for which \( p_i(t, j) = 1 \) with \( (t, j) \in \text{dom} \xi \),

\[ \frac{2\Delta_i \mathcal{N}_{\lambda_i, \mu_i} \ell_{m,i}}{\sqrt{\mu_i} \lambda_i} \leq |e_i(t, j)| \]

Claim 2

\[ \leq \max\{ |y_i|^2, \Delta_{\tilde{\eta}_i}, 1 \}, \]

Claim 1

\[ \leq |e_i(t, j)| \leq |e_i(t, j)| + |e_{q,i}(t, j)| \]

(44)

By means of statement (ii) of Theorem 1 and (10), we obtain that \( |e_i(t, j)| \leq \Delta_{\tilde{\mu}_i}(t, j) \) for all \( (t, j) \in \mathcal{D} \) for which \( p_i(t, j) = 1 \) with \( (t, j) \in \text{dom} \xi \). Combining the latter fact with (44) yields \( |e_{q,i}(t, j)| \geq \frac{2\lambda_i}{\lambda_i} \Delta_{\tilde{\mu}_i}(t, j) \). For all \( (t, j) \in \mathcal{D} \) for which \( p_i(t, j) = 1 \) with \( (t, j) \in \text{dom} \xi \). Since \( \lambda_i \in (0, 1) \), it holds that, for all \( (t, j) \in \mathcal{D}_i \), \( \ell_{m,i} \mu_i^+ \leq \Delta_{\tilde{\mu}_i}(t, j) \geq |e_{q,i}(t, j)| \). Thus, statement (iv) of Corollary 1 is proven.

All the statements of Corollary 1 are now proved.

\section{Proof of Theorem 1. Proof of statement (i)\[\]}

\[ f(t) \geq 0 \] for \( t \in (T_1, T_2) \) for \( T_2 > T_1 > 0 \) and \( f(T_2) = 0 \), then \( f(T_2) = 0 \).
in (45) and thereby constitutes an appropriate candidate ISS Lyapunov function.

**Lemma 2.** For all $\xi \in C \setminus A$, it holds that
\[
\epsilon \left( \varepsilon_x |x|^2 + \sum_{i=1}^{j} 2 \nu_i \gamma_i \phi_i (\tau_i) |e_i|^2 + \sum_{i=1}^{j} \psi_i |\eta_i|^2 \right) \geq \sum_{i=1}^{j} \varepsilon_y_i \Delta^2_{\nu_i}.
\] (48)

**Proof of Lemma 2.** In view of the definition of $A$, it holds for all $\xi \in C \setminus A$ that $R(\xi) \geq c$ with $c$ as defined in Theorem 1, i.e.,
\[
x^T P x + \sum_{i=1}^{j} \left( \gamma_i \phi_i (\tau_i) |e_i|^2 + \eta_i \right) \geq \epsilon \min \{ \varepsilon_x / \lambda_{\max}(P), 2 \min \nu_i, \min \psi_i \} \times \left( \lambda_{\max}(P) |x|^2 + \sum_{i=1}^{j} \left( \gamma_i \phi_i (\tau_i) |e_i|^2 + \eta_i \right) \right) \geq \sum_{i=1}^{j} \varepsilon_y_i \Delta^2_{\nu_i}.
\] (49)

Hence, we deduce from (49) for all $\xi \in C \setminus A$ that
\[
\epsilon \min \{ \varepsilon_x / \lambda_{\max}(P), 2 \min \nu_i, \min \psi_i \} \times \left( \lambda_{\max}(P) |x|^2 + \sum_{i=1}^{j} \left( \gamma_i \phi_i (\tau_i) |e_i|^2 + \eta_i \right) \right) \geq \sum_{i=1}^{j} \varepsilon_y_i \Delta^2_{\nu_i}.
\] (50)

Consequently, by using the fact that for $r \in \mathbb{N}$, $\min_{i \in \{1, 2, \ldots, r\}} \alpha_i \left( \sum_{j=1}^{r} \beta_j \right) \leq \sum_{j=1}^{r} \alpha_j \beta_j$ for all $\alpha_j, \beta_j \in \mathbb{R}_{\geq 0}$, we can conclude that (48) indeed holds for all $\xi \in C \setminus A$. \(\square\)

**Dynamics of $U$ at jumps**

In view of the jump map in (21), we distinguish two type of jumps.

- When $\xi \in D_1$ with $p_i = 0$, $i \in \{1, 2, \ldots, l\}$, and the system jumps according to $\xi^+ = G_i^0 (\xi)$. This case corresponds to an update of the quantizer at node $i$. Since $\tau_i$, $e_i$ and $\eta_i$ are not affected by the jump map $G_i^0$, we obtain
\[
R(G(\xi)) - R(\xi) = \gamma_i \phi_i (\tau_i^+) |e_i^+|^2 + \nu_i^+ - \gamma_i \phi_i (\tau_i) |e_i|^2 - \eta_i \geq 0.
\] (51)

- When $\xi \in D_1$ with $p_i = 1$, $i \in \{1, 2, \ldots, l\}$, and the system jumps according to $\xi^+ = G_i^1 (\xi)$. This case corresponds to a new transmission generated at some node $i$. In view of (20), (21) and by using the fact that $\phi_i (\tau_i^+) = \phi_i (\tau_i)^+ = \lambda_i^{-1}$, we have that
\[
R(G(\xi)) - R(\xi) = \gamma_i \phi_i (\tau_i^+) |e_i^+|^2 + \nu_i^+ - \gamma_i \phi_i (\tau_i) |e_i|^2 - \eta_i \geq 0.
\] (52)

As a result, in view of (51) and (52), we have that when $\xi \in D$, $R(G(\xi)) \leq R(\xi)$. In the view of the definition of $A$, we also have that $G(\xi) \in A$ when $\xi \in A$. Given the latter facts, we can conclude that (46) holds for all $(t, j) \in \text{dom} \xi$.

**Dynamics of $U$ during flows**

Recall that $F(\xi, w) = (A_1 x + B_1 e + E_1 w, A_2 x + B_2 e + E_2 w, 0, 1, \Psi(o), 0)$. Consequently, in view of (20), (24), (28), we obtain for $\xi \in C$ (recall that $L_i = L_i + \nu_i, i \in \{1, 2, \ldots, l\}$)
\[
\langle \nabla R, F(\xi, w) \rangle = \langle \nabla V(x), A_1 x + B_1 e + E_1 w \rangle + \sum_{i=1}^{l} 2 \gamma_i \phi_i (\tau_i) |e_i|^2 + L_i |\tilde{\nu}_i|^2 \leq \sum_{i=1}^{l} \gamma_i^2 |e_i|^2 + \epsilon w |w|^2 + \sum_{i=1}^{l} 2 L_i \gamma_i \phi_i (\tau_i) |e_i|^2
\] + \sum_{i=1}^{l} \gamma_i \phi_i (\tau_i) |e_i| \sup_{|A_2 x + B_2 e + E_2 w|} \|A_2 x + B_2 e + E_2 w\| |
\[
- \frac{1}{2} \sum_{i=1}^{l} \tilde{\omega}_i (\tau_i) \left( 2 L_i \gamma_i \phi_i (\tau_i) + \gamma_i^2 (\phi_i (\tau_i) + 1) \right) |e_i|^2
\] + \sum_{i=1}^{l} \frac{\bar{\psi}_i (y_i, \epsilon, \tau_i, \eta_i)}{\epsilon}
\]

for some $\tilde{\omega}_i (\tau_i) \in \omega_i (\tau_i)$ and $\frac{\bar{\psi}_i (y_i, \epsilon, \tau_i, \eta_i)}{\epsilon} \leq 0$.

By using the fact that $2 \gamma_i \phi_i (\tau_i) |e_i| |A_2 x + B_2 e + E_2 w| \leq \gamma_i^2 \phi_i (\tau_i) |e_i|^2 + |A_2 x + B_2 e + E_2 w|^2$ and since, in view of (25), $\frac{\bar{\psi}_i (y_i, \epsilon, \tau_i, \eta_i)}{\epsilon} \leq \sum_{i=1}^{l} \tilde{\omega}_i (\tau_i) \left( 2 L_i \gamma_i \phi_i (\tau_i) + \gamma_i^2 (\phi_i (\tau_i) + 1) \right) |e_i|^2$
\[
+ \sum_{i=1}^{l} \frac{\bar{\psi}_i (y_i, \epsilon, \tau_i, \eta_i)}{\epsilon} \leq \sum_{i=1}^{l} \tilde{\omega}_i (\tau_i) \left( 2 L_i \gamma_i \phi_i (\tau_i) + \gamma_i^2 (\phi_i (\tau_i) + 1) \right) |e_i|^2
\] + \sum_{i=1}^{l} \frac{\bar{\psi}_i (y_i, \epsilon, \tau_i, \eta_i)}{\epsilon}
\]

for some $\tilde{\omega}_i (\tau_i) \in \omega_i (\tau_i)$.

By recalling that $\gamma_i = \gamma_i^2 + \gamma_i^2 \lambda_i^2 + 2 \gamma_i \lambda_i^2 L_i, L_i = L_i + \nu_i$ and $\phi_i (\tau_i) = \phi_i (T_i) = \lambda_i$ for $\tau_i \geq T_i$ according to (28) and using the fact that $\|y_i\|^2, \Delta^2_{\nu_i}$, we obtain that $\langle \nabla R, F(\xi, w) \rangle \leq -\epsilon_x |x|^2 + \sum_{i=1}^{l} \varepsilon_y_i \Delta^2_{\nu_i} + \epsilon w |w|^2 - \sum_{i=1}^{l} 2 \nu_i \gamma_i \phi_i (\tau_i) |e_i|^2 - \sum_{i=1}^{l} \eta_i \eta_i$. Now by means of Lemma 2, we obtain that for $\epsilon \in (0, 1)$ and for all $(t, j) \in \text{dom} \xi$ for which $\xi(t, j) \in C \setminus A$, it holds that
\[
\langle \nabla R, F(\xi, w) \rangle \leq -\epsilon \epsilon_x |x|^2 + \epsilon w |w|^2 - \epsilon \epsilon_x \sum_{i=1}^{l} 2 \nu_i \gamma_i \phi_i (\tau_i) |e_i|^2 - \epsilon \epsilon_x \sum_{i=1}^{l} \eta_i \eta_i.
\] (53)
By recalling the fact that \( U(\xi) = 0 \) when \( \xi \in A \), we can conclude that for all \((t,j) \in \mathcal{D}\) for which \( \xi(t,j) \in C \)

\[
\left\langle \nabla U(\xi, F(\xi, w)) \right\rangle \leq - \rho U(\xi) + \varepsilon_w |w|^2, \tag{54}
\]

where \( \rho := (1 - \epsilon) \min\{\frac{1}{\lambda_{\max}(P)}, 2 \min \tau_i, \min \bar{\tau}_i\} \) and thus that (47) holds.

**t-completeness of maximal solutions**

We investigate \( t \)-completeness of maximal solutions. To do so, we verify the conditions provided in [29, Proposition 6.10]. First of all, observe that \( G(D) \subseteq C \cup D \) since for all \( \xi \in G(D) \), it holds that \( \tau^*_{\xi} > 0, \eta^*_{\xi} > 0, \mu^*_{\xi} \geq \frac{\bar{\delta}_{\min}}{\eta}, \) and \( p^*_{\xi} \in \{0, 1\} \) due to (20), (25) and property (d) below (38). Next, we show that for any \( \varphi \in C \) there exists a neighborhood \( S \) of \( \xi \) such that, it holds for every \( \varphi \in S \cap C \) that \( F(\varphi, w) \cap T_C(\varphi) \neq \emptyset \), where \( T_C(\varphi) \) is the tangent cone\(^3\) to \( C \) at \( \varphi \). Observe that for each \( \xi \in C \) (recall that \( \xi = (x, e, \mu, \tau, \eta, p) \)), \( T_C(\xi) = \mathbb{R}^n_x \times \mathbb{R}^n_\tau \times T_{2,\geq 0}(\mu_1) \times \cdots \times T_{2,\geq 0}(\mu_i) \times (\bar{T}_{\geq 0}(\tau_1) \times \cdots \times \bar{T}_{\geq 0}(\tau_i)) \). Observe also from (19) that \( C \cup D = \bigcup_{i=1}^{\ell} \{ \xi \in \mathbb{R}^n: p_i = 0 \text{ and } (\tau_i < T_i \text{ or } \eta_i > 0) \} \). Given the facts that, for all \( i \in \{1, 2, \ldots, \ell\}, \tau_i = 0 \) and \( \mu_i = 0 \) due to (21) and that \( \eta_i > 0 \) when \( \tau_i < T_i \) and \( \eta_i = 0 \) due to (9) and (25), it indeed follows that for any \( \xi \in C \) there exists a neighborhood \( S \) of \( \xi \) such that, it holds for every \( \varphi \in S \cap C \) that \( F(\varphi, w) \cap T_C(\varphi) \neq \emptyset \). To conclude the proof for \( t \)-completeness, finite escape speed should be excluded which is the case due to (47). Hence, indeed all maximal solutions (20), (21) with \( \xi(0,0) \in \mathbb{R} \) and \( w \in \mathcal{L}_\infty \) are \( t \)-complete.

**Proof of statement (ii)**

To obtain the upper-bound for \( \kappa_{\text{out},i}(y_i, \mu_i) \), from (54), we derive an upper-bound for \( y_i \). From (54), we obtain that

\[
U(\xi(t,j)) \leq e^{-\rho t} U(\xi(0,0)) + \varepsilon_w \int_0^t e^{-\rho (t-s)} |w(s)|^2 \, ds. \tag{55}
\]

By combining the latter with the fact that \( U(\xi(t,j)) \geq x^T(j) P x(t,j) - c \geq \lambda_{\min}(P) \inf_{\xi(t,j)} (y_{\text{in},i}(y_i, \mu_i)) \), we can conclude that \( \|y_{\text{in},i}\|_{\infty} \leq \bar{y}_i \) if \( \bar{y}_i \) is in (39). Consequently, in view of (31) and the fact \( \lambda_{\min}(P) \geq \mu_i \) for all \( (t,j) \in \mathcal{D} \), we can conclude that the elements of \( \kappa_{\text{out},i}(y_i, \mu_i) \) are upper bounded by \( \kappa_{\text{out},i} \) as in (36).

To obtain the upper-bound for \( \kappa_{\text{in},i}(y_i, \mu_i) \), we first derive an upper-bound for \( \mu_i \). By recalling the fact that \( \mu_i, i \in \{1, 2, \ldots, \ell\} \), is only increased when \( \|y_i\| \geq \ell_{\text{out},i}(\mu_i) \) (see also property (b) below (38), we obtain that \( \|\mu_i\|_{\infty} \leq \mu_i \) as in (37). Given the latter, we can obtain from (31) the elements of \( \kappa_{\text{in},i}(y_i, \mu_i) \) are upper bounded by \( \kappa_{\text{in},i} \) as in (35).

All the statements of Theorem I are now proved.

\(^3\)The tangent cone to a set \( S \subseteq \mathbb{R}^n \) at a point \( x \in \mathbb{R}^n \), denoted \( T_{S}(x) \), is the set of all vectors \( w \in \mathbb{R}^n \) for which there exist \( x_{n} \in S, \tau_i > 0 \) with \( x_{n} \rightarrow x, \tau_i \rightarrow 0 \) as \( i \rightarrow \infty \) such that \( \omega = \lim_{\tau_i \to \infty} (x_{n} - x)/\tau_i \). (See Definition 5.12 in [29]).


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