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Performance analysis of reset control systems

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SUMMARY

In this paper we present a general linear matrix inequality-based analysis method to determine the performance of a SISO reset control system in both the $\mathscr{L}_2$ gain and $\mathscr{H}_2$ sense. In particular, we derive convex optimization problems in terms of LMIs to compute an upperbound on the $\mathscr{L}_2$ gain performance and the $\mathscr{H}_2$ norm, using dissipativity theory with piecewise quadratic Lyapunov functions. The results are applicable to for all LTI plants and linear-based reset controllers, thereby generalizing the available results in the literature. Furthermore, we provide simple though convincing examples to illustrate the accuracy of our proposed $\mathscr{L}_2$ gain and $\mathscr{H}_2$ norm calculations and show that, for an input constrained $\mathscr{H}_2$ problem, reset control can outperform a linear controller designed by a common nonlinear optimization method. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

A reset controller is a linear time-invariant system whose states, or subset of states, reset to zero whenever its input and output satisfy certain conditions. Reset control action has certain resemblance with a number of popular non smooth control strategies including relay control [1], sliding mode control [2] and switching control [3, 4]. A common feature of these methods is a switching surface used to trigger discontinuous changes in the control signal. However, reset control is different if compared to these approaches in the sense that it employs the same (linear) control law on both sides of the switching surface and only resets some of the controller states when the switching surface is reached. The rationale behind reset control can best be understood when considering an integrating controller in particular. An integrator 'sums' the error over time in order to achieve a zero steady-state error. However, when the error becomes zero for the first time, the integrator still has the 'summed' error stored in its states, and the subsequent 'emptying' of the integrator causes the error signal to overshoot. To reduce the overshoot, it might be beneficial to reset the state of this integrator to zero as soon as its input (the error) becomes zero. In this way the integrator state, containing the 'summed' error, is emptied instantaneously, and hence the overshoot is avoided. This concept was indeed validated by simulations in recent publications [5, 6], showing a significant decrease of overshoot in the step response.

The concept of reset control was first introduced in 1958 by means of the resetting integrator of Clegg [7]. The describing function of the Clegg integrator has

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the same magnitude plot as a linear integrator, but its performance limiting phase lag is only 38.1° instead of the normal 90°. However, the use of this Clegg integrator is not straightforward as its overall behavior was hard to analyze. As a consequence, it was not until 1974 that it was first used in a control design procedure in [8]. Subsequently, in [9], a *first-order reset element FORE* was introduced, together with a controller design procedure based on frequency domain techniques. An overview of these results is given in [10].

In the late 90s there has been renewed interest in reset control systems, resulting in various stability analysis techniques. The first results were reported in [6, 11], stating stability criteria for zero-input closed-loops with a second-order plant and a Clegg integrator or a FORE, respectively. However, these criteria involve explicit computation of the reset times and closed-loop solutions, and are hard to generalize to higher order systems.

In the following publications, stability conditions were formulated using Lyapunov-based conditions. This was first done in [12], in which only second-order closed-loops with constant inputs were considered. These results have been extended in [13] to a sufficient criterion for bounded-input bounded-output (BIBO) stability, and later to the so-called H_β-condition [14]. The authors of [12–14] also demonstrated the potential advantages of reset controllers over linear ones both in simulations [5, 6] and experiments [10, 13, 15]. These advantages are especially apparent during the transient response. A clear overview of this line of work on reset systems is provided in [16].

A closer view on the H_β-condition reveals that it is in fact a reformulation of Lyapunov-based stability linear matrix inequalities (LMIs) using the Kalman–Yakubovich–Popov (KYP) lemma, in order to provide computable conditions to check the stability of zero-input reset control systems. The analysis consists of two stability LMIs, one corresponding to the *flowing* of the closed-loop (i.e. smooth evolution of the state) and the other to the *reset* of the controller. These LMIs are coupled as a common quadratic Lyapunov function is employed for both LMIs. Therefore the H_β-condition is rather conservative, and is only necessary and sufficient for *quadratic* stability. Moreover, since the flowing LMI is solved for the complete state-space, it requires the linear part of the closed-loop dynamics to be stable, which already indicates some of the conservatism present in this approach.

This conservatism was reduced in some part by more recent publications [17, 18], where a slightly different resetting condition was suggested. Indeed, the idea to reset when controller in- and output have opposite sign instead of when the input is zero, results in a much smaller flow region. Therefore, the linear closed-loop does not need to be stable anymore and the stability bounds of the reset system are sharpened. Secondly, *piecewise quadratic* (PWQ) Lyapunov functions as in [19] were used, thereby allowing more flexibility in the Lyapunov function to capture a broader class of stability problems. Additionally, the analysis has been extended such that the closed-loop $\mathscr{L}_2$ gain from input to output of a reset control system can be approximated by an upperbound. So far, the work in [17, 18] is the most general analysis framework for reset control systems currently available in literature. However, it is not generally applicable, since it treats only FOREs and Clegg integrators. Furthermore, it does not include a solution to the tracking problem, since its system description assumes a zero reference. In [20], this last issue is partly resolved as constant reference signals are considered.

This paper extends the results in the field of reset control in two directions, namely the work in [17, 18] on $\mathscr{L}_2$ gains of reset control systems is extended on one hand, and completely new contributions are presented in analyzing the $\mathscr{H}_2$ norm on the other. We extend [17, 18] in several ways. First, we generalize the $\mathscr{L}_2$ gain analysis to general reset control systems fitting into the common $\mathscr{H}_\infty$ framework using augmented plants, whereas [17, 18] are restricted to the Clegg integrator and FOREs. Second, arbitrary tracking problems are included in our results, as opposed to [20] which only considers constant reference signals. We will show that this might introduce conservatism, for which a solution will be provided via input filtering.

Another extension is the performance analysis via the computation of the $\mathscr{H}_2$ norm. Although the calculation of the $\mathscr{L}_2$ gain is very useful to express the performance of a reset control system, it is typically a steady-state measure. The advantage of reset control over linear control is especially apparent during the

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transient behavior of constrained problems. Indeed, it has been shown that reset controllers are able to reduce the overshoot of step responses [5, 6], thereby decreasing the total energy of the error signal. This observation shows similarities with one of the interpretations of the $\mathscr{H}_2$ norm, which can be seen as the total output energy (of e.g. the tracking error) of a closed-loop system to either an impulse input or non zero initial values. For this reason, we derive an LMI-based analysis method to calculate upperbounds on the $\mathscr{H}_2$ norm of a closed-loop reset control system. The results can be used to approximate the energy content of the output resulting from specific input signals. Moreover, we provide simple though convincing examples to illustrate the accuracy of our proposed $\mathscr{L}_2$ gain and $\mathscr{H}_2$ norm calculations and show that, for an input constrained $\mathscr{H}_2$ problem, reset control can indeed outperform a linear controller designed by a common nonlinear optimization method.

The paper is organized as follows. The general description of a reset control system is given in Section 3. The $\mathscr{L}_2$ gain analysis results together with an example are presented in Section 4. Section 5 derives the $\mathscr{H}_2$ norm analysis results for reset control systems. In Section 6, we will shortly reflect on synthesis possibilities, the usefulness of $\mathscr{L}_2$ gain and $\mathscr{H}_2$ controllers for reset control systems, and the practical use of reset control in general. Finally, in Section 7 we state the conclusions.

2. NOTATION

The set of real numbers is denoted by $\mathbb{R}$ and the set of nonnegative real numbers is denoted by $\mathbb{R}_+$. The set of real symmetric matrices in $\mathbb{R}^{n\times n}$ is denoted by $\mathbb{S}^{n\times n}$, and the set of real symmetric matrices with nonnegative elements is denoted by $\mathbb{S}_+^{n\times n}$. The identity matrix of dimension $n\times n$ is denoted by $I_n\in\mathbb{R}^{n\times n}$. Given two vectors x_1, x_2 we write (x_1, x_2) to denote $[x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$, and $\langle x_1, x_2\rangle = x_1^{\mathrm{T}} x_2$ denotes the inner product in $\mathbb{R}^n$. A vector $x\in\mathbb{R}^n$ is nonnegative, denoted by $x\geqslant 0$, if its elements x_i satisfy $x_i\geqslant 0$ for $i=1,\ldots,n$. A symmetric matrix $A\in\mathbb{R}^{n\times n}$ is positive definite, denoted by $A\succ 0$, if $x^{\mathrm{T}}Ax>0$ for all $x\in\mathbb{R}^n\backslash\{0\}$. A sequence of scalars $(u^1, u^2, \ldots, u^k)$ is called lexicographically nonnegative, written as $(u^1, u^2, \ldots, u^k)\geqslant_\ell 0$, if $(u^1, u^2, \ldots, u^k)=(0,0,\ldots,0)$ or $u^j>0$ where $j=\min\{p\,|\,u^p\neq 0\}$. For a sequence of vectors $(x^1, x^2, \ldots, x^k)$ with $x^j\in\mathbb{R}^n$, we write $(x^1, x^2, \ldots, x^k)\geqslant_\ell 0$ when $(x_i^1, x_i^2, \ldots, x_i^k)\geqslant_\ell 0$ for all $i=1,\ldots,n$. Likewise, we write $(x^1, x^2, \ldots, x^k)\leqslant_\ell 0$ to denote a lexicographically nonpositive sequence of vectors, meaning that $-(x^1, x^2, \ldots, x^k)\geqslant_\ell 0$.

3. GENERAL SYSTEM DESCRIPTION

In this section we present a mathematical description of the plant, the reset controller, and the resulting closed-loop system. These descriptions are chosen to fit into the common multichannel $\mathscr{H}_\infty$ framework, as depicted in Figure 1. The augmented plant P contains the system to be controlled, together with possible input- and output-weightings. We consider an LTI augmented plant P given by the minimal state-space description

$$\begin{aligned}\dot{x}_p &= Ax_p + Bu + B_w w\\ z &= C_z x_p + D_{zw} w + D_z u\\ y &= Cx_p + D_w w\end{aligned}\tag{1}$$

where $x_p(t)\in\mathbb{R}^{n_p}$ is the augmented plant state, $w(t)\in\mathbb{R}^{n_w}$ and $z(t)\in\mathbb{R}^{n_z}$ are the exogenous input and the controlled output, respectively, at time $t\in\mathbb{R}$ and A, B_w, B, C_z, D_{zw}, D_z, C, D_w are matrices of appropriate dimensions. The controller input and output at time $t\in\mathbb{R}$ are denoted by $y(t)$, $u(t)\in\mathbb{R}$. We assume that there is no direct feedthrough from u to y, as is, for instance, the case for many motion systems. Note that we consider SISO controllers only, since reset control for MIMO systems is still a widely open area as the proper formulation of relevant reset conditions in the MIMO case is far from trivial.

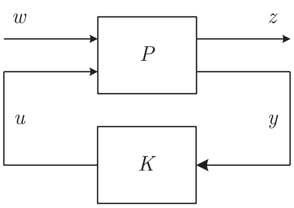

Figure 1. General multichannel closed-loop system.

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3.1. Reset controller

The reset controller K is modeled as a linear controller, which resets whenever its input y and output u satisfy a specific condition. To be more precise, the controller is described by

$$\begin{aligned} \dot{x}_k &= A_K x_k + B_K y \quad &\text{if } (y,u)\in\mathfrak{C}' \\ x_k^+ &= A_r x_k \quad &\text{if } (y,u)\in\mathfrak{D}' \\ u &= C_K x_k + D_K y \end{aligned} \tag{2}$$

where $x_k(t)\in\mathbb{R}^{n_k}$ is the controller state at time $t\in\mathbb{R}$ and A_K, B_K, C_K, D_K are matrices of appropriate dimensions. The superscript $+$ in x_k^+ is used to denote the time instant directly after a reset, since a reset occurs instantaneously, i.e. $x_k^+ := \lim_{\tau\to t,\tau>t} x_k(\tau)$. Given (1) and (2), the closed-loop state becomes $x=(x_p, x_k)$, where $x\in\mathbb{R}^n$ and $n=n_p+n_k$. The reset controller can be considered as a hybrid system with a *flow set* $\mathfrak{C}'$ and a *reset set* $\mathfrak{D}'$ using the framework in [17]. As long as $(y,u)\in\mathfrak{C}'$ the controller behaves linearly and its output u flows conform (A_K, B_K, C_K, D_K). Loosely speaking, when $(y,u)\in\mathfrak{D}'$ the state is changed instantaneously from x_k to x_k^+ by the discrete map corresponding to $A_r\in\mathbb{R}^{n_k\times n_k}$. Various choices for A_r are theoretically possible, but a reasonable and appropriate choice, commonly used in the literature, is

$$A_r = \begin{bmatrix} I_{n_k-n_r} & 0 \\ 0 & 0_{n_r} \end{bmatrix}$$

stating that the last n_r of the n_k controller states are reset to zero, while the others remain unchanged. The reset set $\mathfrak{D}'$ can be defined in various ways, but here we follow [17, 18], where resets occur whenever the controller input and output have opposite signs, i.e. when $yu\leqslant 0$. Compared to [16], where the reset set was given by $e=0$, the choice $yu\leqslant 0$ reduces the size of the flow set and allows a considerable relaxation of the stability and performance conditions as we will show below. As mentioned before, the formulation of suitable reset conditions becomes much more complicated for MIMO systems, where $y\in\mathbb{R}^{n_y}$, $u\in\mathbb{R}^{n_u}$ and possibly $n_y\neq n_u$. In that case there are many possible reset conditions, and at present time it is not clear which choices are meaningful. Therefore, only SISO controllers are considered and the controller flows whenever $y\geqslant 0$, $u\geqslant 0$ or $y\leqslant 0$, $u\leqslant 0$, which means that

$$\mathfrak{C}' := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in\mathbb{R}^2 : E_f \begin{bmatrix} y \\ u \end{bmatrix} \geqslant 0 \quad \text{or} \quad E_f \begin{bmatrix} y \\ u \end{bmatrix} \leqslant 0 \right\} \tag{3a}$$

$$\mathfrak{D}' := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in\mathbb{R}^2 : E_R \begin{bmatrix} y \\ u \end{bmatrix} \geqslant 0 \quad \text{or} \quad E_R \begin{bmatrix} y \\ u \end{bmatrix} \leqslant 0 \right\} \tag{3b}$$

where

$$E_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E_R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The flow set (3a) and reset set (3b) can also be expressed in terms of x and w. Therefore we introduce a transformation matrix T

$$\begin{bmatrix} y \\ u \end{bmatrix} = T \begin{bmatrix} x \\ w \end{bmatrix} = \left[\begin{array}{c|c} T_{yx} & T_{yw} \\ T_{ux} & T_{uw} \end{array}\right] \begin{bmatrix} x_p \\ x_k \\ \hline w \end{bmatrix}$$

where

$$\left[\begin{array}{c|c} T_{yx} & T_{yw} \\ \hline T_{ux} & T_{uw} \end{array}\right] = \left[\begin{array}{cc|c} C & 0 & D_w \\ \hline D_K C & C_K & D_K D_w \end{array}\right]$$

such that

$$\mathfrak{C} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in\mathbb{R}^{n+n_w} : E_f T \begin{bmatrix} x \\ w \end{bmatrix} \geqslant 0 \quad \text{or} \quad E_f T \begin{bmatrix} x \\ w \end{bmatrix} \leqslant 0 \right\} \tag{4a}$$

$$\mathfrak{D} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in\mathbb{R}^{n+n_w} : E_R T \begin{bmatrix} x \\ w \end{bmatrix} \geqslant 0 \quad \text{or} \quad E_R T \begin{bmatrix} x \\ w \end{bmatrix} \leqslant 0 \right\} \tag{4b}$$

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For later reference, we also introduce the decomposition of T as

$$T=\begin{bmatrix} T_x & T_w \end{bmatrix}=\left[\begin{array}{c|c} T_{yx} & T_{yw} \\ T_{ux} & T_{uw} \end{array}\right]$$

The sets $\mathfrak{C}$ and $\mathfrak{D}$ depend on the input w when $D_w \neq 0$, which is a case not considered in [17, 18]. However, this situation is of importance, as typically in tracking problems the case $D_w \neq 0$ occurs. Indeed, consider the problem depicted in Figure 2, where P denotes a dynamical system with input u and output y_p, and K denotes the controller. The signals r and e are the reference signal and the tracking error, respectively. In this case $w=r$, $y=z=e$, and $D_w=1\neq 0$, due to the direct feedthrough of r in e. This means that the flow and reset regions $\mathfrak{C}$ and $\mathfrak{D}$ in (4) explicitly depend on w. Hence, the input r clearly influences the reset moment, since resets are defined to occur at sign changes of u and e. In the analysis of [17, 18] this dependence on w was omitted, since $\mathfrak{C}$ and $\mathfrak{D}$ are only defined in terms of x (e.g. in [17, 18] $\mathfrak{C}:=\{x\in\mathbb{R}^n : x^{\mathrm{T}}Mx \geqslant 0\}$). Therefore, the results from [17, 18] are not applicable for tracking problems, but for input disturbance rejection type of problems only. Although tracking problems are considered in [20], they are restricted to cases with a constant reference signal and a reset controller of type FORE only, whereas our result can handle any reference signal and general reset controllers.

3.2. Closed-loop dynamics

The dynamics of the augmented plant and the reset controller can be combined into one description for the closed-loop dynamics Σ given by

$$\Sigma: \begin{cases} \dot{x} = \mathscr{A}x+\mathscr{B}w & \text{if } (x,w)\in\mathfrak{C} \\ x^{+} = \mathscr{A}_R x & \text{if } (x,w)\in\mathfrak{D} \\ z = \mathscr{C}x+\mathscr{D}w \end{cases} \tag{5}$$

where

$$\left[\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \end{array}\right] = \left[\begin{array}{cc|c} A+BD_KC & BC_K & B_w+BD_KD_w \\ B_KC & A_K & B_KD_w \\ \hline C_z+D_zD_KC & D_zC_K & D_{zw}+D_zD_KD_w \end{array}\right], \quad \mathscr{A}_R=\begin{bmatrix} I_{n_p} & 0 \\ 0 & A_r \end{bmatrix}$$

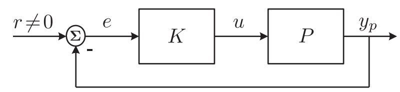

Figure 2. Simple tracking problem.

The linear closed-loop system without resets is given by

$$\begin{aligned} \dot{x} &= \mathscr{A}x+\mathscr{B}w \\ z &= \mathscr{C}x+\mathscr{D}w \end{aligned} \tag{6}$$

and is called the *base linear system*.

The reset controller will be chosen such that multiple resets at one point in time are excluded, in order to guarantee local existence of solutions. For convenience, we consider inputs that belong to the set of real Bohl functions, denoted by $\mathbb{B}$. A function $w:\mathbb{R}\to\mathbb{R}^{n_w}$ is called a real Bohl function if $w(t)=H\mathrm{e}^{Ft}v$ for all $t\in\mathbb{R}$ for some matrices $H\in\mathbb{R}^{n_w\times n_F}$, $F\in\mathbb{R}^{n_F\times n_F}$, and a vector $v\in\mathbb{R}^{n_F}$. The order o_w of a Bohl function $w(t)$ is defined as the minimum value of n_F for which there exist $H\in\mathbb{R}^{n_w\times n_F}$, $F\in\mathbb{R}^{n_F\times n_F}$, and a vector $v\in\mathbb{R}^{n_F}$ such that $w(t)=H\mathrm{e}^{Ft}v$ for all $t\in\mathbb{R}$. To guarantee local solutions, we assume that the closed-loop system can flow after each reset on at least a non trivial time interval. The following assumption formalizes this statement.

Assumption 1

The system (5) is such that for all signals w of interest

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}\in\mathfrak{D} \Rightarrow x^{+}=\mathscr{A}_R x\in\mathscr{F}_{\mathfrak{C}}(w) \tag{7}$$

where $\mathscr{F}_{\mathfrak{C}}(w)$ is given by

$$\begin{aligned} \mathscr{F}_{\mathfrak{C}}(w) := \{x_0\in\mathbb{R}^n \mid \exists\varepsilon>0 \\ \forall\tau\in[0,\varepsilon), (x(\tau,x_0,w), w(\tau))\in\mathfrak{C}\} \end{aligned} \tag{8}$$

where $x(\tau,x_0,w)$ denotes the state trajectory of (5) at time τ with initial state x_0 and input w.

This assumption implies that smooth continuation is possible after a reset from the state x^{+}. The following lemma characterizes the smooth continuation set

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$\mathscr{F}_{\mathfrak{C}}(w)$ using lexicographic orderings, which can be used to verify (7).

Lemma 2
Given the system (5), the set $\mathscr{F}_{\mathfrak{C}}(w)$ as defined in (8), can be characterized for $w \in \mathbb{B}$ by

$$\mathscr{F}_{\mathfrak{C}}(w)=\left\{\begin{array}{ll} x_0\in\mathbb{R}^n: & \left(\begin{bmatrix}\mathscr{O}_y & \mathscr{T}_y\end{bmatrix}\begin{bmatrix}x_0\\ \mathbf{w}_0\end{bmatrix}\succcurlyeq_\ell 0 \text{ and } \begin{bmatrix}\mathscr{O}_u & \mathscr{T}_u\end{bmatrix}\begin{bmatrix}x_0\\ \mathbf{w}_0\end{bmatrix}\succcurlyeq_\ell 0\right) \\ \text{or} & \left(\begin{bmatrix}\mathscr{O}_y & \mathscr{T}_y\end{bmatrix}\begin{bmatrix}x_0\\ \mathbf{w}_0\end{bmatrix}\preccurlyeq_\ell 0 \text{ and } \begin{bmatrix}\mathscr{O}_u & \mathscr{T}_u\end{bmatrix}\begin{bmatrix}x_0\\ \mathbf{w}_0\end{bmatrix}\preccurlyeq_\ell 0\right)\end{array}\right\} \quad (9)$$

where

$$\mathbf{w}_0=\begin{bmatrix} w(0)\\ \dot{w}(0)\\ \ddot{w}(0)\\ \vdots\\ w^{n+o_w-1}(0)\end{bmatrix},\quad \mathscr{O}_y=\begin{bmatrix} T_{yx}\\ T_{yx}\mathscr{A}\\ T_{yx}\mathscr{A}^2\\ \vdots\\ T_{yx}\mathscr{A}^{n+o_w-1}\end{bmatrix},\quad \mathscr{O}_u=\begin{bmatrix} T_{ux}\\ T_{ux}\mathscr{A}\\ T_{ux}\mathscr{A}^2\\ \vdots\\ T_{ux}\mathscr{A}^{n+o_w-1}\end{bmatrix}$$

$$\mathbf{T}_y=\begin{bmatrix} T_{yw} & 0 & 0 & \cdots & 0\\ T_{yw}\mathscr{B} & T_{yw} & \ddots & \ddots & \vdots\\ T_{yw}\mathscr{A}\mathscr{B} & T_{yw}\mathscr{B} & T_{yw} & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ T_{yw}\mathscr{A}^{n+o_w-2}\mathscr{B} & \cdots & T_{yw}\mathscr{A}\mathscr{B} & T_{yw}\mathscr{B} & T_{yw}\end{bmatrix}$$

$$\mathbf{T}_u=\begin{bmatrix} T_{uw} & 0 & 0 & \cdots & 0\\ T_{uw}\mathscr{B} & T_{uw} & \ddots & \ddots & \vdots\\ T_{uw}\mathscr{A}\mathscr{B} & T_{uw}\mathscr{B} & T_{uw} & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ T_{uw}\mathscr{A}^{n+o_w-2}\mathscr{B} & \cdots & T_{uw}\mathscr{A}\mathscr{B} & T_{uw}\mathscr{B} & T_{uw}\end{bmatrix}$$

Proof
The proof is based on inspecting the values of y and u and their derivatives at time $\tau=0$, and combining them with (4a). Utilizing the fact that the functions y and u are Bohl functions of order at most $n+o_w$, the Cayley-Hamilton theorem is used to obtain a finite characterization involving only the first $(n+o_w-1)$ derivatives (see [21] for more details). □

Note that during implementation of the reset controller reset takes precedence over flow. Under Assumption 1 local existence of solutions is guaranteed.

However, in principle the reset times may accumulate (the so-called Zeno behavior in hybrid systems theory). Therefore, we assume that either Zenoness is absent or otherwise, the solution can be continued beyond the accumulation point in such a way that global existence of solutions is guaranteed.

3.3. Preliminaries

In the remainder of this paper, by asymptotic stability of system (5) we mean asymptotic stability of the zero-input system, i.e. when $w=0$. Before we state our main results, we need the following definitions. Let $\|v\|_2$ denote the 2-norm of a signal v, defined by the square root of

$$\|v\|_2^2=\int_0^\infty v^{\mathrm{T}}(t)v(t)\,\mathrm{d}t \tag{10}$$

We say that $v\in\mathscr{L}_2$ if (10) exists and is finite.

Definition 3
The $\mathscr{L}_2$ gain $\|\Sigma\|_\infty$ of Σ in (5) is defined as the square root of

$$\|\Sigma\|_\infty^2=\sup_{0<\|w\|_2<\infty}\frac{\|z\|_2^2}{\|w\|_2^2} \tag{11}$$

where z is the output of system (5) for initial state $x(0)=0$ and input $w\in\mathscr{L}_2$.

It is well known that one of the possible interpretations of the $\mathscr{H}_2$ norm for linear systems is the total energy content of the output z due to an impulsive input w. The response of a linear system to such an impulsive input is equivalent to the response when the system is subjected to the initial condition $x_0=\mathscr{B}_j$ or e.g. $x_0=\mathscr{B}_i+\mathscr{B}_j$ (with $i\neq j$), corresponding to simultaneous impulses on both channels i and j, where $\mathscr{B}_j$ denotes the jth column of $\mathscr{B}$ corresponding to the jth input. In this paper we use the latter interpretation that corresponds to $w=0$, as this transforms to the context of reset control in a straightforward manner.

Definition 4
The squared $\mathscr{H}_2$ norm of reset control system (5) corresponding to an initial value $x_0\in\mathbb{R}^n$ is equal to the squared total output energy and is defined as

$$\|\Sigma\|_{2,x_0}^2=\int_0^\infty z(t)^{\mathrm{T}}z(t)\,\mathrm{d}t \tag{12}$$

where z denotes the output trajectory corresponding to (5) and initial state x_0 (and $w=0$).

Definition 5 ([22])
System (5) with state $x\in\mathbb{R}^n$, input $w\in\mathbb{R}^{n_w}$ and output $z\in\mathbb{R}^{n_z}$ is dissipative w.r.t. a supply function $s:\mathbb{R}^{n_w}\times\mathbb{R}^{n_z}\to\mathbb{R}$ if there exists a positive definite continuous function $V:\mathbb{R}^n\to\mathbb{R}$, sometimes called a storage function, such that $V(0)=0$ and

$$V(x(t_1))-V(x(t_0))\leqslant\int_{t_0}^{t_1}s(w(t),z(t))\,\mathrm{d}t \tag{13}$$

for all $t_1\geqslant t_0$ and all tuples (w,x,z) that satisfy (5).

The following lemma will be of use in the sequel.

Lemma 6
System (5) has a finite $\mathscr{L}_2$ gain from input w to output z smaller than or equal to γ, if system (5) is dissipative w.r.t. the supply function

$$s(w,z)=\gamma^2w^{\mathrm{T}}w-z^{\mathrm{T}}z \tag{14}$$

Proof
If system (5) is dissipative with respect to the supply function (14), then Definition 5 implies that

$$V(x(t_1))-V(x(t_0))\leqslant\int_{t_0}^{t_1}(\gamma^2w^{\mathrm{T}}(t)w(t)-z^{\mathrm{T}}(t)z(t))\,\mathrm{d}t$$

for all $t_1\geqslant t_0$ and all tuples (w,x,z) that satisfy (5). Since $V(0)=0$, taking $t_0=0$ with initial condition $x(0)=0$ in (15) and using that $V(x(t_1))\geqslant 0$, yield

$$\int_0^{t_1}(\gamma^2w^{\mathrm{T}}(t)w(t)-z^{\mathrm{T}}(t)z(t))\,\mathrm{d}t\geqslant 0$$

Using $w\in\mathscr{L}_2$ and letting $t_1\to\infty$ we obtain that

$$\gamma^2\|w\|_2^2-\|z\|_2^2\geqslant 0$$

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for all $w \in \mathscr{L}_2$, or equivalently

$$\sup_{0<\|w\|_2<\infty} \frac{\|z\|_2}{\|w\|_2} \leqslant \gamma$$

which completes the proof. □

4. $\mathscr{L}_2$ GAIN PERFORMANCE ANALYSIS OF RESET CONTROL SYSTEMS

In this section we present a general LMI-based analysis method to determine an upperbound on the $\mathscr{L}_2$ gain performance of a reset control system. These computable and sufficient conditions for $\mathscr{L}_2$ stability, based on PWQ Lyapunov functions, are suitable for all LTI plants and linear-based reset controllers, thereby generalizing the available results in the literature as mentioned in the introduction. Moreover, the results extend the existing literature by providing solutions to arbitrary tracking and measurement noise problems.

4.1. Common Lyapunov function

In order to approximate the $\mathscr{L}_2$ gain of a reset control system we apply Lemma 6 to the closed-loop system (5) by selecting a certain structure for the storage function V. The minimal value of γ for which (13) holds with supply function (14) depends on the particular structure chosen for V. However, at this point, motivated by the linear behavior of the closed-loop in a large part of the state-space, we first restrict our attention to quadratic Lyapunov functions of the form $V(x) = x^{\mathrm{T}} P x$. Using this structure, the following result is obtained.

Theorem 7

The reset control system (5) is globally asymptotically stable with an $\mathscr{L}_2$ gain $\|\Sigma\|_\infty \leqslant \gamma$ if there exist $P \in \mathbb{S}^{n\times n} \succ 0$ and $U, U_R \in \mathbb{S}_+^{2\times 2}$ such that

$$\begin{bmatrix} \mathscr{A}^T P + P\mathscr{A} + T_x^{\mathrm{T}} E_f^{\mathrm{T}} U E_f T_x & P\mathscr{B} + T_x^{\mathrm{T}} E_f^{\mathrm{T}} U E_f T_w & \mathscr{C}^{\mathrm{T}} \\ \mathscr{B}^{\mathrm{T}} P + T_w^{\mathrm{T}} E_f^{\mathrm{T}} U E_f T_x & -\gamma I + T_w^{\mathrm{T}} E_f^{\mathrm{T}} U E_f T_w & \mathscr{D}^{\mathrm{T}} \\ \mathscr{C} & \mathscr{D} & -\gamma I \end{bmatrix} \prec 0 \quad (16a)$$

$$\begin{bmatrix} \mathscr{A}_R^{\mathrm{T}} P \mathscr{A}_R - P + T_x^{\mathrm{T}} E_R^{\mathrm{T}} U_R E_R T_x & T_x^{\mathrm{T}} E_R^{\mathrm{T}} U_R E_R T_w \\ T_w^{\mathrm{T}} E_R^{\mathrm{T}} U_R E_R T_x & T_w^{\mathrm{T}} E_R^{\mathrm{T}} U_R E_R T_w \end{bmatrix} \preccurlyeq 0 \quad (16b)$$

Proof

First of all, note that the storage function $V(x) = x^{\mathrm{T}} P x$ is continuously differentiable. Moreover, since $P \succ 0$, $V(x) > 0$ for $x \neq 0$ and thus, V is positive definite. Furthermore,

$$\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{C} \Rightarrow \begin{bmatrix} x \\ w \end{bmatrix}^{\mathrm{T}} T^{\mathrm{T}} E_f^{\mathrm{T}} U E_f T \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x \\ w \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} T_x^{\mathrm{T}} E_f^{\mathrm{T}} U E_f T_x & T_x^{\mathrm{T}} E_f^{\mathrm{T}} U E_f T_w \\ T_w^{\mathrm{T}} E_f^{\mathrm{T}} U E_f T_x & T_w^{\mathrm{T}} E_f^{\mathrm{T}} U E_f T_w \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \geqslant 0 \quad (17a)$$

$$\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{D} \Rightarrow \begin{bmatrix} x \\ w \end{bmatrix}^{\mathrm{T}} T^T E_R^T U_R E_R T \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x \\ w \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} T_x^{\mathrm{T}} E_R^{\mathrm{T}} U_R E_R T_x & T_x^{\mathrm{T}} E_R^{\mathrm{T}} U_R E_R T_w \\ T_w^{\mathrm{T}} E_R^{\mathrm{T}} U_R E_R T_x & T_w^{\mathrm{T}} E_R^{\mathrm{T}} U_R E_R T_w \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \geqslant 0 \quad (17b)$$

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since U, $U_R \in \mathbb{S}_+^{2\times 2}$ only have nonnegative elements. Since (16a) is strict, we have that $\mathscr{A}^\mathrm{T}P+P\mathscr{A}+T_x^\mathrm{T}E_f^\mathrm{T}UE_fT_x^\mathrm{T}\prec 0$ and thus, since $V(x)=x^\mathrm{T}Px$, we have that

$$\left\langle \frac{\partial V}{\partial x}, \mathscr{A}x \right\rangle < -\varepsilon V \quad \text{if } \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{C}, \quad x\neq 0 \tag{18}$$

for some $\varepsilon>0$. Combining (17b) with (16b) gives

$$x^\mathrm{T}(\mathscr{A}_R^\mathrm{T}P\mathscr{A}_R-P)x\leqslant 0 \quad \text{if } \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{D} \tag{19}$$

and thus, since $V(x)=x^\mathrm{T}Px$, we have that

$$V(x^+)\leqslant V(x) \quad \text{when } \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{D} \tag{20}$$

Continuous differentiability and positive definiteness of V together with (18) and (20), and the fact that V is radially unbounded imply that system (5) with input $w=0$ is globally asymptotically stable [23].

To prove that $\|\Sigma\|_\infty\leqslant\gamma$ we will show that the inequalities in (16) imply, for $V(x)=x^TPx$ and $s(w,z)=\gamma^2w^Tw-z^Tz$, that

$$\left\langle \frac{\partial V}{\partial x}, \mathscr{A}x+\mathscr{B}w \right\rangle \leqslant s(w,z) \quad \text{when } \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{C} \tag{21}$$

Indeed, if (21) holds, then, since (20) also holds, we have that for all $t_1\geqslant t_0$

$$V(x(t_1))-V(x(t_0))\leqslant \int_{t_0}^{t_1} s(w(t),z(t))\,\mathrm{d}t \tag{22}$$

showing that (5) is dissipative with respect to the supply rate $s(w,z)$. Invoking Lemma 6 would then show that (5) has an $\mathscr{L}_2$ gain $\|\Sigma\|_\infty\leqslant\gamma$.

To show that (21) holds, combine (17a) with the Schur complement of (16a) and employ the S-procedure, to derive

$$\begin{bmatrix} x \\ w \end{bmatrix}^\mathrm{T} \begin{bmatrix} \mathscr{A}^\mathrm{T}P+P\mathscr{A}+\mathscr{C}^\mathrm{T}\mathscr{C} & P\mathscr{B}+\mathscr{C}^\mathrm{T}\mathscr{D} \\ \mathscr{B}^\mathrm{T}P+\mathscr{D}^\mathrm{T}\mathscr{C} & \mathscr{D}^\mathrm{T}\mathscr{D}-\gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0 \quad \text{if } \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{C}, \quad x\neq 0 \tag{23}$$

Since $V(x)=x^\mathrm{T}Px$, (23) is just a reformulation of (21), which completes the proof. □

4.2. PWQ Lyapunov function

As stated in the previous section, the minimal value of γ for which (13) holds depends on the chosen structure of V, i.e. the value of γ will in general only be an upper-bound on the actual value of $\|\Sigma\|_\infty$ when imposing a structure on V. Therefore, the result in Theorem 7 can be conservative. Since we are dealing with a nonlinear closed-loop system, the V that yields the smallest value of γ might be a very complicated function. For the sake of computability we choose to use *PWQ storage functions* [18, 19]. This choice is motivated by the flexibility of PWQ functions, since they can approximate many functions arbitrarily close by using an increasing number of regions, while still resulting in LMIs to check the dissipativity inequality (13). These LMIs have the advantage that they can be efficiently solved. Nevertheless, some conservatism remains given a finite number of regions, which implies that the obtained minimal value of γ will only be an upperbound on the actual $\mathscr{L}_2$ gain.

The PWQ storage functions are obtained by partitioning the flow set $\mathfrak{C}'$ into smaller regions $\mathfrak{C}'_i$ and assigning a different quadratic storage function $V_i(x)=x^\mathrm{T}P_ix$ to each of them [18]. There are many ways the flow set can be partitioned. Here we choose the partitioning as depicted in Figure 3. The angles θ_i and θ_{i-1} uniquely define two lines,

$$u\cos(\theta_{i-1})=y\sin(\theta_{i-1}), \quad u\cos(\theta_i)=y\sin(\theta_i)$$

which bound each region $\mathfrak{C}'_i$ as

$$\left.\begin{aligned} u&\geqslant\frac{\sin(\theta_{i-1})}{\cos(\theta_{i-1})}y \\ u&\leqslant\frac{\sin(\theta_i)}{\cos(\theta_i)}y \end{aligned}\right\} \Rightarrow \begin{bmatrix} -\sin(\theta_{i-1}) & \cos(\theta_{i-1}) \\ \sin(\theta_i) & -\cos(\theta_i) \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = E_i \begin{bmatrix} y \\ u \end{bmatrix} \geqslant 0 \tag{24}$$

These angles should be chosen such that $0=\theta_0<\theta_1<\ldots<\theta_N=\pi/2$. Here we choose to distribute θ_i equidistantly, so $\theta_i=i/N\pi/2$, where $i=0,\ldots,N$

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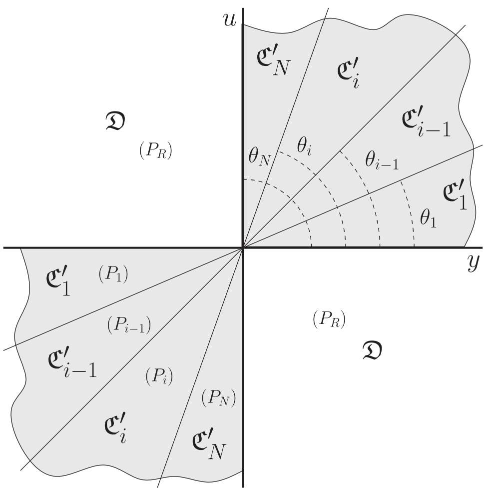

Figure 3. Partitioning of the (y, u)-space.

and N is the number of regions. Using the coordinate transformation matrix T, we can now define regions $\mathfrak{C}_i$ and $\mathfrak{D}$ as

$$\mathfrak{C}_i := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{n+n_w} : E_i T \begin{bmatrix} x \\ w \end{bmatrix} \geqslant 0 \text{ or } E_i T \begin{bmatrix} x \\ w \end{bmatrix} \leqslant 0 \right\} \tag{25a}$$

$$\mathfrak{D} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{n+n_w} : E_R T \begin{bmatrix} x \\ w \end{bmatrix} \geqslant 0 \text{ or } E_R T \begin{bmatrix} x \\ w \end{bmatrix} \leqslant 0 \right\} \tag{25b}$$

where

$$E_i = \begin{bmatrix} -\sin(\theta_{i-1}) & \cos(\theta_{i-1}) \\ \sin(\theta_i) & -\cos(\theta_i) \end{bmatrix}, \quad E_R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using a symmetric matrix $U_i \in \mathbb{S}_+$, a region can be defined with the following quadratic form:

$$\begin{bmatrix} x \\ w \end{bmatrix}^{\mathrm{T}} T^{\mathrm{T}} E_i^{\mathrm{T}} U_i E_i T \begin{bmatrix} x \\ w \end{bmatrix} \geqslant 0$$

Note that, due to the positive elements of U_i, this region always includes the region $\mathfrak{C}_i$ in the sense that

$$\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{C}_i \Rightarrow \begin{bmatrix} x \\ w \end{bmatrix}^{\mathrm{T}} T^{\mathrm{T}} E_i^{\mathrm{T}} U_i E_i T \begin{bmatrix} x \\ w \end{bmatrix} \geqslant 0 \tag{26}$$

A similar argument holds for the region $\mathfrak{D}$. Moreover, using $T = [T_x | T_w]$, we introduce

$$E_i T = [E_i T_x \;\; E_i T_w] = [E_{x,i} \;\; E_{w,i}] \tag{27a}$$

$$E_R T = [E_R T_x \;\; E_R T_w] = [E_{x,R} \;\; E_{w,R}] \tag{27b}$$

The boundaries of the regions are defined by

$$[-\sin(\theta_i) \;\; \cos(\theta_i)] T \begin{bmatrix} x \\ w \end{bmatrix} = \Phi_i \begin{bmatrix} x \\ w \end{bmatrix} = 0 \tag{28}$$

whose solutions are in the kernel of Φ_i. We can also use an image representation for these boundaries using full column rank matrices $W_{\Phi i} \in \mathbb{R}^{(n+n_w)\times(n+n_w-1)}$ such that $\mathrm{im}(W_{\Phi i}) = \ker(\Phi_i)$, where $\mathrm{im}(W_{\Phi i})$ denotes the image of $W_{\Phi i}$.

In some situations however, such as tracking problems as in Figure 2, the introduction of continuous PWQ storage functions does not result in less conservatism due to the fact that $D_w \neq 0$ in those situations. Namely, each region $\mathfrak{C}_i$ has its own Lyapunov function V_i, which solely depends on x, while (25) shows that the region itself is defined in terms of both x and w. Figure 4 illustrates this for the simple case where $x, w \in \mathbb{R}$. Since V only depends on x (depicted by the dashed vertical lines), continuity of V across the border between Ω_1 and Ω_2 requires that $V_1(*) = V_2(*)$ for all (x, w) on the boundary, i.e. for all x. Hence, in such situations the continuity requirement causes the PWQ function to reduce to a quadratic one. This issue arises in any situation where $D_w \neq 0$, including measurement noise attenuation and tracking problems. This drawback can be avoided by forcing $D_w = 0$, which can be done by including strictly proper input filters for exogenous signals that enter the closed-loop before the controller. This is illustrated in Figure 5, where we have exogenous signals $\bar{r}$, $\bar{\eta}$, replacing the original reference r and noise η. Since these strictly proper filters have no direct feedthrough of the input, there is also no direct feedthrough from w (containing $\bar{r}$ and $\bar{\eta}$) to y in

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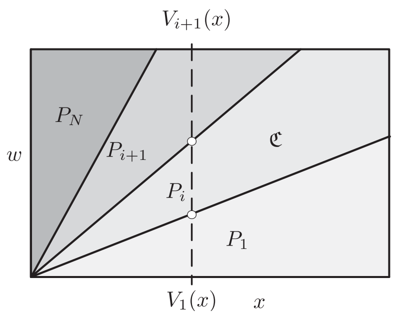

Figure 4. Common Lyapunov function.

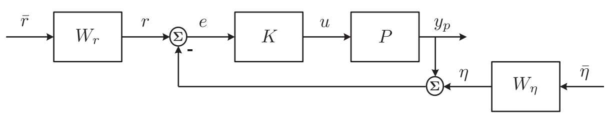

Figure 5. Closed-loop with filtered inputs.

the augmented plant in (1), so $D_w = 0$. By including input filters in the augmented plant we assume to have a priori knowledge of these inputs (as is often the case in practice), which is then via the filter states contained inside the state vector x_p. Possible input filters include:

- unit step: $W(s) = \frac{1}{(s+\varepsilon)}$,
- unit ramp: $W(s) = \frac{1}{(s+\varepsilon)^2}$,
- sine wave with frequency ω: $W(s) = \frac{\omega}{(s+\varepsilon)^2+\omega^2}$,

where s is the Laplace variable and $\varepsilon > 0$ is a small offset. This offset is standard in $\mathscr{H}_\infty$ and $\mathscr{H}_2$ problems for linear systems to ensure closed-loop stability, see e.g. [24]. Using these filters, the Lyapunov function V also depends on the input knowledge (as reflected in the filter state variables corresponding to w that are now included in x_p and thus in x), while $\mathfrak{C}_i$ and $\mathfrak{D}$ no longer depend on w, i.e.

$$\mathfrak{C}_i := \{x \in \mathbb{R}^n : E_i T_x x \geqslant 0 \quad \text{or} \quad E_i T_x x \leqslant 0\} \tag{29a}$$

$$\mathfrak{D} := \{x \in \mathbb{R}^n : E_R T_x x \geqslant 0 \quad \text{or} \quad E_R T_x x \leqslant 0\} \tag{29b}$$

Using this partitioning we can formulate the following result on the calculation of an upperbound on the $\mathscr{L}_2$ gain including also tracking problems using input filtering. This result adopts PWQ storage functions.

Theorem 8

The reset control system (5) with $D_w = 0$ and a partitioning of the flow set given by (29) is globally asymptotically stable with an $\mathscr{L}_2$ gain $\|\Sigma\|_\infty \leqslant \gamma$ if, for a given N, there exists $P_i, P_R \in \mathbb{S}^{n\times n}$ and $U_i, U_{R_0}, U_{R_i}, U_{RI_i}, V_i, V_R \in \mathbb{S}_+^{2\times 2}$, $i = 1, \ldots, N$ such that

$$\begin{bmatrix} \mathscr{A}^{\mathrm{T}} P_i + P_i \mathscr{A} + E_{x,i}^{\mathrm{T}} U_i E_{x,i} & P_i \mathscr{B} & \mathscr{C}^{\mathrm{T}} \\ \mathscr{B}^{\mathrm{T}} P_i & -\gamma I & \mathscr{D}^{\mathrm{T}} \\ \mathscr{C} & \mathscr{D} & -\gamma I \end{bmatrix} \prec 0$$

$$i = 1, \ldots, N \tag{30a}$$

$$\mathscr{A}_R^{\mathrm{T}} P_R \mathscr{A}_R - P_R + E_{x,R}^{\mathrm{T}} U_{R_0} E_{x,R} \preccurlyeq 0 \tag{30b}$$

$$\mathscr{A}_R^{\mathrm{T}} P_i \mathscr{A}_R - P_R + E_{x,R}^{\mathrm{T}} U_{R_i} E_{x,R} + \mathscr{A}_R^{\mathrm{T}} E_{x,i}^{\mathrm{T}} U_{RI_i} E_{x,i} \mathscr{A}_R \preccurlyeq 0, \quad i = 1, \ldots, N \tag{30c}$$

$$P_i - E_{x,i}^{\mathrm{T}} V_i E_{x,i} \succ 0, \quad i = 1, \ldots, N \tag{30d}$$

$$P_R - E_{x,R}^{\mathrm{T}} V_R E_{x,R} \succ 0 \tag{30e}$$

$$\overline{W}_{\Phi i_x}^{T} (P_i - P_{i+1}) \overline{W}_{\Phi i_x} = 0, \quad i = 1, \ldots, N-1 \tag{30f}$$

$$\overline{W}_{\Phi 0_x}^{\mathrm{T}} (P_R - P_1) \overline{W}_{\Phi 0_x} = 0 \tag{30g}$$

$$\overline{W}_{\Phi N_x}^{\mathrm{T}} (P_N - P_R) \overline{W}_{\Phi N_x} = 0 \tag{30h}$$

where $\mathrm{im}(\overline{W}_{\Phi i}) = \ker([-\sin(\theta_i)\ \cos(\theta_i)] T_x)$.

Proof

The proof is based on showing that hypotheses (30) imply that the storage function V (defined as $V(x) = V_i(x) := x^{\mathrm{T}} P_i x$ when $x \in \mathfrak{C}_i$ and $V(x) = x^{\mathrm{T}} P_R x$ when $x \in \mathfrak{D}$) is continuous, positive definite and radially unbounded, and that for $s(w, z) = \gamma^2 w^{\mathrm{T}} w - z^{\mathrm{T}} z$,

$$\left\langle \frac{\partial V_i}{\partial x}, \mathscr{A}x + \mathscr{B}w \right\rangle \leqslant s(w, z) \quad \text{if } x \in \mathfrak{C}_i \tag{31a}$$

$$V(x^+) - V(x) \leqslant 0 \quad \text{if } x \in \mathfrak{D} \tag{31b}$$

$$\frac{\partial V_i}{\partial x} \mathscr{A} x < -\varepsilon V_i \quad \text{if } x \in \mathfrak{C}_i, \quad x \neq 0 \tag{31c}$$

for some $\varepsilon > 0$. Indeed, continuity together with the positive definiteness and radially unboundedness of V and

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(31b) and (31c) imply global asymptotic stability [23] as V is a Lyapunov function. Moreover, if (31a) and (31b) hold, then for all $t_1 \geqslant t_0$

$$V(x(t_1)) - V(x(t_0)) \leqslant \int_{t_0}^{t_1} s(w(t), z(t))\,\mathrm{d}t \tag{32}$$

showing that (5) is dissipative w.r.t. the supply function $s(w,z) = \gamma^2 w^{\mathrm{T}} w - z^{\mathrm{T}} z$, and hence that $\|\Sigma\|_\infty \leqslant \gamma$ by virtue of Lemma 6.

Continuity of the PWQ function V follows from constraints (30f), (30g) and (30h). Furthermore, V is positive definite. To show this, note that since V_i, $V_R \in \mathbb{S}_+^{2\times 2}$ have only non-negative elements it holds that

$$x \in \mathfrak{C}_i \Rightarrow x^{\mathrm{T}} E_{x,i}^{\mathrm{T}} V_i E_{x,i} x \geqslant 0 \tag{33a}$$

$$x \in \mathfrak{D} \Rightarrow x^{\mathrm{T}} E_{x,R}^{\mathrm{T}} V_R E_{x,R} x \geqslant 0 \tag{33b}$$

Therefore, for $x \in \mathfrak{C}_i$ (hence not necessarily for all x) it holds that for $x \neq 0$

$$V(x) = x^{\mathrm{T}} P_i x \overset{(30\mathrm{d})}{>} x^{\mathrm{T}} E_{x,i}^{\mathrm{T}} V_i E_{x,i} x \geqslant 0 \tag{34}$$

The same applies for $x \in \mathfrak{D}$ using (30e). Combining (33a) with the Schur complement of (30a) via the S-procedure yields that for $x \in \mathfrak{C}_i$

$$\begin{bmatrix} x \\ w \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathscr{A}^{\mathrm{T}} P_i + P_i \mathscr{A} + \mathscr{C}^{\mathrm{T}} \mathscr{C} & P_i \mathscr{B} + \mathscr{C}^{\mathrm{T}} \mathscr{D} \\ \mathscr{B}^{\mathrm{T}} P_i + \mathscr{D}^{\mathrm{T}} \mathscr{C} & \mathscr{D}^{\mathrm{T}} \mathscr{D} - \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leqslant 0 \tag{35}$$

which is equivalent to (31a). Note that (30a) implicitly implies that $x^{\mathrm{T}}(\mathscr{A}^{\mathrm{T}} P_i + P_i \mathscr{A} + E_{x,i}^{\mathrm{T}} U_i E_{x,i})x < -\varepsilon x^{\mathrm{T}} P_i x$ for some $\varepsilon > 0$ which implies (31c).

It only remains to be shown that (31b) holds. The state x^+ after reset is either an element of $\mathfrak{D}$ or $\mathfrak{C}_i$. In case $x^+ \in \mathfrak{D}$, (30b) yields

$$x^{\mathrm{T}}(\mathscr{A}_R^{\mathrm{T}} P_R \mathscr{A}_R - P_R)x \leqslant 0 \quad \text{if } x, x^+ \in \mathfrak{D} \tag{36}$$

In case $x^+ \in \mathfrak{C}_i$, (30c) implies that

$$x^{\mathrm{T}}(\mathscr{A}_R^{\mathrm{T}} P_i \mathscr{A}_R - P_R)x \leqslant 0 \quad \text{if } x \in \mathfrak{D}, x^+ \in \mathfrak{C}_i \tag{37}$$

The combination of (36) and (37) is equivalent to (31b), which completes the proof. □

Theorem 8 extends the result in [18], which also applied PWQ functions in the context of $\mathscr{L}_2$ gain analysis. Our result is applicable to all possible LTI plants and reset controllers, which fit the $\mathscr{H}_\infty$ framework, as long as $D_w = 0$. In contrast to [18] it can cope with measurement noise attenuation and tracking problems, as long as these inputs are filtered with a strictly proper filter. Since the analysis in Theorem 8 encompasses a priori knowledge of the inputs, it can be performed for a broad class of desired input types (and corresponding input filters).

4.3. Example

In order to illustrate the relevance of the derived analysis method, the above result is applied to a simulation example. The system used in this example is taken from [14, 17], and extended with a tracking problem as depicted in Figure 2. Consider a second order LTI plant, represented by $G(s) = (s+1)/(s(s+0.2))$. This system should track a step reference $r(t) = 1(t)$, which can be represented by the input filter $W_r(s) = 1/(s+\varepsilon)$. The system and the filter form the augmented plant (1). Similarly as in [14, 17], we only allow first order low-pass controllers K to achieve this. The goal is to compare the tracking performance of a linear controller of the form $K(s) = 1/(s+\beta)$, where β is a free variable, with its resetting counterpart, also known as a FORE. This reset controller K is given by (2) with

$$A_K = \beta, \quad B_K = C_K = 1, \quad D_K = A_r = 0$$

yielding closed-loop matrices of the form

$$\mathscr{A} = \begin{bmatrix} A & B \\ C & \beta \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} B_w \\ 0 \end{bmatrix},$$

$$\mathscr{C} = [C \ \ 0], \quad \mathscr{D} = 0$$

The closed-loop layout is shown in Figure 6. The $\mathscr{L}_2$ gain of both the linear and the reset closed-loop are compared in Figure 7(a) for varying pole values β. The linear control curve is obtained using standard $\mathscr{H}_\infty$

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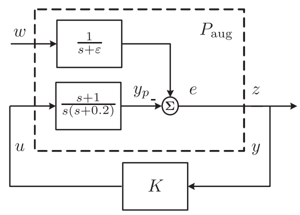

Figure 6. Closed-loop layout.

techniques, i.e. by minimizing γ in the LMI

$$\begin{bmatrix} \mathscr{A}^T P + P\mathscr{A} & P\mathscr{B} & \mathscr{C}^T \\ \mathscr{B}^T P & -\gamma I & \mathscr{D}^T \\ \mathscr{C} & \mathscr{D} & -\gamma I \end{bmatrix} \prec 0 \tag{38}$$

where $P \succ 0$. Theorem 8 is used for the reset control curve (with $N = 100$).

To verify the accuracy of the LMI analysis from Theorem 8, a describing function analysis of the closed-loop behavior is performed (a technique already used by Clegg [7]). Therefore, pure sinusoidal inputs w of various frequencies at each value of β are applied to the closed-loop, and the amplitude of the fundamental frequency present in the output z is identified. The peak amplitude over all frequencies is then used as an estimate $\gamma_{\min}$ of the $\mathscr{L}_2$ gain. Figure 7(b) shows this $\gamma_{\min}$ of the describing function. From this figure we can conclude that the describing function analysis matches our LMI $\mathscr{L}_2$ analysis very well, underlining the accuracy of Theorem 8. Figure 7(a) does not necessarily imply that reset control outperforms any linear controller in $\mathscr{H}_\infty$ sense. We have restricted ourselves to linear controllers K with transfer function $\frac{1}{s+\beta}$, but of course one can find other controllers with lower $\mathscr{L}_2$ gains (e.g. with a larger controller gain or of a higher order).

5. $\mathscr{H}_2$ PERFORMANCE ANALYSIS OF RESET CONTROL SYSTEMS

The focus in this section is to determine the performance of a SISO reset control system in $\mathscr{H}_2$ sense. We derive convex optimization problems in terms of LMIs to compute an upperbound on the $\mathscr{H}_2$ norm, using dissipativity theory with PWQ Lyapunov functions.

5.1. *System description*

Reset control has been shown to be advantageous when the transient response to specific input signals (such as step functions) is considered [5, 6]. Reset controllers show less overshoot than their linear counterparts and hence, are expected to have less energy in the closed-loop output, here being the tracking error. Therefore, in this section we do not consider the general system description as in Section 3 but we focus on the augmented layout depicted in Figure 8, consisting of various input filters, a linear plant P and a reset controller K. The goal is to calculate the total output energy of the unfiltered output z, consisting of the signals e (tracking error), u_p (control) and y_p (plant output) or a subset of these signals, subject to certain *specific* inputs, i.e. the signals r (reference), d (disturbance) and η (measurement noise) or a subset of these inputs. Again, we only consider SISO plants and controllers. The inputs are assumed to be known a priori, and this knowledge is captured in the input filters W_r, W_d and W_η, whose inputs are bundled in the exogenous input w. Possible input filters have already been discussed in Section 4.2. Note that when an impulse input is applied to the mentioned filters in Section 4.2, their outputs indeed approximate a step, ramp or sine wave, respectively. Hence, the total energy in z as a result of such specific signals in r, d or η gets arbitrarily close to the total output energy of the impulse response from w to z, which is an interpretation of the $\mathscr{H}_2$ norm. It is well known that the $\mathscr{H}_2$ norm of a linear system is bounded only if the corresponding closed-loop transfer matrix between w and z is strictly proper [25, 26]. As in the closed-loop system the transfer matrix between r and e is defined by the sensitivity function, which is biproper, a strictly proper filter W_r is needed to obtain a strictly proper transfer matrix from $\bar{r}$ to e. A similar reasoning can be applied to the other input signals, which implies that the following assumption is required to ensure a bounded $\mathscr{H}_2$ norm.

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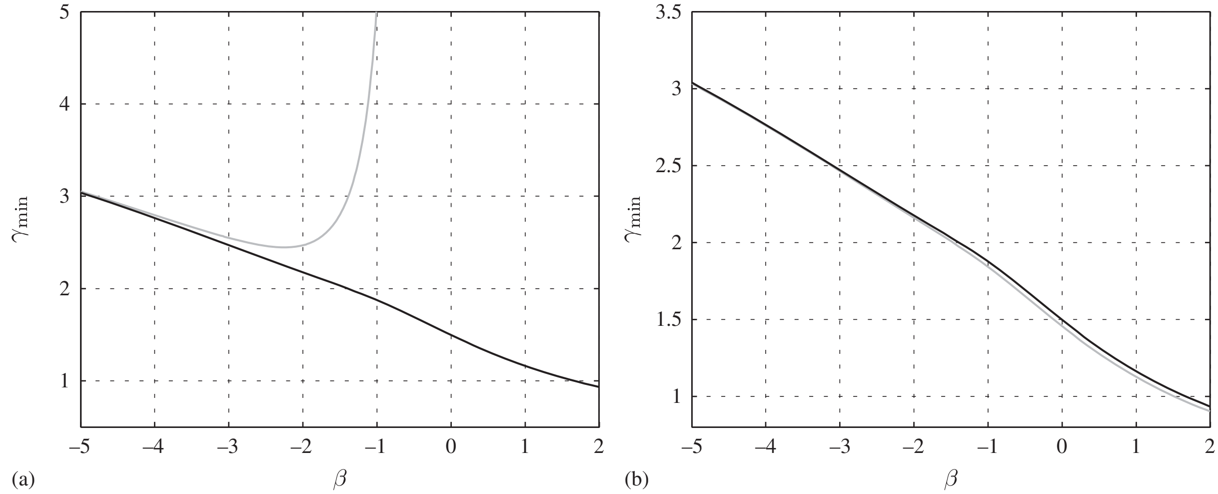

Figure 7. (a) Estimated closed-loop $\mathscr{L}_2$ gains using linear (grey) and reset control (black), as a function of β, (b) results using LMI analysis (black) and describing function analysis (grey).

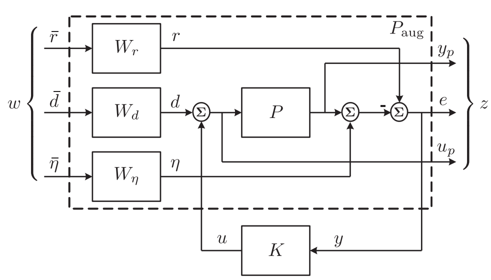

Figure 8. Closed-loop layout with input w and output z.

Assumption 9
The input filters W_r, W_d and W_η are all strictly proper.

The strict properness of these filters is quite natural in practice, as is also indicated by the three examples in Section 4.2. As discussed in Section 3.3, to compute the $\mathscr{H}_2$ norm we will use the initial condition setting and assume $w = 0$. However, in order to be able to select appropriate initial states $x_0 = \mathscr{B}_j$ later on, we will elaborate on the plant and controller dynamics without this assumption for now.

The dynamics of the reset controller is discussed in Section 3.1, but here we make a slight adjustment. The strict properness of the weighting functions implies that there is no direct feedthrough between the input w and the controller input signal e (i.e. $D_w = 0$) or between w and z (i.e. $D_{zw} = 0$). This simplifies (4a) and (4b) to

$$\mathfrak{C} := \{x \in \mathbb{R}^n : E_f T_x x \geqslant 0 \quad \text{or} \quad E_f T_x x \leqslant 0\} \tag{39a}$$

$$\mathfrak{D} := \{x \in \mathbb{R}^n : E_R T_x x \geqslant 0 \quad \text{or} \quad E_R T_x x \leqslant 0\} \tag{39b}$$

which now only depend on x.

We combine the augmented plant and the reset controller into one closed-loop system, described by Σ as

$$\Sigma : \begin{cases} \dot{x} = \mathscr{A} x + \mathscr{B} w & \text{if } x \in \mathfrak{C} \\ x^+ = \mathscr{A}_R x & \text{if } x \in \mathfrak{D} \\ z = \mathscr{C} x + \mathscr{D} w \end{cases} \tag{40}$$

where, using $D_w = 0$ and $D_{zw} = 0$,

$$\left[\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \end{array}\right] = \left[\begin{array}{cc|c} A + B D_K C & B C_K & B_w \\ B_K C & A_K & 0 \\ \hline C_z + D_z D_K C & D_z C_K & 0 \end{array}\right]$$

$$\mathscr{A}_R = \begin{bmatrix} I_{n_p} & 0 \\ 0 & A_r \end{bmatrix}$$

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At this point we return to the assumption that $w=0$ and consider non zero initial values of the input filters. Hence, we take $w=0$ and $x_0=\mathscr{B}_j$. Note that $x_0=\mathscr{B}_i+\mathscr{B}_j$ where $i\neq j$ is also a valid initial condition corresponding to simultaneous impulses on both channels i and j. Again, to guarantee local existence of solutions, we adopt Assumption 1, and Lemma 2 reduces to:

Lemma 10
Given system (40), the set $\mathscr{F}_{\mathfrak{C}}$ in Assumption 1 can be characterized by two lexicographic orderings, i.e.

$$\begin{aligned}\mathscr{F}_{\mathfrak{C}}=\{x_0\in\mathfrak{C}:&(E_fT_xx_0,E_fT_x\mathscr{A}x_0,\\ &E_fT_x\mathscr{A}^2x_0,\ldots,E_fT_x\mathscr{A}^{n-1}x_0)\geqslant_\ell 0\\ &\text{or }(E_fT_xx_0,E_fT_x\mathscr{A}x_0,\\ &E_fT_x\mathscr{A}^2x_0,\ldots,E_fT_x\mathscr{A}^{n-1}x_0)\leqslant_\ell 0\}\end{aligned}$$

We are now ready to present the main results on the LMI-based calculation of an upperbound on the $\mathscr{H}_2$ norm of reset control systems.

5.2. Common Lyapunov function

Although a reset control system behaves in a hybrid manner, the mathematical description of its dynamics (40) shows that both the flow and the reset part can be described in a linear fashion. This motivates our choice to focus first on a common quadratic Lyapunov function $V(x)=x^TPx$ as was also done in Section 4.1.

Theorem 11
Consider the reset control system (40) with $\mathfrak{C}$ and $\mathfrak{D}$ as defined in (39). This system is globally asymptotically stable and its $\mathscr{H}_2$ norm $\|\Sigma\|_{2,x_0}\leqslant\gamma$ if there exist $P\succ 0$ and $U_f,U_R\in\mathbb{S}_+^{2\times 2}$ such that

$$\mathscr{A}^TP+P\mathscr{A}+\mathscr{C}^T\mathscr{C}+T_x^TE_f^TU_fE_fT_x\prec 0 \quad (41a)$$

$$\mathscr{A}_R^TP\mathscr{A}_R-P+T_x^TE_R^TU_RE_RT_x\preccurlyeq 0 \quad (41b)$$

$$\gamma^2-x_0^TPx_0\geqslant 0 \quad (41c)$$

Proof
The storage function $V(x)=x^TPx$ is continuously differentiable. Since $P\succ 0$, $V(x)>0$ for $x\neq 0$ and $V(0)=0$. Hence, V is positive definite. Note that

$$x\in\mathfrak{C}\Rightarrow x^TT_x^TE_f^TU_fE_fT_xx\geqslant 0 \quad (42a)$$

$$x\in\mathfrak{D}\Rightarrow x^TT_x^TE_R^TU_RE_RT_xx\geqslant 0 \quad (42b)$$

since U_f, $U_R\in\mathbb{S}_+^{2\times 2}$ only have nonnegative elements. Furthermore, (41a) is strict, which implies that $\mathscr{A}^TP+P\mathscr{A}+\mathscr{C}^T\mathscr{C}+T_x^TE_f^TU_fE_fT_x\prec 0$ and thus

$$\left\langle\frac{\partial V}{\partial x},\mathscr{A}x\right\rangle<-\varepsilon V \quad \text{if } x\in\mathfrak{C},\quad x\neq 0 \quad (43)$$

for some $\varepsilon>0$. Combining (42b) with (41b) gives

$$x^T(\mathscr{A}_R^TP\mathscr{A}_R-P)x\leqslant 0 \quad \text{if } x\in\mathfrak{D} \quad (44)$$

and thus

$$V(x^+)\leqslant V(x) \quad \text{when } x\in\mathfrak{D} \quad (45)$$

Continuous differentiability and positive definiteness of V together with (43) and (45), and the fact that V is radially unbounded imply that system (40) is globally asymptotically stable [23].

The proof that $\|\Sigma\|_{2,x_0}\leqslant\gamma$ is based on showing that (41) imply, for $V(x)=x^TPx$ and $s(w,z)=-z^Tz$, that

$$\left\langle\frac{\partial V}{\partial x},\mathscr{A}x\right\rangle\leqslant s(w,z) \quad \text{when } x\in\mathfrak{C}\backslash\{0\} \quad (46)$$

Indeed, if (46) would hold then, since (45) also holds, we have that for all $t_1\geqslant t_0$

$$V(x(t_1))-V(x(t_0))\leqslant\int_{t_0}^{t_1}s(w(t),z(t))\,\mathrm{d}t \quad (47)$$

showing that (40) is dissipative w.r.t. the supply function $s(w,z)=-z^Tz$. Using that $V(x(t_1))\geqslant 0$ and letting $t_1\to\infty$, the combination of (47) and (41c) yields

$$\begin{aligned}\|\Sigma\|_{2,x_0}^2&=\int_0^\infty z^T(t)z(t)\,\mathrm{d}t\\ &=-\int_0^\infty s(w(t),z(t))\,\mathrm{d}t\leqslant V(x(0))\leqslant\gamma^2 \quad (48)\end{aligned}$$

To show (46), combine (42a) with (41a) to obtain that

$$x^T(\mathscr{A}^TP+P\mathscr{A}+\mathscr{C}^T\mathscr{C})x<0 \quad \text{if } x\in\mathfrak{C}\backslash\{0\} \quad (49)$$

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which is just a reformulation of (46). Hence, the proof is complete. □

5.3. PWQ Lyapunov function

Although Theorem 11 provides an easy way to determine an upperbound on the $\mathscr{H}_2$ norm for a specific x_0, it can be conservative for the same reasons as discussed in Section 4.2. Therefore, we propose here also the use of PWQ Lyapunov functions based on the same partitioning of the flow set as in Section 4.2. Using the partitioning (29), the calculation of an $\mathscr{H}_2$ upperbound can be performed using the following theorem.

Theorem 12

The reset control system (40) is globally asymptotically stable with an $\mathscr{H}_2$ norm $\|\Sigma\|_{2,x_0}\leqslant\gamma$ if, for a given N, there exists $P_i,\,P_R\in\mathbb{S}^{n\times n}$ and $U_i,\,U_{R_0},\,U_{R_i},\,U_{RI_i},\,V_i,\,V_R\in\mathbb{S}_+^{2\times 2}$ such that

$$\mathscr{A}^{\mathrm{T}}P_i+P_i\mathscr{A}+\mathscr{C}^{\mathrm{T}}\mathscr{C}+E_{x,i}^{\mathrm{T}}U_iE_{x,i}\prec 0$$

$$i=1,\ldots,N \tag{50a}$$

$$\mathscr{A}_R^{\mathrm{T}}P_R\mathscr{A}_R-P_R+E_{x,R}^{\mathrm{T}}U_{R_0}E_{x,R}\preceq 0 \tag{50b}$$

$$\mathscr{A}_R^{\mathrm{T}}P_i\mathscr{A}_R-P_R+E_{x,R}^{\mathrm{T}}U_{R_i}E_{x,R}$$

$$+\mathscr{A}_R^{\mathrm{T}}E_{x,i}^{\mathrm{T}}U_{RI_i}E_{x,i}\mathscr{A}_R\preceq 0,\quad i=1,\ldots,N \tag{50c}$$

$$P_i-T_x^TE_i^TV_iE_iT_x\succ 0,\quad i=1,\ldots,N \tag{50d}$$

$$P_R-T_x^{\mathrm{T}}E_R^{\mathrm{T}}V_RE_RT_x\succ 0 \tag{50e}$$

$$W_{\Phi i}^{\mathrm{T}}(P_i-P_{i+1})W_{\Phi i}=0,\quad i=1,\ldots,N-1 \tag{50f}$$

$$W_{\Phi 0}^{\mathrm{T}}(P_R-P_1)W_{\Phi 0}=0 \tag{50g}$$

$$W_{\Phi N}^{\mathrm{T}}(P_N-P_R)W_{\Phi N}=0 \tag{50h}$$

$$\gamma^2-x_0^{\mathrm{T}}P_jx_0\geqslant 0,\quad j\in I(x_0) \tag{50i}$$

where $I(x_0):=\{i\,|\,x_0\in\mathfrak{C}_i\}$ denotes the indices of the regions that contain x_0.

Proof

The function V defined as $V(x)=V_i(x):=x^TP_ix$ when $x\in\mathfrak{C}_i$ and $V(x)=x^TP_Rx$ when $x\in\mathfrak{D}$, is a continuous PWQ function due to the continuity constraints (50f)–(50h) and is thus locally Lipschitz continuous as can be proven similarly as in the proof of Theorem 8. Furthermore, V is positive definite. To show this, note that for $x\in\mathfrak{C}_i\backslash\{0\}$ (not necessarily for all x) it holds that

$$V(x)=x^{\mathrm{T}}P_ix\overset{(50\mathrm{d})}{>}x^{\mathrm{T}}T_x^{\mathrm{T}}E_i^{\mathrm{T}}V_iE_iT_xx\geqslant 0 \tag{51}$$

as $x\in\mathfrak{C}_i\Rightarrow x^{\mathrm{T}}T_x^{\mathrm{T}}E_i^{\mathrm{T}}V_iE_iT_xx\geqslant 0$ due to the fact that $V_i\in\mathbb{S}_+^{2\times 2}$ has nonnegative elements. The same applies for $x\in\mathfrak{D}$ using (50e). Using (50a) and (50b) similarly as in the proof of Theorem 11 again shows global asymptotic stability, as V is proven to be a Lyapunov function for the reset system.

Furthermore, applying (50a)–(50c) similarly as in the dissipativity part of the proof of Theorems 11 gives

$$\left\langle\frac{\partial V_i}{\partial x},\mathscr{A}x\right\rangle<-z^{\mathrm{T}}z\quad\text{if } x\in\mathfrak{C}_i,\,x\neq 0 \tag{52a}$$

$$V(x^+)-V(x)\leqslant 0\quad\text{if } x\in\mathfrak{D} \tag{52b}$$

Clearly, (52a) and (52b) guarantee that

$$V(x(t_1))-V(x(t_0))\leqslant\int_{t_0}^{t_1}s(w(t),z(t))\,\mathrm{d}t \tag{53}$$

which in turn yields that $\|\Sigma\|_{2,x_0}^2\leqslant V(x(0))\leqslant\gamma^2$ as in the proof of Theorem 11. □

This upperbound can be lowered by increasing the number of subregions N, thereby increasing the tightness of the approximation of the actual $\mathscr{H}_2$ norm. Hence, as the solution to the LMIs in Theorem 11 is always a solution to (50) for some U_i,U_R,V_i,V_R, Theorem 12 is always less conservative than Theorem 11.

5.4. Example

As mentioned before, performance improvement by using reset control is especially apparent in the transient closed-loop behavior of constrained problems, which motivates our choice to consider the $\mathscr{H}_2$ norm of reset control systems. Indeed, in this section we show, by means of an input constrained $\mathscr{H}_2$ problem, that reset control can outperform linear control. Consider a closed-loop system with an integrator plant, $G(s)=\frac{1}{s}$, which should track a unit step reference $r(t)=1(t)$. The goal is to minimize the energy in the tracking error e for

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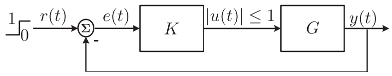

Figure 9. Closed-loop for design problem (54).

this specific reference, subject to a constrained control signal u to the plant (see Figure 9), as is usually the case in practical situations. Hence, the design problem is

$$\min_K \sqrt{\int_0^\infty e^2\,\mathrm{d}t} \tag{54a}$$

$$\text{subject to } |u(t)|\leqslant 1 \quad \forall t\geqslant 0 \tag{54b}$$

The theoretically best non linear controller for this specific problem is described by the discontinuous feedback law

$$u=\text{sign}(e) \tag{55}$$

since this controller produces the maximum control signal as long as possible, and vanishes as soon as the plant output reaches the desired value. This way the plant reacts as fast as possible, without any overshoot, realizing a minimal amount of energy in e, i.e. $\sqrt{1/3}\approx 0.577$. The closed-loop response resulting from this discontinuous feedback is depicted in Figure 10. We now try to approach this optimal performance by using both linear and reset control. First, a well-performing linear controller will be designed. To this end, we use the numerical optimization tool `fminsearch` in Matlab to specify the constraint and objective of (54) in the time-domain. For now, only first-order controllers are considered. After numerous iterations with various initial conditions, the solution eventually converged to the controller parameterized by

$$\left[\begin{array}{c|c} A_{K,\text{lin}} & B_{K,\text{lin}} \\ \hline C_{K,\text{lin}} & D_{K,\text{lin}} \end{array}\right]=\left[\begin{array}{c|c} -1.4775 & 0.6602 \\ \hline 2.4794 & 0.9259 \end{array}\right] \tag{56}$$

with an $\mathscr{H}_2$ norm of 0.599. The closed-loop response with this controller is shown in Figure 11 by the grey line.

A similar approach can be used to find a first-order reset controller. Again using various initial conditions,

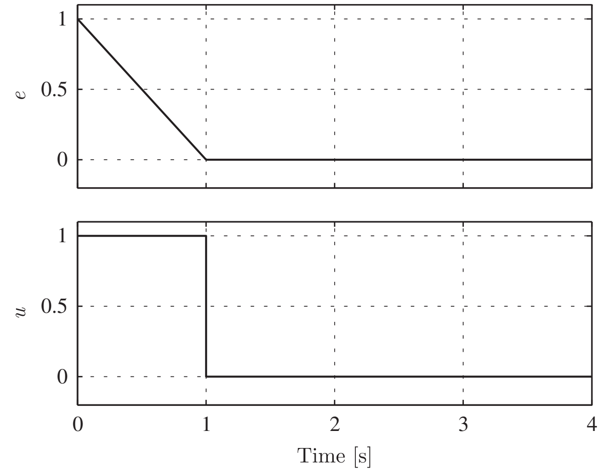

Figure 10. Closed-loop response using the hybrid feedback.

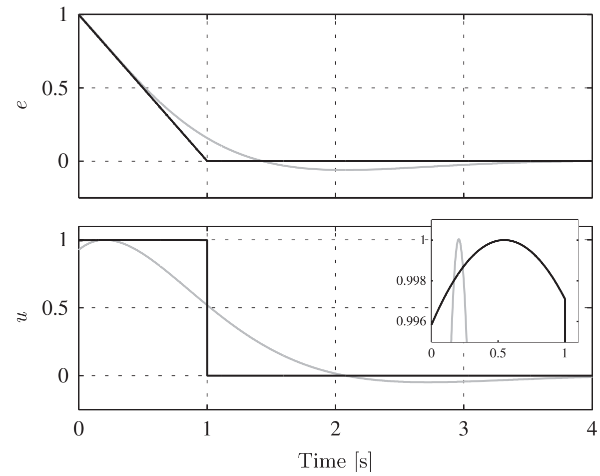

Figure 11. Closed-loop responses u and e using linear controller (56) (grey), and the reset controller (57) (black). The inlay shows a close-up of the flow behavior of the reset controller in u.

the solution converged to the controller parameterized by $A_r=0$ and

$$\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]=\left[\begin{array}{c|c} 0.9831 & -0.3843 \\ \hline -2.6201 & 0.9958 \end{array}\right] \tag{57}$$

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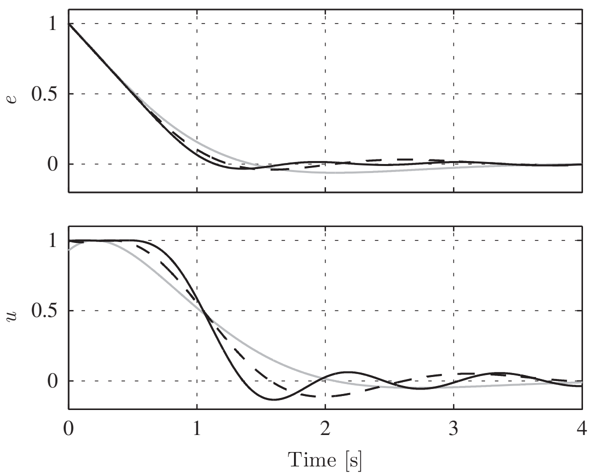

Figure 12. Closed-loop responses u and e using linear controllers of order 1 (solid grey), 2 (dashed black), and 4 (solid black).

The black lines in Figure 11 show its closed-loop response. Theorem 12 can now be applied to approximate the $\mathscr{H}_2$ norm, using the plant parameters

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -\varepsilon \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C = C_z = [-1 \;\; 1], \quad D_z = 0$$

By increasing the number of subregions N, the upper-bound on the $\mathscr{H}_2$ norm becomes less conservative, and for $N = 30$ we obtain $\gamma = 0.579$. The actual energy content in e obtained by the reset controller, calculated by numerical integration of the energy in e, equals 0.578, which shows the accuracy of Theorem 12. It can be seen in Figure 11 that the time-domain performance of the reset controller is comparable to the discontinuous controller (55). When the order of the linear controller increases, the response starts to resemble the hybrid response, see Figure 12. Although the $\mathscr{H}_2$ norm decreases with increasing controller order, the first-order reset controller still yields a lower one. All results are summarized in Table I. Hence, when considering only low-order controllers, the obtained reset controller clearly outperforms the linear controller.

We emphasize that it is not guaranteed that the constructed controllers are the optimal ones for problem (54), since the nonlinear optimization routine is not necessarily converging to the global optimum.

6. DISCUSSION: CONTROLLER SYNTHESIS AND IMPLEMENTATION ISSUES

Previous publications have shown in simulations [5, 6] and experiments [10, 13, 15] that reset control can outperform linear control in some situations. The $\mathscr{H}_\infty$ and $\mathscr{H}_2$ analysis tools presented in Sections 4 and 5, respectively, can be very useful to quantify this ability. The main problem, however, is the actual synthesis of a reset controller that outperforms a linear controller. In this section, we reflect on the possibilities for controller synthesis, and on practical implementation issues.

6.1. Synthesis

Using standard LMI-based $\mathscr{H}_\infty$ and $\mathscr{H}_2$ analysis techniques, it is possible to design the corresponding optimal linear controllers for an LTI system, e.g. by using the synthesis method from [26]. The synthesis of a reset controller, however, is still an open question. With the currently available knowledge it seems impossible to formulate $\mathscr{H}_\infty$ or $\mathscr{H}_2$ synthesis LMIs to find the optimal reset controller. This problem is to a large extend caused by the $E_{\cdot,\cdot}^T U_{\cdot} E_{\cdot,\cdot}$ terms in the analysis LMIs in Theorems 7, 8, 11, and 12 which introduce bilinear combinations of design variables. Unfortunately, these nonlinearities cannot be eliminated using the procedure from [27] or linearized via the change of variables introduced in [26]. Therefore, it is not straightforward to arrive at synthesis LMIs. Resorting to bilinear matrix inequality (BMI) solvers, of which some are available without guarantees on finding solutions if they exist, is one option. An alternative way can be to use recent results on optimization of matrix valued polynomials by sum-of-squares methods [28]. Furthermore, it is questionable whether reset control can ever outperform linear controllers in terms of the $\mathscr{L}_2$ gain or $\mathscr{H}_2$ performance. Indeed, as is shown in [29, 30], for LTI plants there exists no nonlinear (possibly time-varying) controller that yields a lower $\mathscr{L}_2$ gain than the optimal linear controller. Unfortunately, we expect reset control to be no exception to

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Table I. $\mathscr{H}_2$ norms for various controllers.

	Hybrid	Linear control order			Reset control			
		1	2	4	$N=3$	$N=5$	$N=10$	$N=30$
Numerical int.	0.577	0.599	0.586	0.580	0.578	0.578	0.578	0.578
LMI approx.	—	0.599	0.586	0.580	0.665	0.606	0.584	0.579

this. We therefore point out that Examples 2 and 3 in [17] do not provide a fair comparison between linear and reset controllers; in both examples it is easy to find linear controllers with lower $\mathscr{H}_\infty$ norms, and hence better performance when quantified in terms of the $\mathscr{L}_2$ gain. However, the performance improvement of reset controllers with respect to linear controllers can be shown by transient response performance qualifiers such as rise-time, overshoot, and settling-time (see e.g. [31]). Therefore, it is still of interest to be able to also characterize the $\mathscr{L}_2$ gain and $\mathscr{H}_2$ performance of reset control systems as is done in this paper.

6.2. *Design issues*

Although reset control might seem very advantageous, its usage and implementation is far from straightforward. To illustrate this, reconsider the relaxation of the resetting condition, i.e. reset when $yu\leqslant 0$ instead of when $y=0$. For some controllers, like low-pass filters, these two conditions are equivalent, but for others the former performs much worse in an implementation than the latter. This is especially true for controllers with a derivative action, such as lead filters. This derivative action acts like a 'brake' since it reverses the sign of the control signal u when the desired setpoint is approached. The $yu\leqslant 0$ reset condition then requires a state reset, but also resets this 'braking action', causing a large overshoot instead of preventing it. Hence, $y=0$ is preferred for implementation reasons, while $yu\leqslant 0$ clearly has advantages for analysis of the $\mathscr{L}_2$ gain and $\mathscr{H}_2$ norm (note that an analysis with $yu\leqslant 0$ does not guarantee anything for the $y=0$ implementation). Still, even with an $y=0$ implementation, reset actions tend to counteract derivative controllers, thereby diminishing damping effects. Resets change the dynamical behavior of the (otherwise linear) controller drastically. This observation makes the design of reset controller even more complicated. That makes the 32 year old frequency-based design of Horowitz [9] still the only currently available synthesis method for reset controllers, besides the BMI approach discussed in the previous section.

7. CONCLUSIONS

Motivated by recent publications on the potential advantages of reset control, we have developed a set of analysis LMIs with which the $\mathscr{L}_2$ gain and the $\mathscr{H}_2$ norm of any reset control system that fits into the linear-based generalized plant framework can be approximated. With respect to the $\mathscr{L}_2$ gain, the work in [17, 18] is generalized, while the results on the $\mathscr{H}_2$ norm have not been reported in literature before. The proposed analysis methods can also be applied to tracking and measurement noise problems, which could not be solved by the approaches in [17, 18]. The possible conservatism present in the $\mathscr{L}_2$ gain approximation for these tracking problems can be removed by including strictly proper input filters. The introduction of PWQ Lyapunov functions, which are much more flexible than quadratic ones, results in tighter approximations. Examples have shown the accuracy of the proposed calculations of the $\mathscr{L}_2$ gain and $\mathscr{H}_2$ norm. Furthermore, we have presented an example which shows that reset control can be close to the performance of the optimal (discontinuous) controller for a constrained $\mathscr{H}_2$ problem, while a common optimization method to design a linear controller provides a worse $\mathscr{H}_2$ performance. Finally, we reflected on synthesis possibilities and practical design issues.

The initial goal of studying reset control was to formulate a method with which reset controllers

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could be synthesized. Unfortunately, it turned out that the standard methods to arrive at controller synthesis inequalities, i.e. parameter elimination and the linearizing change of variables, are not applicable to the specific structure of the reset analysis LMIs. Hence, the 'synthesis conditions' provided in this paper are in the form of BMIs. As such, possible solution directions for synthesis are the use of BMI solvers and sum-of-squares decompositions of matrix valued polynomials. However, taking into account the results in [29, 30], it is not likely that there exists a reset controller yielding a lower $\mathscr{L}_2$ gain than the optimal linear controller. Therefore, any synthesis method resulting in a reset controller that outperforms the optimal linear controller in some sense should not only be based on the $\mathscr{L}_2$ gain, but should also take performance measures into account that reflect the advantages of reset control, e.g. transient measures such as overshoot and settling-time. Since this is not a trivial procedure even for linear control systems, it is not clear at this point how to proceed for reset controllers. Concluding, the analysis results that are derived in this part of the thesis are well suited to determine (upperbounds on) the $\mathscr{L}_2$ gain and $\mathscr{H}_2$ norm of a closed-loop reset system with a given reset controller, however, systematic reset controller synthesis is still a widely open issue.

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