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An LMI-based $\mathcal{L}_2$ gain performance analysis for reset control systems

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***Abstract*— In this paper we present a general LMI-based analysis method to determine an upperbound on the $\mathcal{L}_2$ gain performance of a reset control system. These computable sufficient conditions for $\mathcal{L}_2$ stability, based on piecewise quadratic Lyapunov functions, are suitable for all LTI plants and linear-based reset controllers, thereby generalizing the results available in literature. Our results furthermore extend the existing literature by including tracking and measurement noise problems by using strictly proper input filters. We illustrate the approach by a numerical example.**

***Index Terms*— Hybrid systems, reset control, linear matrix inequality, Lyapunov stability, $\mathcal{L}_2$ gain, tracking.**

I. Introduction

Linear controllers are known to be subject to certain fundamental performance limitations [1], [2]. To overcome these limitations, various nonlinear feedback controllers for linear time-invariant (LTI) plants were proposed in the literature [3]. The reset controller is one of such controllers, which is basically a linear controller whose states (or subset of states) are reset to zero when its input and output satisfy certain conditions. The first resetting element was introduced in 1958, when Clegg proposed an integrator which resets whenever the input is zero [4]. However, the use and effect of this Clegg integrator is not straightforward, and consequently, its first use in a control design procedure [5] was not until 1974. Subsequently, a *first order reset element (FORE)* was introduced in [6], together with a controller design procedure based on frequency domain techniques.

After that, it would take another two decades until the interest in reset control was again renewed, by means of various publications on stability analysis techniques for reset control systems, in the late '90s. In a series of publications, highlighted especially by [7] and [8], the involved authors formulated computable stability conditions based on quadratic Lyapunov functions. Their main result is the so called H_β-condition, a stability test applicable to zero-input reset control systems, which can, to some extend, also be used for BIBO (bounded input bounded output) stability and tracking problems. A more detailed literature overview regarding this stability condition is provided in [9].

A closer view on the H_β-condition reveals that it is in fact a reformulation of Lyapunov based stability LMIs (linear matrix inequalities) using the Kalman-Yakubovich-Popov Lemma. The analysis consists of two stability LMIs, one corresponding to the *flowing* of the closed loop (i.e. smooth evolution of the state) and the other to the *reset* of the controller. These LMIs are coupled as a common quadratic Lyapunov function is employed. Therefore the H_β-condition is rather conservative, and is only necessary and sufficient for *quadratic* stability. Moreover, since the flowing LMI is solved for the complete state space, it requires the linear part of the closed-loop dynamics to be stable, which indicates some of the conservatism present in this approach.

This conservatism was reduced in some part by more recent publications [10], [11], where the authors suggested a slightly different resetting condition. Indeed, their idea to reset when controller in- and output have opposite sign instead of when the input is zero results in a much smaller flow region. Therefore, the linear closed-loop does not need to be stable anymore and the stability bounds of the reset system are sharpened. Second, the authors allowed *piecewise quadratic (PWQ)* Lyapunov functions, thereby approximating higher order Lyapunov functions to capture a broader class of stability problems. On top of this, the stability analysis was extended to $\mathcal{L}_2$ stability, such that the closed loop $\mathcal{L}_2$-gain from input to output of a reset control system could be approximated by an upperbound.

So far, the work in [10], [11] is the most general analysis framework for reset control systems currently available in literature. However, it is not generally applicable, since it treats only FOREs and Clegg integrators. Furthermore, it does not include a solution to the tracking problem, since its system description assumes a zero reference. Our work will be an extension of [10], [11] in several directions. First, it generalizes the $\mathcal{L}_2$-gain analysis to general reset control systems fitting into the common $\mathcal{H}_\infty$ framework using augmented plants. Second, tracking problems are included in our results, as opposed to [12] which only considers constant reference signals. We will show that possible conservatism can be reduced via input filtering and an example is presented illustrating the approach.

This paper is organized as follows. First, Section II introduces the general $\mathcal{H}_\infty$ framework for reset control systems and describes the dynamics of the plant and the controller. In Section III our main results are derived, as well as a solution to deal with tracking problems. Next, an example is provided in Section IV. We conclude in Section V.

Notation. The set of real numbers is denoted by $\mathbb{R}$. The set of real symmetric matrices is denoted by $\mathbb{S}$, the set of real symmetric matrices with nonnegative elements is denoted by $\mathbb{S}_+$. The identity matrix of dimension $n \times n$ is denoted by $I_n \in \mathbb{R}^{n \times n}$. Given two vectors x_1, x_2 we write (x_1, x_2) to denote $[x_1^T, x_2^T]^T$. A vector $x \in \mathbb{R}^n$ is nonnegative, denoted by $x \geq 0$, if its elements $x_i \geq 0$ for $i = 1, \ldots, n$. A symmetric matrix $A \in \mathbb{S}^{n \times n}$ is positive definite, denoted by $A \succ 0$ if $x^T A x > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$.

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II. General system description

In this section we present a mathematical description of the reset controller and the resulting closed-loop. These descriptions are chosen to fit into the common multichannel $\mathcal{H}_\infty$ framework, as depicted in Figure 1. The augmented plant P, with state $x_p \in \mathbb{R}^{n_p}$, contains the system to be controlled, together with possible input- and output-weightings. The reset controller is denoted by K, whose states are given by $x_k \in \mathbb{R}^{n_k}$. The closed-loop state is defined by $x \in \mathbb{R}^n$ with $x = (x_p, x_k)$. Moreover, $w \in \mathbb{R}^{n_w}$ and $z \in \mathbb{R}^{n_z}$ denote the exogenous inputs and the to be controlled output, and $y, u \in \mathbb{R}$ denote the controller input and output, respectively. Here we consider SISO plants and controllers only, since a suitable formulation of reset controllers for MIMO systems is still a widely open issue.

In the remainder of this paper we will consider LTI augmented plants P, whose dynamics are described by

$$\begin{aligned} \dot{x}_p &= Ax_p + Bu + B_w w \\ z &= C_z x_p + D_{zw} w + D_z u \\ y &= Cx_p + D_w w. \end{aligned} \qquad (1)$$

We consider no direct feedthrough from u to y, as is e.g. the case for many motion systems. Furthermore, we assume that (1) is a minimal realization of the augmented plant P.

A. Reset controller

The reset controller K is described by a linear system whose (subset of) states are reset whenever its input y and output u satisfy a certain condition, i.e.

$$\begin{aligned} \dot{x}_k &= A_K x_k + B_K y && \text{if } (y,u) \in \mathfrak{C}' \\ x_k^+ &= A_r x_k && \text{if } (y,u) \in \mathfrak{D}' \\ u &= C_K x_k + D_K y \end{aligned} \qquad (2)$$

This reset controller can thus be seen as a hybrid system with a *flow set* $\mathfrak{C}'$ and a *reset set* $\mathfrak{D}'$ using the framework in [10]. As long as $(y,u) \in \mathfrak{C}'$ the controller behaves linearly and its output u flows conform (A_K, B_K, C_K, D_K). Loosely speaking, when $(y,u) \in \mathfrak{D}'$ the state is changed instantaneously from x_k to x_k^+ by the discrete map corresponding to $A_r \in \mathbb{R}^{n_k \times n_k}$. Various choices for A_r are theoretically possible, but a reasonable and appropriate choice, commonly used in literature, is

$$A_r = \begin{bmatrix} I_{n_k - n_r} & 0 \\ 0 & 0_{n_r} \end{bmatrix},$$

stating that the last n_r of the n_k controller states are reset to zero, while the others remain unchanged. The reset set $\mathfrak{D}'$ can be defined in various ways, but here we follow [10], [11], where resets occur whenever input and output have opposite sign, i.e. $yu \leq 0$. Compared to [8] this choice reduces the size of the flow set and allows a considerable relaxation of the stability and performance conditions later on. Hence, the controller flows whenever $y \geq 0, u \geq 0$ or $y \leq 0, u \leq 0$, which means that

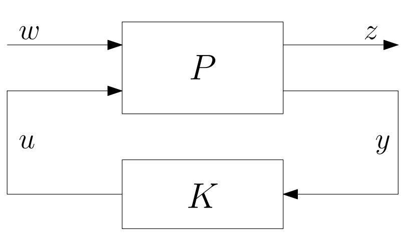

Fig. 1. General multichannel closed-loop system

$$\mathfrak{C}' := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \mathbb{R}^2 : E_f \begin{bmatrix} y \\ u \end{bmatrix} \geq 0 \quad \text{or} \quad E_f \begin{bmatrix} y \\ u \end{bmatrix} \leq 0 \right\} \qquad (3a)$$

$$\mathfrak{D}' := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \mathbb{R}^2 : E_R \begin{bmatrix} y \\ u \end{bmatrix} \geq 0 \quad \text{or} \quad E_R \begin{bmatrix} y \\ u \end{bmatrix} \leq 0 \right\}, \qquad (3b)$$

where

$$E_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E_R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The flow set (3a) and reset set (3b) can also be expressed in terms of x and w. Therefore we introduce a transformation matrix $T = \left[\begin{array}{c|c} T_x & T_w \end{array}\right]$ defined as

$$\begin{bmatrix} y \\ u \end{bmatrix} = T \begin{bmatrix} x \\ w \end{bmatrix} = \left[\begin{array}{cc|c} C & 0 & D_w \\ D_K C & C_K & D_K D_w \end{array}\right] \begin{bmatrix} x_p \\ x_k \\ w \end{bmatrix},$$

such that

$$\mathfrak{C} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{n+n_w} : E_f T \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \text{ or } E_f T \begin{bmatrix} x \\ w \end{bmatrix} \leq 0 \right\} \qquad (4a)$$

$$\mathfrak{D} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{n+n_w} : E_R T \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \text{ or } E_R T \begin{bmatrix} x \\ w \end{bmatrix} \leq 0 \right\}. \qquad (4b)$$

For later reference we introduce

$$\begin{aligned} E_f T &= \left[\begin{array}{c|c} E_f T_x & E_f T_w \end{array}\right] = \left[\begin{array}{c|c} E_{x,f} & E_{w,f} \end{array}\right], && (5a) \\ E_R T &= \left[\begin{array}{c|c} E_R T_x & E_R T_w \end{array}\right] = \left[\begin{array}{c|c} E_{x,R} & E_{w,R} \end{array}\right]. && (5b) \end{aligned}$$

Remark 1 In case $D_w \neq 0$, $\mathfrak{C}$ and $\mathfrak{D}$ also depend on the input w, which is a case not considered in [10], [11]. However, this situation is of importance, as typically in tracking problems the case $D_w \neq 0$ occurs. Indeed, consider the problem depicted in Figure 2, where P denotes a dynamical system with input u and output y_p, and K denotes the controller. The signals r and e are the reference signal and the tracking error, respectively. In this case $w = r$,

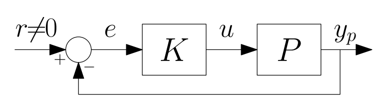

Fig. 2. Simple tracking problem

$y = z = e$, and $D_w = 1 \neq 0$, due to the direct feedthrough of r in e. This means that the flow and reset regions $\mathfrak{C}$ and $\mathfrak{D}$ in (4) explicitly depend on w. Hence, the input r clearly influences the reset moment, since resets are defined to occur at sign changes of u and e. In the analysis of [10], [11] this dependency on w was omitted, since the definitions of $\mathfrak{C}$ and $\mathfrak{D}$ only depend on x (e.g. $\mathfrak{C} := \{x : x^T M x \geq 0\}$. Therefore, the results from [10], [11] are not applicable for tracking problems, but for disturbance rejection type of problems only. Furthermore, [12] does only consider tracking examples with a constant reference, whereas our result can handle any reference signal. □

B. Closed-loop dynamics

The dynamics of the augmented plant and the reset controller can be combined into one description for the closed-loop dynamics Σ

$$\Sigma : \begin{cases} \dot{x} &= \mathcal{A}x + \mathcal{B}w & \text{if } (x,w) \in \mathfrak{C} \\ x^+ &= A_R x & \text{if } (x,w) \in \mathfrak{D} \\ z &= \mathcal{C}x + \mathcal{D}w \end{cases} \tag{6}$$

where

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{cc|c} A + BD_K C & BC_K & B_w + BD_K D_w \\ B_K C & A_K & B_K D_w \\ \hline C_z + D_z D_K C & D_z C_K & D_{zw} + D_z D_K D_w \end{array}\right]$$
$$A_R = \begin{bmatrix} I_{n_p} & 0 \\ 0 & A_r \end{bmatrix}.$$

The linear closed-loop system without resets (i.e. $\mathfrak{D} = \emptyset$ and $\mathfrak{C} = \mathbb{R}^{n+n_w}$), is called the *base linear system*. The reset controller will be chosen such that multiple resets at one point in time are excluded, in order to guarantee local existence of solutions. To guarantee local solutions, we assume that the closed-loop system can flow after each reset on at least a non-trivial time interval. The following assumption formalizes this statement.

Assumption 2 *The system (6) is such that for all signals w of interest*

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in \mathfrak{D} \quad \Rightarrow \quad x^+ = A_R x \in \mathcal{F}_{\mathfrak{C}}(w), \tag{7}$$

where $\mathcal{F}_{\mathfrak{C}}(w)$ is given by

$$\mathcal{F}_{\mathfrak{C}}(w) := \{x_0 \in \mathbb{R}^n : \exists \epsilon > 0 \quad \forall \tau \in [0,\epsilon) \; (x(\tau), w(\tau)) \in \mathfrak{C}\} \tag{8}$$

where $(x(\tau), w(\tau))$ denotes the state/input trajectory of (6) from initial state x^+.

This assumption implies that smooth continuation is possible after a reset from the state x^+. The smooth continuation set set $\mathcal{F}_{\mathfrak{C}}(w)$ can be explicitly characterized using lexicographic orderings, analogous to the work in [9]. Note however that in this case we require a priori knowledge of the input w and its time derivatives to do this. Finally, in theory the reset times may accumulate (so called Zeno behavior). Therefore, at this point we assume that either Zenoness is absent or we can continue beyond the accumulation point such that global existence of solutions is guaranteed.

III. Main results

In this section we present our main results on $\mathcal{L}_2$ stability, applicable to any LTI plant (1) and any reset controller (2). In the remainder of the paper, by asymptotic stability of system (6) we mean asymptotic stability of the zero-input system, i.e. when $w = 0$. Furthermore, we use the following definitions [13], [14].

Definition 3 *The $\mathcal{L}_2$-gain $||\Sigma||_\infty$ of Σ in (6), with input $w \in \mathcal{L}_2$, output $z \in \mathcal{L}_2$ and $x(0) = 0$, is defined as the square root of*

$$||\Sigma||_\infty^2 = \sup_{0<||w||_2<\infty} \frac{||z||_2^2}{||w||_2^2}, \tag{9}$$

where $||v||_2$ denotes the 2-norm of a signal $v(t)$, defined by the square root of

$$||v||_2^2 = \int_0^\infty v^T(t)\, v(t)\, dt. \tag{10}$$

Definition 4 *System (6) with state $x \in \mathbb{R}^n$, input $w \in \mathbb{R}^{n_w}$ and output $z \in \mathbb{R}^{n_z}$ is dissipative w.r.t. a supply function $s : \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \to \mathbb{R}$ if there exists a storage function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$ and*

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t))dt \tag{11}$$

for all $t_1 \geq t_0$ and all signals w, x, and z that satisfy (6).

If V is differentiable, then (11) is equal to

$$\frac{d}{dt}V(x) \leq s(w(t), z(t)). \tag{12}$$

The following lemma will be of use in the sequel.

Lemma 5 *System (6) has an $\mathcal{L}_2$ gain from input w to output z smaller than or equal to γ if system (6) is asymptotically stable and dissipative w.r.t. the supply function*

$$s(w,z) = \gamma^2 w^T w - z^T z. \tag{13}$$

A. $\mathcal{L}_2$ analysis

In order to approximate the $\mathcal{L}_2$ gain of a reset system we apply Lemma 5 to the closed-loop system (6). Note that the minimal value of γ for which (11) holds depends on the particular structure of V. Since we are dealing with a nonlinear closed-loop system, the V that yields the smallest value of γ might be a very complicated function. At this point however, motivated by the linear behavior of the closed-loop in a large part of the state space, we first restrict ourselves to quadratic Lyapunov functions of the form $V(x) = x^T P x$. Using this structure, the following result is obtained.

Theorem 6 *The reset control system (6) is globally asymptotically stable with an $\mathcal{L}_2$ gain $||\Sigma||_\infty \leq \gamma$ if there exist $P \in \mathbb{S}^{n\times n} \succ 0$ and $U, U_R \in \mathbb{S}_+^{2\times 2}$ such that*

$$\begin{bmatrix} \mathcal{A}^T P + P\mathcal{A} + E_{x,f}^T U E_{x,f} & P\mathcal{B} + E_{x,f}^T U E_{w,f} & \mathcal{C}^T \\ \mathcal{B}^T P + E_{w,f}^T U E_{x,f} & -\gamma I + E_{2,f}^T U E_{w,f} & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & -\gamma I \end{bmatrix} \prec 0, \tag{14a}$$

$$\begin{bmatrix} \mathcal{A}_R^T P \mathcal{A}_R - P + E_{x,R}^T U_R E_{x,R} & E_{x,R}^T U_R E_{w,R} \\ E_{w,R}^T U_R E_{x,R} & E_{w,R}^T U_R E_{w,R} \end{bmatrix} \preceq 0. \tag{14b}$$

Proof: To prove that $||\Sigma||_\infty \leq \gamma$ we will show that the inequalities in (14) imply, for $V(x) = x^T P x$ and $s(w,z) = \gamma^2 w^T w - z^T z$, that system (6) is globally asymptotically stable and that

$$\frac{d}{dt}V(x) \leq s(w,z) \quad \text{when } \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{C}, \tag{15a}$$

$$V(x^+) \leq V(x) \quad \text{when } \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{D}. \tag{15b}$$

Indeed, if (15) holds and system (6) is asymptotically stable then for all $t_1 \geq t_0$

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t))dt \qquad (16)$$

showing that (6) is dissipative with respect to $s(w, z)$, and according to Lemma 5 has an $\mathcal{L}_2$ gain $||\Sigma||_\infty \leq \gamma$.
The storage function $V(x) = x^T P x$ is continuously differentiable. Since $P \succ 0$, $V(x) > 0$ for $x \neq 0$ and hence V is positive definite. To show that (15a) and (15b) hold, note that

$$\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{C} \quad \Rightarrow \quad \begin{bmatrix} x \\ w \end{bmatrix}^T T^T E_f^T U E_f T \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \qquad (17a)$$

$$\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{D} \quad \Rightarrow \quad \begin{bmatrix} x \\ w \end{bmatrix}^T T^T E_R^T U_R E_R T \begin{bmatrix} x \\ w \end{bmatrix} \geq 0, \qquad (17b)$$

since $U, U_R \in \mathbb{S}_+^{2\times 2}$ only have non-negative elements. Combining (17a) with the Schur complement of (14a) and employing the S-procedure, yields if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{C}, x \neq 0$

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} \mathcal{A}^T P + P\mathcal{A} + \mathcal{C}^T\mathcal{C} & P\mathcal{B} + \mathcal{C}^T\mathcal{D} \\ \mathcal{B}^T P + \mathcal{D}^T\mathcal{C} & \mathcal{D}^T\mathcal{D} - \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0, \qquad (18)$$

and combining (17b) with (14b) gives

$$x^T(\mathcal{A}_R^T P \mathcal{A}_R - P)x \leq 0 \qquad \text{if } \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{D}. \qquad (19)$$

Since $V(x) = x^T P x$, (18) and (19) are just reformulations of (15a) and (15b). Furthermore, (14a) is strict and implies that $\mathcal{A}^T P + P\mathcal{A} + T_x^T E_f^T U E_f T_x^T \prec 0$ which is equivalent to

$$\frac{d}{dt}V(x) < 0 \quad \text{if } \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{C}, x \neq 0, w = 0. \qquad (20)$$

Continuity and positive definiteness of V together with (20) and the fact that V is radially unbounded implies that system (6) is globally asymptotically stable [15], which completes the proof. ∎
The analysis in Theorem 6 is capable of providing an upperbound on the actual $\mathcal{L}_2$ gain of any closed-loop reset control system.

B. Reducing conservatism

As stated in the previous section, the minimal value of γ for which (11) holds depends on the chosen structure of V. Therefore, the result in Theorem 6 can be conservative. To reduce this conservatism, we choose to use continuous *piecewise quadratic (PWQ) storage functions* [11], [16]. This choice is motivated by the flexibility of PWQ functions, since they can be arbitrarily complex with increasing number of regions, while still resulting in LMIs to check the dissipativity inequality. The PWQ storage functions are obtained by partitioning the flow set $\mathfrak{C}'$ into smaller regions $\mathfrak{C}'_i$ and assigning a different quadratic Lyapunov function $V_i(x) = x^T P_i x$ to each of them [11], see Figure 3. Each region $\mathfrak{C}'_i$ is bounded by two lines uniquely defined by the angles θ_i and θ_{i-1}. These angles should be chosen such that $0 = \theta_0 < \theta_1 < \ldots < \theta_N = \frac{\pi}{2}$, e.g. equidistantly distributed as $\theta_i = \frac{i}{N}\frac{\pi}{2}$, where $i = 0, \ldots, N$ and N is the number of regions. Using the coordinate transformation matrix T, we can now define regions $\mathfrak{C}_i$ as

$$\mathfrak{C}_i := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{n+n_w} : E_i T \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \text{ or } E_i T \begin{bmatrix} x \\ w \end{bmatrix} \leq 0 \right\}, \qquad (21)$$

where

$$E_i = \begin{bmatrix} -\sin(\theta_{i-1}) & \cos(\theta_{i-1}) \\ \sin(\theta_i) & -\cos(\theta_i) \end{bmatrix}.$$

The boundaries of the regions are defined by

$$\begin{bmatrix} -\sin(\theta_i) & \cos(\theta_i) \end{bmatrix} T \begin{bmatrix} x \\ w \end{bmatrix} = \Phi_i \begin{bmatrix} x \\ w \end{bmatrix} = 0, \qquad (22)$$

whose solutions are in the kernel of Φ_i. We can also use an image representation for these boundaries using full column rank matrices $W_{\Phi i} \in \mathbb{R}^{(n+n_w)\times(n+n_w-1)}$ such that $\mathrm{im}(W_{\Phi i}) = \ker(\Phi_i)$, where $\mathrm{im}(W_{\Phi i})$ denotes the image of $W_{\Phi i}$.
In some situations however, such as tracking problems as in Figure 2, the introduction of continuous PWQ storage functions does not result in less conservatism due to the fact that $D_w \neq 0$ in those situations. Namely, each region $\mathfrak{C}_i$ has its own Lyapunov function V_i, which solely depends on x, while (21) shows that the region itself is defined in terms of both x and w. Figure 4 illustrates this for the simple case where $x, w \in \mathbb{R}$. Since V only depends on x (depicted by the dashed vertical lines), continuity of V across the border between Ω_1 and Ω_2 requires that $V_1 = V_2$ for all (x, w) on the boundary, i.e., for all x.
This issue arises in any situation with $D_w \neq 0$, including measurement noise attenuation and tracking problems. This drawback can be avoided by forcing $D_w = 0$, which can be done by including strictly proper input filters for exogenous signals that enter the closed-loop before the controller, see Figure 5. Since these strictly proper filters have no direct feedthrough of the input, there is also no direct feedthrough from w (containing $\bar{r}$ and $\bar{\eta}$) to y in the augmented plant in (1), so $D_w = 0$. By including input filters in the augmented plant we assume to have a priori knowledge of these inputs (as is often the case in practice), which is then via the filter states contained inside the state vector x_p. Possible input filters include:

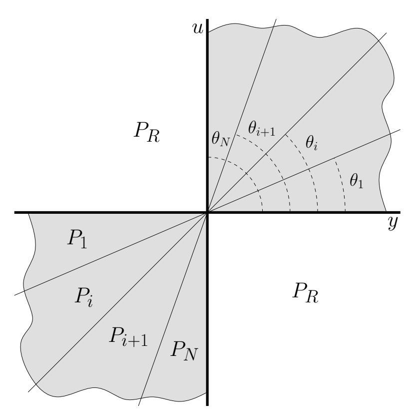

Fig. 3. Partitioning of the (y, u)-space

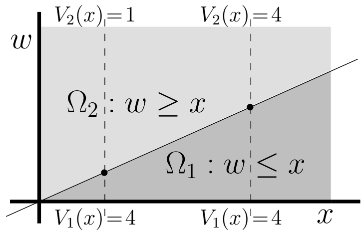

Fig. 4. The case $D_1 \neq 0$.

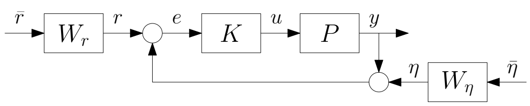

Fig. 5. Closed-loop with filtered inputs

- unit step: $W(s) = \frac{1}{(s+\varepsilon)}$;
- unit ramp: $W(s) = \frac{1}{(s+\varepsilon)^2}$;
- sine wave with frequency ω: $W(s) = \frac{\omega}{(s+\varepsilon)^2+\omega^2}$;

where s is the Laplace variable and $\varepsilon > 0$ is a small offset. This offset is standard in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems for linear systems to ensure closed-loop stability. Using these filters, the Lyapunov function V also depends on the input knowledge (as the state variables corresponding to w are now included in x_p and thus in x), while $\mathfrak{C}_i$ and $\mathfrak{D}$ no longer depend on w, i.e.

$$\mathfrak{C}_i := \{x \in \mathbb{R}^n : E_{x,i}x \geq 0 \quad \text{or} \quad E_{x,i}x \leq 0\} \tag{23a}$$
$$\mathfrak{D} := \{x \in \mathbb{R}^n : E_{x,R}x \geq 0 \quad \text{or} \quad E_{x,R}x \leq 0\}, \tag{23b}$$

where $E_{x,i} = E_i T_x$. Using this partitioning we can formulate the following result on the calculation of an upperbound on the $\mathcal{L}_2$ gain for tracking problems.

Theorem 7 *The reset control system (6) with $D_w = 0$ and a partitioning of the flow set given by (23) is globally asymptotically stable with an $\mathcal{L}_2$ gain $||\Sigma||_\infty \leq \gamma$ if, for a given N, there exists $P_i, P_R \in \mathbb{S}^{n\times n}$ and $U_i, U_{R_0}, U_{R_i}, U_{RI_i}, V_i, V_R \in \mathbb{S}_+^{2\times 2}$, $i = 1, \ldots, N$ such that*

$$\begin{bmatrix} \mathcal{A}^T P_i + P_i\mathcal{A} + E_{x,i}^T U_i E_{x,i} & P_i\mathcal{B} & \mathcal{C}^T \\ \mathcal{B}^T P_i & -\gamma I & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & -\gamma I \end{bmatrix} \prec 0, \qquad i = 1, \ldots, N \tag{24a}$$
$$A_R^T P_R A_R - P_R + E_{x,R}^T U_{R_0} E_{x,R} \preceq 0 \tag{24b}$$
$$\mathcal{A}_R^T P_i \mathcal{A}_R - P_R + E_{x,R}^T U_{R_i} E_{x,R} + \ldots$$
$$\mathcal{A}_R^T E_{x,i}^T U_{RI_i} E_{x,i} \mathcal{A}_R \preceq 0, \qquad i = 1, \ldots, N \tag{24c}$$
$$P_i - E_{x,i}^T V_i E_{x,i} \succ 0, \qquad i = 1, \ldots, N \tag{24d}$$
$$P_R - E_{x,R}^T V_R E_{x,R} \succ 0 \tag{24e}$$
$$\bar{W}_{\Phi i}^T (P_i - P_{i+1}) \bar{W}_{\Phi i} = 0, \qquad i = 1, \ldots, N-1 \tag{24f}$$
$$\bar{W}_{\Phi 0}^T (P_R - P_1) \bar{W}_{\Phi 0} = 0 \tag{24g}$$
$$\bar{W}_{\Phi N}^T (P_N - P_R) \bar{W}_{\Phi N} = 0, \tag{24h}$$

where $\mathrm{im}(\bar{W}_{\Phi i}) = \ker([\, -\sin(\theta_i) \;\; \cos(\theta_i) \,]\, T_x)$.

Proof: The proof is based on showing that hypotheses (24) imply that the storage function V (defined as $V(x) = V_i(x) := x^T P_i x$ when $x \in \mathfrak{C}_i$ and $V(x) = x^T P_R x$ when $x \in \mathfrak{D}$) is continuous, positive definite and radially unbounded, and that for $s(w,z) = \gamma^2 w^T w - z^T z$,

$$\frac{\partial V_i}{\partial x}(\mathcal{A}x + \mathcal{B}w) \leq s(w,z) \qquad \text{if } x \in \mathfrak{C}_i \tag{25a}$$
$$V(x^+) - V(x) \leq 0 \qquad \text{if } x \in \mathfrak{D} \tag{25b}$$
$$\frac{\partial V_i}{\partial x}\mathcal{A}x < 0 \qquad \text{if } x \in \mathfrak{C}_i, x \neq 0 \tag{25c}$$

Indeed, continuity together with the positive definiteness and radially unboundedness of V imply global asymptotic stability [15]. Moreover, if (25a) and (25b) hold, then for all $t_1 \geq t_0$

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t))dt \tag{26}$$

showing that (6) is dissipative w.r.t. the supply function $s(w,z) = \gamma^2 w^T w - z^T z$, and hence that $||\Sigma||_\infty \leq \gamma$ by virtue of Lemma 5.
Continuity of the piecewise quadratic function V follows from constraints (24f), (24g) and (24h). Furthermore, V is positive definite. To show this, note that since $V_i, V_R \in \mathbb{S}_+^{2\times 2}$ have only non-negative elements it holds that

$$x \in \mathfrak{C}_i \quad \Rightarrow \quad x^T E_{x,i}^T V_i E_{x,i} x \geq 0 \tag{27a}$$
$$x \in \mathfrak{D} \quad \Rightarrow \quad x^T E_{x,R}^T V_R E_{x,R} x \geq 0. \tag{27b}$$

Therefore, for $x \in \mathfrak{C}_i$ (hence not necessarily for all x) it holds that for $x \neq 0$

$$V(x) = x^T P_i x \overset{(24d)}{>} x^T E_{x,i}^T V_i E_{x,i} x \geq 0. \tag{28}$$

The same applies for $x \in \mathfrak{D}$ using (24e). Combining (27a) with the Schur complement of (24a) via the S-procedure yields that for $x \in \mathfrak{C}_i$

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} \mathcal{A}^T P_i + P_i\mathcal{A} + \mathcal{C}^T\mathcal{C} & P_i\mathcal{B} + \mathcal{C}^T\mathcal{D} \\ \mathcal{B}^T P_i + \mathcal{D}^T\mathcal{C} & \mathcal{D}^T\mathcal{D} - \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0, \tag{29}$$

which is equivalent to (25a). Note that (24a) implicitly implies that $x^T \left(\mathcal{A}^T P_i + P_i\mathcal{A} + E_{x,i}^T U_i E_{x,i}\right) x < 0$ which implies (25c).
It only remains to be shown that (25b) holds. The state x^+ after reset is either an element of $\mathfrak{D}$ or $\mathfrak{C}_i$. In case $x^+ \in \mathfrak{D}$, (24b) is equivalent to

$$x^T(\mathcal{A}_R^T P_R \mathcal{A}_R - P_R)x \leq 0 \qquad \text{if } x, x^+ \in \mathfrak{D}. \tag{30}$$

In case $x^+ \in \mathfrak{C}_i$, (24c) implies that

$$x^T(\mathcal{A}_R^T P_i \mathcal{A}_R - P_R)x \leq 0 \qquad \text{if } x \in \mathfrak{D}, x^+ \in \mathfrak{C}_i. \tag{31}$$

The combination of (30) and (31) is equivalent to (25b), which completes the proof. ■

Remark 8 Theorem 7 is similar to the result in [11]. However, our result is applicable to all possible LTI plants and reset controllers which fit the augmented plant description, as long as $D_w = 0$. In contrast to [11] it can cope with measurement noise attenuation and tracking problems, as long as these inputs are filtered with a strictly proper filter.□

IV. Example

In order to illustrate the derived analysis method the above result is applied to a simulation example. The system used in this example is taken from [10], [17], and extended with a tracking problem. Consider a second order LTI plant, represented by $G(s) = \frac{s+1}{s(s+0.2)}$. This system should track a step reference $r(t) = 1(t)$, which can be represented by the input filter $W_r(s) = \frac{1}{s+\varepsilon}$. Similarly as in [10], [17], we allow only first order low-pass controllers K to achieve this. The goal is to compare the tracking performance of a linear controller of this form, i.e. $K(s) = \frac{1}{s+\beta}$, where β is a free variable, with its resetting counterpart, also known as a FORE. This reset controller K is characterized by

$$A_K = \beta, \quad B_K = C_K = 1, \quad D_K = A_r = 0,$$

yielding closed-loop matrices of the form

$$\mathcal{A} = \begin{bmatrix} A & B \\ C & \beta \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B_w \\ 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \quad \mathcal{D} = 0.$$

The closed-loop layout is shown in Figure 6. The $\mathcal{L}_2$ gain

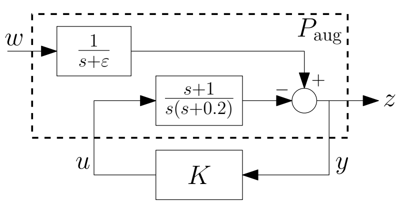

Fig. 6. Closed-loop layout.

of both the linear and the reset closed-loop are compared in Figure 7 for varying pole values β. The linear control curve is obtained using standard $\mathcal{H}_\infty$ techniques, i.e. by minimizing γ in the LMI

$$\begin{bmatrix} \mathcal{A}^T P + P\mathcal{A} & P\mathcal{B} & \mathcal{C}^T \\ \mathcal{B}^T P & -\gamma I & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & -\gamma I \end{bmatrix} \prec 0, \tag{32}$$

where $P \succ 0$. Theorem 7 is used for the reset control curve (with $N = 100$). The $\mathcal{L}_2$ gains for this tracking problem differ significantly from the result in [10], which only considered disturbance rejection.

Figure 7 does not necessarily imply that reset control outperforms any linear controller in $\mathcal{H}_\infty$ sense. We have restricted ourselves to linear controllers K with transfer function $\frac{1}{s+\beta}$, but of course one can find other controllers with lower $\mathcal{L}_2$ gains (e.g. with larger controller gain or higher order).

V. Conclusions

In this paper we have derived a set of LMIs with which the $\mathcal{L}_2$ gain of any reset control system which fits into a augmented plant description can be upperbounded, thereby generalizing the work in [11]. Our analysis can also be applied to tracking and measurement noise problems. Conservatism in these cases is removed by including strictly proper input filters which enables the use of PWQ storage functions.

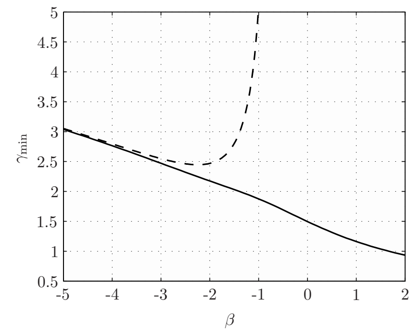

Fig. 7. Estimated closed-loop $\mathcal{L}_2$ gains using linear (dashed) and reset control (solid), as a function of β.

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References

[1] R. Middleton, "Trade-offs in linear control system design," *Automatica*, vol. 27, no. 2, pp. 281–292, March 1991.

[2] M. Seron, J. Braslavsky, and G. Goodwin, *Fundamental Limitations in Filtering and Control*. London: Springer-Verlag, 1997.

[3] A. Feuer, G. Goodwin, and M. Saldago, "Potential benefits of hybrid control for linear time invariant plants," in *Proc. Amer. Contr. Conf.*, Albuquerque, NM, June 1997, pp. 2790–2794.

[4] J. Clegg, "A nonlinear integrator for servomechanisms," *Transactions A.I.E.E. Part II*, vol. 77, pp. 41–42, March 1958.

[5] K. Krishnan and I. Horowitz, "Synthesis of a non-linear feedback system with significant plant-ignorance for prescribed system tolerances," *Int. J. Contr.*, vol. 19, pp. 689–706, April 1974.

[6] I. Horowitz and P. Rosenbaum, "Non-linear design for cost of feedback reduction in systems with large parameter uncertainty," *Int. J. Contr.*, vol. 21, pp. 977–1001, June 1975.

[7] Y. Chait and C. Hollot, "On Horowitzs contributions to reset control," *Int. J. Rob. Nonlin. Contr.*, vol. 12, no. 4, pp. 335–355, 2002.

[8] O. Beker, C. Hollot, Y. Chait, and H. Han, "Fundamental properties of reset control systems," *Automatica*, vol. 40, no. 6, pp. 905–915, 2004.

[9] G. Witvoet, W. Aangenent, W. Heemels, M. van de Molengraft, and M. Steinbuch, "$\mathcal{H}_2$ performance analysis of reset control systems," in *Proc. 46th Conf. Decis. Contr.*, New Orleans, LA, December 2007, pp. 3278–3284.

[10] D. Nešić, L. Zaccarian, and A. Teel, "Stability properties of reset systems," in *IFAC World Congress*, Prague (CZ), 2005.

[11] L. Zaccarian, D. Nešić, and A. Teel, "First order reset elements and the Clegg integrator revisited," in *Proc. Amer. Contr. Conf.*, Portland, OR, June 2005, pp. 563–568.

[12] ——, "Set-point stabilization of SISO linear systems using first order reset elements," in *Proc. Amer. Contr. Conf.*, New York City, NY, July 2007, pp. 5808–5809.

[13] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994.

[14] J. Willems, "Dissipative dynamical systems, part I: General theory," *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 321–351, January 1972.

[15] W. Haddad, "Nonlinear impulsive dynamical systems, part i: stability and dissipativity," *Int. Journal of Control*, vol. 74, no. 17, pp. 1631–1658, 2001.

[16] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 43, no. 4, pp. 555–559, April 1998.

[17] C. Hollot, O. Beker, Y. Chait, and Q. Chen, "On establishing classic performance measures for reset control systems," in *Perspectives in Robust Control*, S. Moheimani, Ed. Springer-Verlag, April 2001, pp. 123–148.